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An extension of the univalence criteria of Nehari and Ozaki

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Abstract

In this paper, we obtain a sufficient condition for the univalence of analyticfunctions in the open unit disk $\mathbb{U}$. *This condition involves two arbitrary functions* $g(z)$ *and* $h(z)$ *analytic in* $\mathbb{U}$. *Replacing* $g(z)$ *and* $h(z)$ *by some particular functions, we find the well-known conditions for univalency established by Z.Nehari* (Bull. Amer. Math. Soc. **55**(1949)) *and S.Ozaki* (Proc. Amer. Math. Sco. **33**(1972)). *Likewise we find other new sufficient conditions.*

1 Introduction

We denote by $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$ the disk of z-plane, where $r \in (0,1]$, $\mathbb{U}_1 = \mathbb{U}$ and $I = [0, \infty)$. Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in $\mathbb{U}$ with the normalizations $f(0) = 0$ and $f'(0) = 1$. In the present paper, we consider the following conditions for univalency of functions $f(z)$ belonging to the class $\mathcal{A}$.

Theorem 1.1. ([1]) *Let* $f(z) \in \mathcal{A}$. *If, for all* $z \in \mathbb{U}$, $f(z)$ *satisfies*

$$|\{f; z\}| \leqq \frac{2}{(1-|z|^2)^2}, \tag{1.1}$$

where

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2},\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{1.2}$$

then the function $f(z)$ *is univalent in* $\mathbb{U}$.

Theorem 1.2. ([2]) *Let* $f(z) \in \mathcal{A}$. *If, for all* $z \in \mathbb{U}$, $f(z)$ *satisfies*

$$\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| < 1, \tag{1.3}$$

then the function $f(z)$ *is univalent in* $\mathbb{U}$.

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Example 1.1. *If we take Koebe functon* $f(z) = \dfrac{z}{(1-z)^2}$ *which is the extremal function for the class of starlike functions in* $\mathbb{U}$*, then*

$$\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| = |-z^2| < 1 \qquad (z \in \mathbb{U}).$$

2 Preliminaries

Our considerations are based on the theory of Löwner chains. We first recall here the following basic result of this theory by Pommerenke.

Theorem 2.1. ([4]) *Let* $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots,\ a_1(t) \neq 0$ *be analytic in* $\mathbb{U}_r$ *for all* $t \in I$*, locally absolutely continuous in* I*, and locally uniform with respect to* $\mathbb{U}_r$*. For almost all* $t \in I$ *suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \qquad (\forall z \in \mathbb{U}_r),$$

where $p(z,t)$ *is analytic in* $\mathbb{U}$ *and satisfies the condition* $\operatorname{Re} p(z,t) > 0$ *for all* $z \in \mathbb{U},\ t \in I$*. If* $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and* $\{L(z,t)/a_1(t)\}$ *forms a normal family in* $\mathbb{U}_r$*, then, for each* $t \in I$*, the function* $L(z,t)$ *has an analytic and univalent extension to the whole disk* $\mathbb{U}$*.*

3 Main results

Main theorem of our paper is contained in

Theorem 3.1. *Let* $f(z) \in \mathcal{A}$*. If, for some analytic functions* $g(z) = 1 + b_1 z + \dots$ *and* $h(z) = c_0 + c_1 z + \dots$ *in* $\mathbb{U}$*, the following inequalities*

$$\left|\frac{f'(z)}{g(z)} - 1\right| < 1, \tag{3.1}$$

and

$$\left|\left(\frac{f'(z)}{g(z)} - 1\right)|z|^4 + z(1-|z|^2)|z|^2\left(2\frac{f'(z)h(z)}{g(z)} + \frac{g'(z)}{g(z)}\right)\right.$$
$$\left. + z^2(1-|z|^2)^2\left(\frac{f'(z)h(z)^2}{g(z)} + \frac{g'(z)h(z)}{g(z)} - h'(z)\right)\right| \leqq |z|^2 \tag{3.2}$$

hold true for all $z \in \mathbb{U}$*, then the function* $f(z)$ *is univalent in* $\mathbb{U}$*.*

Proof. Let us consider the function $h_1(z,t)$ given by

$$h_1(z,t) = 1 + (e^t - e^{-t})zh(e^{-t}z).$$

For all $t \in I$ and $z \in \mathbb{U}$ we have $e^{-t}z \in \mathbb{U}$ and from the analyticity of $h(z)$ in $\mathbb{U}$ it follows that $h_1(z,t)$ is also analytic in $\mathbb{U}$. Since $h_1(0,t) = 1$, there exists a disk $\mathbb{U}_r$, $0 < r < 1$ in which $h_1(z,t) \neq 0$ for all $t \in I$. Then the function $L(z,t)$ defined by

$$L(z,t) = f(e^{-t}z) + \frac{(e^t - e^{-t})zg(e^{-t}z)}{1 + (e^t - e^{-t})zh(e^{-t}z)}$$

is analytic in $\mathbb{U}_r$ for all $t \in I$ and has the following form

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots ,$$

where $a_1(t) = e^t$, $a_1(t) \neq 0$ for all $t \in I$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.
From the analyticity of $L(z,t)$ in $\mathbb{U}_r$, it follows that there exists a number r_1, $0 < r_1 < r$, and a constant $K = K(r_1)$ such that

$$| L(z,t)/a_1(t) | < K \qquad (\forall z \in \mathbb{U}_{r_1}, t \in I).$$

In consequence, the family $\{ L(z,t)/a_1(t) \}$ is normal in $\mathbb{U}_{r_1}$. From the analyticity of $\dfrac{\partial L(z,t)}{\partial t}$, for all fixed numbers $T > 0$ and r_2, $0 < r_2 < r_1$, there exists a constant $K_1 > 0$ (that depends on T and r_2) such that

$$\left| \frac{\partial L(z,t)}{\partial t} \right| < K_1 \qquad (\forall z \in \mathbb{U}_{r_2}, t \in [0,T]).$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to $\mathbb{U}_{r_2}$. Let us define the functions $p(z,t)$ and $w(z,t)$ by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}$$

and

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}.$$

Then the function $p(z,t)$ is analytic in $\mathbb{U}_{r_3}$, $0 < r_3 < r_2$, and the function $p(z,t)$ has an analytic extension with positive real part in $\mathbb{U}$, for all $t \in I$, if the function $w(z,t)$ can be continued analytically in $\mathbb{U}$ and $|w(z,t)| < 1$ for all $z \in \mathbb{U}$ and $t \in I$.
After simple computation, we obtain that

$$w(z,t) = \left(\frac{f'(e^{-t}z)}{g(e^{-t}z)} - 1\right) e^{-2t} + (1 - e^{-2t})e^{-t}z \left(\frac{2f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)}{g(e^{-t}z)}\right)$$

$$+(1 - e^{-2t})^2 z^2 \left(\frac{f'(e^{-t}z)h(e^{-t}z)^2}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z)\right). \tag{3.3}$$

From (3.1) and (3.2), we deduce that $g(z) \neq 0$ for all $z \in \mathbb{U}$ and then the function $w(z,t)$ is analytic in $\mathbb{U}$. In view of (3.1) and (3.3), we have

$$w(0,t) = 0 \qquad \text{and} \qquad |w(z,0)| = \left| \frac{f'(z)}{g(z)} - 1 \right| < 1 . \tag{3.4}$$

If $t > 0$ is a fixed number and $z \in \mathbb{U}$, $z \neq 0$, then the function $w(z,t)$ is analytic in $\bar{\mathbb{U}}$ because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{\mathbb{U}}$, and it is known that

$$|w(z,t)| = \max_{|\zeta|=1} |w(\zeta,t)| = |w(e^{i\theta},t)|, \quad \theta = \theta(t) \in \mathcal{R}. \tag{3.5}$$

Let us denote by $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and, from (3.3), we get

$$|w(e^{i\theta},t)| = \left| \left(\frac{f'(u)}{g(u)} - 1\right) |u|^2 + (1-|u|^2)u\left(\frac{2f'(u)h(u)}{g(u)} + \frac{g'(u)}{g(u)}\right)\right.$$
$$\left. +(1-|u|^2)^2\frac{u^2}{|u|^2}\left(\frac{f'(u)h(u)^2}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u)\right)\right| .$$

Since $u \in \mathbb{U}$, the relation (3.2) implies $|w(e^{i\theta},t)| \le 1$ and, from (3.4) and (3.5), we conclude that $|w(z,t)| < 1$ for all $z \in \mathbb{U}$ and $t \in I$. This gives us that $L(z,t)$ is the Löwner chain and hence the function $L(z,0) = f(z)$ is univalent in $\mathbb{U}$. □

We can get some corollaries for special cases of functions $g(z)$ and $h(z)$. So in the particular case $g(z) = f'(z)$ as a direct consequence of Theorem 3.1, we get

Theorem 3.2. *Let $f \in \mathcal{A}$. If, for an analytic function $h(z) = c_0 + c_1 z + \ldots$ in $\mathbb{U}$, $f(z)$ satisfies*

$$\left| (1-|z|^2)|z|^2\left(2h(z) + \frac{f''(z)}{f'(z)}\right)\right.$$

$$\left. +z(1-|z|^2)^2\left(h(z)^2 + \frac{f''(z)h(z)}{f'(z)} - h'(z)\right)\right| \leqq |z| \tag{3.6}$$

for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

If we take

$$h(z) = -\frac{1}{2}\frac{f''(z)}{f'(z)} \tag{3.7}$$

in Theorem 3.2, then we have

Corollary 3.1. ([1]) *If $f(z) \in \mathcal{A}$ satisfies the inequality* (1.1) *for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.*

Proof. For the function $h(z)$ defined by (3.7), the Schwartzian derivative (1.2) shows that

$$h(z)^2 + \frac{f''(z)h(z)}{f'(z)} - h'(z) = \frac{1}{2}\left[\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right] = \frac{1}{2}\{f;z\}.$$

and then the inequality (3.6) becomes (1.1). □

In the particular case $g(z) = \left(\frac{f(z)}{z}\right)^2$ in Theorem 3.1, we have

Theorem 3.3. *Let $f(z) \in \mathcal{A}$. If, for an analytic function $h(z) = c_0 + c_1 z + \ldots$ in $\mathbb{U}$, $f(z)$ satisfies*

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 1 \tag{3.8}$$

and

$$\left| \left(\frac{z^2 f'(z)}{f(z)^2} - 1 \right) |z|^4 + 2z(1-|z|^2)|z|^2 \left(\frac{z^2 f'(z) h(z)}{f(z)^2} + \frac{f'(z)}{f(z)} - \frac{1}{z} \right) \right.$$

$$\left. + z^2(1-|z|^2)^2 \left[\frac{z^2 f'(z) h(z)^2}{f(z)^2} + 2h(z) \left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) - h'(z) \right] \right| \leqq |z|^2 \tag{3.9}$$

for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

We remark that the inequality (3.8) is just the inequality (1.3) and we will get Ozaki's univalent criterion for a particular choise of the function $h(z)$. So, if we take in Theorem 3.3

$$h(z) = \frac{1}{z} - \frac{f(z)}{z^2}, \tag{3.10}$$

then we obtain

Corollary 3.2. ([2]) *If $f(z) \in \mathcal{A}$ satisfies the inequality (1.3) for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.*

Proof. For the function $h(z)$ defined by (3.10), we see that

$$\frac{z^2 f'(z) h(z)}{f(z)^2} + \frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{z f'(z)}{f(z)^2} - \frac{1}{z}$$

and

$$\frac{z^2 f'(z) h(z)^2}{f(z)^2} + 2h(z) \left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) - h'(z) = \frac{f'(z)}{f(z)^2} - \frac{1}{z^2}.$$

The inequality (3.9) becomes

$$\left| \left(\frac{z^2 f'(z)}{f(z)^2} - 1 \right) \left(|z|^4 + 2|z|^2(1-|z|^2) + (1-|z|^2)^2 \right) \right| \leqq |z|^2,$$

and then

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| \leqq |z|^2. \tag{3.11}$$

It is easy to prove that if the inequality (1.3) is true, then the inequality (3.11) is also true. Indeed, if we put

$$w(z) = \frac{z^2 f'(z)}{f(z)^2} - 1 \,,$$

then the function $w(z)$ is analytic in $\mathbb{U}$ and, since $f(z) \in \mathcal{A}$, we observe that

$$w(z) = d_2 z^2 + d_3 z^3 + \dots \,,$$

which shows that $w(0) = w'(0) = 0$. By inequality (1.3), we have $|w(z)| < 1$. Thus th Schwartz's lemma gives us that $|w(z)| < |z|^2$. □

Finally, we give a example for Corollary 3.2.

Example 3.1. *Let us consider the function $f(z)$ given by*

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} \frac{1}{n(n^2-1)} z^n}.$$

Then we have that

$$\frac{z^2 f'(z)}{f(z)^2} - 1 = -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} z^n \,,$$

which gives that

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1.$$

Therefore, the function $f(z)$ is univalent in $\mathbb{U}$.

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