数理解析研究所講究録 1341 巻 2003 年 31-44



# SOME FAMILIES OF MEROMORPHIC MULTIVALENT FUNCTIONS INVOLVING CERTAIN LINEAR OPERATOR

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**Abstract.** Let $\sum_p$ denote the class of functions of the form $f(z)=z^{-p}+\sum_{k=0}^{\infty}a_kz^k$ $(p\in N=\{1,2,3,\cdots\})$ which are analytic and $p-$valent in the punctured disc $D=\{z:0<|z|<1\}$. We introduce and study some new families of meromorphic multivalent functions defined by certain linear operator. A number of useful characteristics of functions in these families are obtained. In particular, some properties of neighborhoods of functions in these families are given.

**Key words and phrases.** Meromorphic, neighborhood, operator, partial sum.
**2000 Mathematics Subject Classification. Primary 30C45; Secondary 30D30, 33C20.**

## 1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z)=z^{-p}+\sum_{k=0}^{\infty}a_kz^k \quad (p\in N=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic and $p-$valent in the punctured disc $D=\{z:0<|z|<1\}$. Define a linear operator as following

$$D^0f(z)=f(z)$$

$$D^1 f(z) = z^{-p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + \cdots$$

$$= \frac{(z^{p+1} f(z))'}{z^p}$$

$$D^2 f(z) = D(D^1 f(z))$$

and for $n = 1,2,\cdots$

$$D^n f(z) = D(D^{n-1} f(z)) = z^{-p} + \sum_{k=0}^{\infty} (p+k+1)^n a_k z^k$$

$$= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}. \tag{1.2}$$

It is easy to see that

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z) . \tag{1.3}$$

When $p = 1$, Uralegaddi and Somanatha[20] investigated certain properties of the operator $D^n$. Recently, Aouf and Hossen [2] showed some results of the operator $D^n$ for $p \in N = \{1,2,3,\cdots\}$.

Let $-1 \le B < A \le 1$. A function $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \sum_p$ is said to be in the class $T_n(A,B)$ if it satisfies the condition

$$\left| \frac{z(D^n f(z))' + pD^n f(z)}{Bz(D^n f(z))' + ApD^n f(z)} \right| < 1 \tag{1.4}$$

for all $z \in E = \{z : |z| < 1\}$.

Furthermore, a function $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \in \sum_p$ is said to be in the class $T_n^*(A,B)$ if it satisfies the condition (1.4).

It should be remarked in passing that the definition (1.4) is motivated essentially by the recent work of Mogra [13] and Liu and Srivastava [12]. The special class $T_0(A,B)$ was studied by Mogra [13]. Another subclass associated with the linear operator $D^n$ was considered recently by Liu and Srivastava[12].

In recent years, many important properties and characteristics of various interesting subclasses of the class $\sum_p$ of meromorphically $p-$valent functions were investigated extensively by (among others) Aouf et al. ([2] and [3]), Joshi and Srivastava [8], Kulkarni et al.[9], Liu and Srivastava ([10],[11] and [12]), Mogra ([13] and [14]), Owa et al.[15], Srivastava et al.[18], Uralegaddi and Somanatha ([20] and [21]), and Yang ([22]). The main object of this paper is to present several inclusion and other properties of functions in the classes $T_n(A,B)$ and $T_n^*(A,B)$ which we have introduced here. We also apply the familiar concept of neighborhoods of analytic functions (see [6] and [16]) to meromorphically $p-$valent functions in the class $\sum_p$ .

## 2. Properties of the class $T_n(A,B)$

In proving our results, we shall need the following lemma which is due to Jack [7].

**Lemma.** Let $w(z)$ be non-constant analytic in $E=\{z:|z|<1\}$, $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at $z_0$, we have $z_0w'(z_0)=kw(z_0)$, where $k$ is a real number and $k\geq 1$.

**Theorem 2.1.** Let $1+B\geq p(A-B)$, then $T_{n+1}(A,B)\subset T_n(A,B)$.

**Proof.** Let $f(z)\in T_{n+1}(A,B)$. Suppose that

$$\frac{z(D^n f(z))'}{D^n f(z)}=-p\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.1}$$

where $w(0)=0$.

By using (2.1) and (1.3), we have

$$\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)}=-p\frac{1+Aw(z)}{1+Bw(z)}-\frac{p(A-B)zw'(z)}{(1+Bw(z))\{1+[B-p(A-B)]w(z)\}}. \tag{2.2}$$

Suppose now that, for $z_0\in E$, $\max\limits_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$. Applying Lemma, we have $z_0w'(z_0)=kw(z_0)$, $k\geq 1$. Writing $w(z_0)=e^{i\theta}$ and putting $z=z_0$ in (2.2), we obtain

$$\left|\frac{z(D^{n+1} f(z))'+pD^{n+1} f(z)}{Bz(D^{n+1} f(z))'+ApD^{n+1} f(z)}\right|^2_{z=z_0}-1$$

$$=\left|\frac{(k+1)+[B-p(A-B)]e^{i\theta}}{1-[B(k-1)+p(A-B)]e^{i\theta}}\right|^2-1$$

$$=\frac{H_1(\cos\theta)}{|1-[B(k-1)+p(A-B)]e^{i\theta}|^2}, \qquad (2.3)$$

where

$$H_1(\cos\theta)=k^2(1-B^2)+2k[1+B^2-pB(A-B)]+2k[2B-p(A-B)]\cos\theta.$$

Since $1+B\geq p(A-B)$, then

$$H_1(1)=k^2(1-B^2)+2k(1+B)[(1+B)-p(A-B)]\geq 0$$

and

$$H_1(-1)=k^2(1-B^2)+2k(1-B)[(1-B)+p(A-B)]\geq 0.$$

This shows that $H(\cos\theta)\geq 0$ for all $\theta$ $(0\leq\theta<2\pi)$ and it follows that (2.3) contradicts the hypothesis $f(z)\in T_{n+1}(A,B)$. Hence $|w(z)|<1$ for all $z\in E$ and (2.1) shows that $f(z)\in H_n(A,B)$. □

**Theorem 2.2** Let $\alpha>0$ and let $\lambda$ be a complex number such that $\operatorname{Re}\lambda\geq p\alpha\frac{1+A}{1+B}$. If $f(z)\in T_n(A,B)$, then the function $g(z)$ defined by

$$D^n g(z)=\left\{\frac{\lambda-p\alpha}{z^\lambda}\int_0^z t^{\lambda-1}(D^n f(t))^\alpha\,dt\right\}^{\frac{1}{\alpha}} \qquad (2.4)$$

also belongs to $T_n(A,B)$.

**Proof.** Put

$$\frac{z(D^n g(z))'}{D^n g(z)}=-p\frac{1+Aw(z)}{1+Bw(z)}, \qquad (2.5)$$

where $w(0)=0$.

Using (2.4) and (2.5) together with some computations, it follows that

$$\frac{z(D^n f(z))'}{D^n f(z)}=-p\frac{1+Aw(z)}{1+Bw(z)}-\frac{p(A-B)zw'(z)}{(1+Bw(z))[(\lambda-p\alpha)+(B\lambda-Ap\alpha)w(z)]}. \qquad (2.6)$$

The remaining part of the proof of Theorem 2.2 is similar to that of Theorem 2.1 and so is omitted. □

## 3. Properties of the class $T_n^*(A,B)$

In this section, we assume that $A+B\le 0$.

**Theorem 3.1** Let $f(z)=z^{-p}+\sum_{k=p}^{\infty}|a_k|z^k$ be analytic and $p-$valent in $D=\{z:0<|z|<1\}$. Then $f(z)\in T_n^*(A,B)$ if and only if

$$\sum_{k=p}^{\infty}[p(1-A)+k(1-B)](p+k+1)^n|a_k|\le p(A-B). \qquad (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-p}+\frac{p(A-B)}{(p+k+1)^n[p(1-A)+k(1-B)]}z^k \qquad (k\ge p). \qquad (3.2)$$

**Proof.** Let $f(z)=z^{-p}+\sum_{k=p}^{\infty}|a_k|z^k\in T_n^*(A,B)$. Then

$$\left|\frac{z(D^nf(z))'+pD^nf(z)}{Bz(D^nf(z))'+ApD^nf(z)}\right|=\left|\frac{\sum_{k=p}^{\infty}(p+k)(p+k+1)^n|a_k|z^{k+p}}{p(A-B)+\sum_{k=p}^{\infty}(Ap+Bk)(p+k+1)^n|a_k|z^{k+p}}\right|$$

$$<1. \qquad (3.3)$$

Since $|\operatorname{Re} z|\le|z|$ for any $z$, choosing $z$ to be real and letting $z\to 1^-$ through real values, (3.3) yields

$$\sum_{k=p}^{\infty}(p+k)(p+k+1)^n|a_k|\le p(A-B)+\sum_{k=p}^{\infty}(Ap+Bk)(p+k+1)^n|a_k|,$$

which gives (3.1).

On the other hand, we have that

$$\left|\frac{z(D^nf(z))'+pD^nf(z)}{Bz(D^nf(z))'+ApD^nf(z)}\right|\le\frac{\sum_{k=p}^{\infty}(p+k)(p+k+1)^n|a_k|}{p(A-B)+\sum_{k=p}^{\infty}(Ap+Bk)(p+k+1)^n|a_k|}$$

$$<1.$$

This shows that $f(z)\in T_n^*(A,B)$.□

Next, we prove the following growth and distortion property for the class $T_n^*(A,B)$.

**Theorem 3.2** Let $0\le m<p$. If $f(z)\in T_n^*(A,B)$, then for $0<|z|=r<1$,

$$\left\{\frac{(p+m-1)!}{(p-1)!}-\frac{(A-B)p!}{(2-(A+B))(2p+1)^n(p-m)!}r^{2p}\right\}r^{-(p+m)}$$

$$\le |f^{(m)}(z)|$$

$$\le\left\{\frac{(p+m-1)!}{(p-1)!}+\frac{(A-B)p!}{(2-(A+B))(2p+1)^n(p-m)!}r^{2p}\right\}r^{-(p+m)} .$$

The results are sharp for the function $f(z)$ given by

$$f(z)=z^{-p}+\frac{A-B}{(2-(A+B))(2p+1)^n}z^p .$$

**Proof.** Let $f(z)\in T_n^*(A,B)$. Then we find from Theorem 3.1 that

$$\frac{p(2-(A+B))(2p+1)^n}{p!}\sum_{k=p}^{\infty}k!|a_k|\le\sum_{k=p}^{\infty}[p(1-A)+k(1-B)](p+k+1)^n|a_k|$$

$$\le p(A-B),$$

which yields

$$\sum_{k=p}^{\infty}k!|a_k|\le\frac{(A-B)p!}{(2-(A+B))(2p+1)^n} . \tag{3.4}$$

Now by differentiating $f(z)$ $m$ times, we have

$$f^{(m)}(z)=(-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-p-m}+\sum_{k=p}^{\infty}\frac{k!}{(k-m)!}|a_k|z^{k-m} . \tag{3.5}$$

Hence Theorem 3.2 would follow from (3.4) and (3.5).□

Finally, we determine the radius of meromorphically $p$-valent starlikeness and convexity for functions in the class $T_n^*(A,B)$.

**Theorem 3.3** Let $f(z)\in T_n^*(A,B)$. Then

(i) $f(z)$ is meromorphically $p$-valent starlike of order $\delta$ in $|z|<r_1$, that is

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<-p\delta \qquad (|z|<r_1) \tag{3.6}$$

where $0\le\delta<1$ and

$$r_1 = \inf_{k \ge p} \left\{ \frac{(1-\delta)[p(1-A)+k(1-B)](p+k+1)^n}{(A-B)(k+p\delta)} \right\}^{\frac{1}{k+p}} .$$

(ii) $f(z)$ is meromorphically $p-$valent convex of order $\delta$ in $|z|<r_2$, that is

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < -p\delta \qquad (|z|<r_2) , \tag{3.7}$$

where $0 \le \delta < 1$ and

$$r_2 = \inf_{k \ge p} \left\{ \frac{p(1-\delta)[p(1-A)+k(1-B)](p+k+1)^n}{k(k+p\delta)(A-B)} \right\}^{\frac{1}{k+p}} .$$

Each of these results is sharp for the function $f(z)$ given by (3.2).

**Proof.** (i) From Theorem 3.1, we have

$$\sum_{k=p}^{\infty} \frac{k+p\delta}{p(1-\delta)} |a_k| |z|^{k+p} < \sum_{k=p}^{\infty} \frac{[p(1-A)+k(1-B)](p+k+1)^n}{p(A-B)} |a_k|$$

$$\le 1 \qquad (|z|<r_1).$$

Therefore for $|z|<r_1$

$$\left| \frac{zf'(z)/f(z)+p}{zf'(z)/f(z)-p(1-2\delta)} \right| \le \frac{\sum_{k=p}^{\infty} (k+p)|a_k||z|^{k+p}}{2p(1-\delta)-\sum_{k=p}^{\infty}[k-p(1-2\delta)]|a_k||z|^{k+p}}$$

$$<1 ,$$

which shows that (3.6) is true.

(ii) It follows from Theorem 3.1 that

$$\sum_{k=p}^{\infty} \frac{k(k+p\delta)}{p^2(1-\delta)} |a_k| |z|^{k+p} < \sum_{k=p}^{\infty} \frac{[p(1-A)+k(1-B)](p+k+1)^n}{p(A-B)} |a_k|$$

$$\le 1 \qquad (|z|<r_2).$$

Thus for $|z|<r_2$, we obtain

$$\left| \frac{1+zf''(z)/f'(z)+p}{1+zf''(z)/f'(z)-p(1-2\delta)} \right| \le \frac{\sum_{k=p}^{\infty} k(k+p)|a_k||z|^{k+p}}{2p^2(1-\delta)-\sum_{k=p}^{\infty}k[k-p(1-2\delta)]|a_k||z|^{k+p}}$$

$$<1,$$

which shows that (3.7) is true.

Sharpness can be verified easily. □

## 4. Neighborhoods and partial sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman[6] and Ruscheweyh[16], and(more recently) by Altintas and Owa [1] and Liu and Srivastava ([10] and [12]), we begin by introducing here the $\delta$ – neighborhood of a function $f \in \sum_p$ of the form (1.1) by means of the definition:

$$N_\delta(f) = \left\{ g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k \in \sum_p : \right.$$

$$\left. \sum_{k=0}^{\infty} \frac{[p(1-A)+k(1-B)]}{p(A-B)} (p+k+1)^n \,|\, b_k - a_k \,| \le \delta, -1 \le B < A \le 1; \delta \ge 0 \right\}.$$

Making use of the definition, we now prove

**Theorem 4.1** Let $\delta > 0$ and $-1 < A \le 0$. If $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \sum_p$ satisfies the condition

$$\frac{f(z)+\varepsilon z^{-p}}{1+\varepsilon} \in T_n(A,B) \tag{4.1}$$

for any complex number $\varepsilon$ such that $|\varepsilon| < \delta$, then $N_\delta(f) \subset T_n(A,B)$.

**Proof.** It is obvious from (1.4) that $g(z) \in T_n(A,B)$ if and only if for any complex number $\sigma$ with $|\sigma| = 1$

$$\frac{z(D^n g(z))' + pD^n g(z)}{Bz(D^n g(z))' + ApD^n g(z)} \ne \sigma \qquad (z \in E),$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-p}} \ne 0 \qquad (z \in E), \tag{4.2}$$

where

$$h(z) = z^{-p} + \sum_{k=0}^{\infty} c_k z^k$$

$$= z^{-p} + \sum_{k=0}^{\infty} \frac{[(p+k) - \sigma(pA+kB)]}{p\sigma(B-A)} (p+k+1)^n z^k \tag{4.3}$$

From (4.3), we have

$$|c_k| = \left|\frac{[(p+k)-\sigma(pA+kB)]}{p\sigma(B-A)}(p+k+1)^n\right|$$

$$\le \frac{p(1-A)+k(1-B)}{p(A-B)}(p+k+1)^n .$$

If $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \sum_p$ satisfies the condition (4.1), then (4.2) yields

$$\left|\frac{f(z)*h(z)}{z^{-p}}\right| \ge \delta \qquad (z \in E). \tag{4.4}$$

Now let $p(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k \in N_\delta(f)$, then

$$\left|\frac{(p(z)-f(z))*h(z)}{z^{-p}}\right| = \left|\sum_{k=0}^{\infty}(b_k - a_k)c_k z^{k+p}\right|$$

$$\le |z| \sum_{k=0}^{\infty} \frac{[p(1-A)+k(1-B)]}{p(A-B)}(p+k+1)|b_k - a_k|$$

$$< \delta .$$

Thus for any complex number $\sigma$ such that $|\sigma| = 1$, we have

$$\frac{p(z)*h(z)}{z^{-p}} \ne 0 \qquad (z \in E),$$

which implies that $p(z) \in T_n(A,B)$.□

**Theorem 4.2** Let $-1 < A \le 0$. Let $f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \in \sum_p$ , $s_1(z) = z^{-p}$

and $s_m(z) = z^{-p} + \sum_{k=0}^{m-2} a_k z^k \quad (m \ge 2)$. Suppose that

$$\sum_{k=0}^{\infty} c_k |a_k| \le 1 \tag{4.5}$$

where

$$c_k = \frac{p(1-A)+k(1-B)}{p(A-B)}(p+k+1)^n .$$

Then we have

(i) $f(z) \in T_n(A,B)$;

(ii) $\mathrm{Re}\left\{\dfrac{f(z)}{s_m(z)}\right\} > 1-\dfrac{1}{c_{m-1}}$ (4.6)

and

$$\mathrm{Re}\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{c_{m-1}}{1+c_{m-1}} . \tag{4.7}$$

The estimates are sharp.

**Proof.** (i) It is obvious that $z^{-p} \in T_n(A,B)$. Thus from Theorem 4.1 and the condition (4.5), we have $N_1(z^{-p}) \subset T_n(A,B)$. This gives $f(z) \in T_n(A,B)$.

(ii)It is easy to see that $c_{k+1} > c_k > 1$. Thus

$$\sum_{k=0}^{m-2} |a_k| + c_{m-1}\sum_{k=m-1}^{\infty} |a_k| \le \sum_{k=0}^{\infty} c_k |a_k| \le 1 . \tag{4.8}$$

Let

$$h_1(z) = c_{m-1}\left\{\frac{f(z)}{s_m(z)} - (1-\frac{1}{c_{m-1}})\right\}$$

$$= 1 + \frac{c_{m-1}\sum_{k=m-1}^{\infty} a_k z^{k+p}}{1+\sum_{k=0}^{m-2} a_k z^{k+p}} .$$

It follows from (4.8) that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \le \frac{c_{m-1}\sum_{k=m-1}^{\infty}|a_k|}{2-2\sum_{k=0}^{m-2}|a_k| - c_{m-1}\sum_{k=m-1}^{\infty}|a_k|} \le 1 \qquad (z \in E).$$

From this we obtain the inequality (4.6).

If we take

$$f(z) = z^{-p} - \frac{z^{m-1}}{c_{m-1}} , \tag{4.9}$$

then

$$\frac{f(z)}{s_m(z)} = 1 - \frac{z^{p+m-1}}{c_{m-1}} \to 1 - \frac{1}{c_{m-1}} \qquad \text{as} \quad z \to 1^- .$$

This shows that the bound in (4.6) is best possible for each $m$.

Similarly, if we take

$$h_2(z) = (1+c_{m-1})\left\{\frac{s_m(z)}{f(z)} - \frac{c_{m-1}}{1+c_{m-1}}\right\}$$

$$= 1 - \frac{(1+c_{m-1})\sum_{k=m-1}^{\infty} a_k z^{k+p}}{1+\sum_{k=0}^{\infty} a_k z^{k+p}},$$

then we deduce that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \le \frac{(1+c_{m-1})\sum_{k=m-1}^{\infty} |a_k|}{2-2\sum_{k=0}^{m-2}|a_k| + (1-c_{m-1})\sum_{k=m-1}^{\infty}|a_k|}$$

$$\le 1 \qquad (z \in E),$$

which yields (4.7). The estimate (4.7) is sharp with the extremal function $f(z)$ given by (4.9).□

For $\delta \ge 0, -1 \le B < A \le 1$ and $f(z) = z^{-p} + \sum_{k=0}^{\infty} |a_k| z^k \in \sum_p$, we define neighborhood of $f(z)$ by

$$N_\delta^*(f) = \left\{ g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \in \sum_p : \right.$$

$$\left. \sum_{k=p}^{\infty} \frac{[p(1-A)+k(1-B)]}{p(A-B)} (p+k+1)^n \big\| b_k| - |a_k\big\| \le \delta \right\}.$$

**Theorem 4.3** Let $A+B \le 0$. If $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \in T_{n+1}^*(A,B)$, then $N_\delta^*(f) \subset T_n^*(A,B)$, where $\delta = \frac{2p}{1+2p}$. The result is sharp.

**Proof.** Using the same method as in Theorem 4.1, we would have

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} c_k z^k$$

$$= z^{-p} + \sum_{k=p}^{\infty} \frac{(p+k)-\sigma(pA+kB)}{p\sigma(B-A)} (p+k+1)^n z^k.$$

Under the hypothesis $A+B \le 0$, we obtain that

$$\left|\frac{f(z)*h(z)}{z^{-p}}\right| = \left|1+\sum_{k=p}^{\infty} c_k \, | a_k | \, z^{k+p}\right|$$

$$\geq 1 - \frac{1}{1+2p}\sum_{k=p}^{\infty}\frac{[p(1-A)+k(1-B)]}{p(A-B)}(p+k+1)^{n+1} \, | a_k | \, .$$

From Theorem 3.1, we get

$$\left|\frac{f(z)*h(z)}{z^{-p}}\right| \geq \frac{2p}{1+2p} = \delta \, .$$

The remaining part of the proof is similar to that of Theorem 4.1.

To show the sharpness, we consider the function

$$f(z) = z^{-p} + \frac{A-B}{(2-(A+B))(1+2p)^{n+1}} z^p \in T^*_{n+1}(A,B)$$

and

$$g(z) = z^{-p} + \left[\frac{A-B}{(2-(A+B))(1+2p)^{n+1}} + \frac{(A-B)\delta'}{(2-(A+B))(1+2p)^n}\right] z^p \, ,$$

where $\delta' > \dfrac{2p}{1+2p}$. Then the function $g(z)$ belongs to $N^*_{\delta'}(f)$.

On the other hand, we find from Theorem 3.1 that $g(z)$ is not in $T^*_n(A,B)$. Now the proof is complete. □

**Acknowledgements**

The authors warmly thank the referee for his suggestions and criticisms which have essentially improved our original paper.

## References


[1] O.Altintas and S.Owa, Neighborhoods of certain analytic functions with negative coefficients, *Internat.J.Math.Math.Sci.*19(1996), 797-800.

[2] M.K.Aouf and H.M.Hossen, New criteria for meromorphic $p$ – valent starlike functions, *Tsukuba J.Math.*, 17(1993), 481-486.

[3] M.K.Aouf and H.M.Srivastava, A new criterion for meromorphically $p$ – valent convex functions of order alpha, *Math.Sci.Res.Hot-Line* 1(8) (1997), 7-12.

[4] M.P.Chen, H.Irmak and H.M.Srivastava, Some families of multivalently analytic functions with negative coefficients, *J.Math.Anal.Appl.*214(1997), 674-690.

[5] N.E.Cho and S.Owa, On certain classes of meromorphically $p$ – valent starlike functions, in New Developments in Univalent Function Theory (Kyoto; August 4-7, 1992) (S.Owa, Editor), *Surikaisekikenkyusho Kokyuroku*, Vol.821, pp.159-165, *Research Institute for Mathematical Sciences, Kyoto University, Kyoto*, 1993.

[6] A.W.Goodman, Univalent functions and nonanalytic curves, *Proc.Amer.Math.Soc.*8(1957), 598-601.

[7] I.S.Jack, Functions starlike and convex of order $\alpha$, *J.London Math.Soc.*3(1971),469-474.

[8] S.B.Joshi and H.M.Srivastava, A certain family of meromorphically multivalent functions, *Comput.Math.Appl.*38(3-4)(1999), 201-211.

[9] S.R.Kulkarni, U.H.Naik and H.M.Srivastava, A certain class of meromorphically $p$ – valent quasi-convex functions, *Pan. Amer.Math.J.*8(1)(1998), 57-64.

[10] J.-L. Liu and H.M.Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J.Math.Anal.Appl.* 259(2001), 566-581.

[11] J.-L. Liu and H.M.Srivastava, Some convolution conditions for starlikeness and convexity of meromorphically multivalent functions, *Appl.Math.Lett.*16(2003), 13-16.

[12] J.-L. Liu and H.M.Srivastava, Subclasses of meromorphically multivalent functions associated with a certain linear operator, *preprint*.

[13] M.L.Mogra, Meromorphic multivalent functions with positive coefficients I, *Math.Japonica* 35(1990), 1-11.

[14] M.L.Mogra, Meromorphic multivalent functions with positive coefficients II, *Math,Japonica* 35(1990), 1089-1098.

[15] S.Owa, H.E.Darwish and M.A.Aouf, Meromorphically multivalent functions with positive and fixed second coefficients, *Math.Japon.*46(1997), 231-236.

[16] S.Ruscheweyh, Neighborhoods of univalent functions, *Proc.Amer.Math.Soc.*81(1981),521-527.

[17] H.Saitoh, A linear operator and its applications of first order differential

subordinations, *Math.Japonica* 44(1996), 31-38.

[18] H.M.Srivastava, H.M.Hossen and M.K.Aouf, A unified presentation of some classes of meromorphically multivalent functions, *Comput.Math.Appl.*38(11-12)(1999), 63-70.

[19] H.M.Srivastava and S.Owa (Editors), Current Topics in Analytic Function Theory, *World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.*

[20] B.A.Uralegaddi and C.Somanatha, New criteria for meromorphic starlike univalent functions, *Bull.Austral.Math.Soc.*43(1991), 137-140.

[21] B.A.Uralegaddi and C.Somanatha, Certain classes of meromorphic multivalent functions, *Tamkang J.Math.*23(1992), 223-231.

[22] D.G.Yang, On new subclasses of meromorphic $p$-valent functions, *J.Math.Res.Expo.*15(1995), 7-13.