数理解析研究所講究録 1341 巻 2003 年 94-101

Some Criteria for univalence of certain integral operators

Virgil Pescar and Shigeyoshi Owa

Abstract

In this work, we derive some criteria for univalence of certain integral operators for analytic functions in the open unit disk.

1 Introduction

Let $\mathcal{A}$ be the class of the functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and $f(0)=f'(0)-1=0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)\in\mathcal{A}$ which are univalent in $\mathbb{U}$. Miller and Mocanu [1] have considered many integral operators for functions $f(z)$ belonging to the class $\mathcal{A}$. In this paper, we consider the following integral operators

$$F_\alpha(z)=\left\{\frac{1}{\alpha}\int_0^z f(u)^{\frac{1}{\alpha}}u^{-1}\,du\right\}^\alpha \qquad (z\in\mathbb{U}) \tag{1.1}$$

for $f(z)\in\mathcal{A}$ and for some $\alpha\in\mathbb{C}$. It is well-known that $F_\alpha(z)\in\mathcal{S}$ for $f(z)\in\mathcal{S}^*$ and $\alpha>0$, where $\mathcal{S}^*$ denots the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathbb{U}$.

2 PRELIMINARY RESULTS

To discuss about our integral operators, we need the following theorems.

Theorem 2.1 ([3]) *Let α be a complex number with* $\mathrm{Re}(\alpha)>0$, *and $f(z)\in\mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right|\leqq 1, \tag{2.1}$$

for all $z\in\mathbb{U}$, then the following integral operator

$$G_\alpha(z)=\left\{\alpha\int_o^z u^{\alpha-1}f'(u)du\right\}^{\frac{1}{\alpha}} \tag{2.2}$$

is in the class $\mathcal{S}$.

2000 Mathematics Subject Classification. Primary 30C45.
Key Words and Phrases. Analytic function, univalent function, integral operator, Schwarz lemma.

Theorem 2.2 ([4]) *Let α be a complex number with $Re(\alpha) > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1-|z|^{2Re(\alpha)}}{Re(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \tag{2.3}$$

for all $z \in \mathbb{U}$, then, for any complex number β with $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$, the integral operator

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u)\, du\right\}^{\frac{1}{\beta}} \tag{2.4}$$

is in the class $\mathcal{S}$.

Example 2.3 Defining the function $f(z)$ by

$$f(z) = \int_0^z \left(\frac{1+u^{\mathrm{Re}(\alpha)}}{1-u^{\mathrm{Re}(\alpha)}}\right)^{\frac{1}{2}} du$$

with $\mathrm{Re}(\alpha) \geqq 1$, we have that

$$\frac{1-z^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left(\frac{zf''(z)}{f'(z)}\right) = z^{\mathrm{Re}(\alpha)-1}.$$

Thus the function $f(z)$ satisfies the condition of Theorem 2.2. Therefore, for $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$,

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1}\left(\frac{1+u^{\mathrm{Re}(\alpha)}}{1-u^{\mathrm{Re}(\alpha)}}\right)^{\frac{1}{2}} du\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Teorem 2.4 [2] *If the function $g(z)$ is regular in $\mathbb{U}$, then, for all $\xi \in \mathbb{U}$ and $z \in \mathbb{U}$, $g(z)$ satisfies*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \leqq \left|\frac{\xi - z}{1-\overline{z}\xi}\right| \tag{2.5}$$

and

$$|g'(z)| \leqq \frac{1-|g(z)|^2}{1-|z|^2}. \tag{2.6}$$

The equalities hold only in the case $g(z) = \varepsilon\frac{z+u}{1+\overline{u}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 2.5 ([2]) For $z = 0$, from inequality (2.5)

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leqq |\xi| \tag{2.7}$$

and, hence

$$|g(\xi)| \leqq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}. \tag{2.8}$$

Considering $g(0) = a$ and $\xi = z$, we see that

$$|g(z)| \leqq \frac{|z| + |a|}{1 + |a|\,|z|} \tag{2.9}$$

for all $z \in \mathbb{U}$.

Schwarz Lemma ([2]) *If the function $g(z)$ is regular in $\mathbb{U}, g(0) = 0$ and $|g(z)| \leqq 1$ for all $z \in \mathbb{U}$, then*

$$|g(z)| \leqq |z|, \tag{2.10}$$

for all $z \in \mathbb{U}$, and $|g'(0)| \leqq 1$. The equality in (2.10) for $z \neq 0$ holds only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3 Main results

Theorem 3.1 *Let α be a complex number with $\mathrm{Re}\left(\frac{1}{\alpha}\right) = a > 0$ and the function $g(z) \in \mathcal{A}$ satisfy*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leqq 1 \qquad (z \in \mathbb{U}). \tag{3.1}$$

Then, for

$$|\alpha| \geqq \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}, \tag{3.2}$$

the integral operator

$$F_\alpha(z) = \left\{\frac{1}{\alpha}\int_0^z g(u)^{\frac{1}{\alpha}} u^{-1} du\right\}^\alpha \tag{3.3}$$

is in the class S.

Proof Let $\frac{1}{\alpha} = \beta$. Then we have

$$F_{\frac{1}{\beta}}(z) = \left\{\beta\int_0^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^\beta du\right\}^{\frac{1}{\beta}}. \tag{3.4}$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^\beta du. \tag{3.5}$$

Then the function

$$h(z) = \left(\frac{1}{|\beta|}\right)\frac{zf''(z)}{f'(z)} \tag{3.6}$$

is regular in U and the constant $|\beta|$ satisfies the inequality

$$|\beta| \leqq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}. \tag{3.7}$$

From (3.5) and (3.6), we have that

$$h(z) = \frac{\beta}{|\beta|}\left(\frac{zg'(z)}{g(z)} - 1\right). \tag{3.8}$$

Using (3.8) and (3.1), we obtain

$$|h(z)| \leqq 1 \qquad (z \in \mathbb{U}). \tag{3.9}$$

Noting that $h(0) = 0$ and applying Schwarz - Lemma for $h(z)$, we get

$$\frac{1}{|\beta|}\left|\frac{zf''(z)}{f'(z)}\right| \leqq |z| \qquad (z \in \mathbb{U}), \tag{3.10}$$

and hence, we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leqq |\beta|\left(\frac{1-|z|^{2a}}{a}\right)|z| \qquad (z\in\mathbb{U}). \tag{3.11}$$

Because

$$\max_{|z|\leq 1}\left(\frac{1-|z|^{2a}}{a}|z|\right) = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$$

from (3.11) and (3.7), we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \tag{3.12}$$

for $z \in \mathbb{U}$. From (3.12) and Theorem 2.1, it follows that

$$G_\beta(z) = \left\{\beta\int_0^z u^{\beta-1}f'(u)\,du\right\}^{\frac{1}{\beta}} \tag{3.13}$$

belongs to the class $\mathcal{S}$.

By means of (3.13) and (3.5), we have the integtal operator $F_{\frac{1}{\beta}}(z)$ is in the class $\mathcal{S}$, and hence, we conclude that the integral operator $F_\alpha(z)$ is in the class $\mathcal{S}$.

Example 3.2 If we take the function $g(z) = ze^z$ and $\alpha = \dfrac{1}{a} > 0$, then

$$g(z) = z + a_2z^2 + a_3z^3 + \cdots$$

is analytic in $\mathbb{U}$ and

$$\left|\frac{zg'(z)}{g(z)} - 1\right| = |z| < 1 \qquad (z\in\mathbb{U}).$$

Since the function $g(z)$ satisfies the condition of Theorem 3.1, we have

$$T_\alpha(z) = \left\{\frac{1}{\alpha}\int_0^z e^{\frac{1}{\alpha u}}\, u^{\frac{1}{\alpha}-1}\,du\right\}^\alpha \in \mathcal{S}.$$

Theorem 3.3 *Let α,β be complex numbers with* $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha) > 0$ *and the function* $g(z) \in \mathcal{A}$ *satisfy*

$$\left|\frac{zg'(z) - g(z)}{z\,g(z)}\right| \leqq 1 \qquad (z\in\mathbb{U}). \tag{3.14}$$

Then, for

$$|\alpha| \geqq \max_{|z|\leqq 1}\left\{\left(\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2||z|}\right)\right\}, \tag{3.15}$$

the integral operator

$$F_{\alpha,\beta}(z) = \left\{\beta\int_0^z g(u)^{\frac{1}{\alpha}}u^{\beta-\frac{1}{\alpha}-1}\,du\right\}^{\frac{1}{\beta}} \tag{3.16}$$

is in the class $\mathcal{S}$.

Proof We have

$$F_{\alpha,\beta}(z) = \left\{\beta\int_0^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}}du\right\}^{\frac{1}{\beta}}. \tag{3.17}$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}}du. \tag{3.18}$$

which is regular in $\mathbb{U}$. The function

$$p(z) = |\alpha|\frac{f''(z)}{f'(z)}, \tag{3.19}$$

where the constant $|\alpha|$ satisfies the inequality (3.15), is regular in $\mathbb{U}$. From (3.19) and (3.18), we obtain

$$p(z) = \frac{|\alpha|}{\alpha}\left\{\frac{zg'(z)-g(z)}{zg(z)}\right\} \tag{3.20}$$

and using (3.14) we have

$$|p(z)| < 1 \qquad (z \in \mathbb{U}) \tag{3.21}$$

and $|p(0)| = |a_2|$. Applying Remark 2.5, we obtain

$$\left|\alpha\frac{f''(z)}{f'(z)}\right| \leq \frac{|z|+|a_2|}{1+|a_2||z|} \qquad (z \in \mathbb{U}). \tag{3.22}$$

It follows that

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leqq \left(\frac{1}{|\alpha|}\right)\left(\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2||z|}\right) \tag{3.23}$$

for all $z \in \mathbb{U}$. Let us consider the function

$$Q(x) = \left(\frac{1-x^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\right)x\left(\frac{x+|a_2|}{1+|a_2|x}\right) \qquad (x=|z|; x \in [0,1]).$$

Because $Q\left(\frac{1}{2}\right) > 0$, $Q(x)$ satisfies

$$\max_{x\in[0,1]}Q(x) > 0 \tag{3.24}$$

Using this fact, (3.23) gives us that

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|z\frac{f''(z)}{f'(z)}\right| \leqq \frac{1}{|\alpha|}\max_{|z|\leqq 1}\left\{\left(\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2|\,|z|}\right)\right\}. \tag{3.25}$$

From (3.25) and (3.15), we obtain

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \qquad (z \in \mathbb{U}). \tag{3.26}$$

Using (3.26) and Theorem 2.2, we obtain that the integral operator

$$G_\beta(z) = \left\{\beta\int_0^z u^{\beta-1}f'(u)\,du\right\}^{\frac{1}{\beta}} \tag{3.27}$$

belongs to the class $\mathcal{S}$. Therefore, it follows from (3.27) and (3.18), that $F_{\alpha,\beta}(z)$ is in the class $\mathcal{S}$.

Corollary 3.4 *Let α be a complex number with* $\mathrm{Re}(\alpha) > 0$ *and the function $g(z) \in \mathcal{A}$ satisfy*

$$\left|\frac{zg'(z)-g(z)}{zg(z)}\right| \leqq 1 \qquad (z \in \mathbb{U}). \tag{3.28}$$

Then, for

$$\max_{|z|\leqq 1}\left\{\left(\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\right)|z|\left(\frac{|z|+|a_2|}{1+|a_2|\,|z|}\right)\right\} \leqq |\alpha| \leqq 1, \tag{3.29}$$

the integral operator

101

$$F_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z g(u)^{\frac{1}{\alpha}} u^{-1}\, du \right\}^\alpha \tag{3.30}$$

is in the class $\mathcal{S}$.

Proof From Theorem 3.3 for $\beta = \frac{1}{\alpha}$, the condition $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha) > 0$, is identical with $|\alpha| < 1$ and we have $F_{\alpha,\beta}(z) = F_\alpha(z)$.

References

[1] S.S.Miller and P.T. Mocanu, *Differential subordinations* ,Pure and Applied Mathematics 225, Marcel Dekker, 2000.

[2] Z. Nehari, *Conformal mapping*, Mc Graw-Hill Book Comp., New York, Toronto, London, 1952.

[3] N.N. Pascu, *On a univalence criterion II,* Inter. Seminar on Functional Equations Approximation and Convexity, Cluj - Napoca, Preprint No. 6, 1985, 153-154.

[4] N.N.Pascu, *An improvement of Becker's univalence criterion.* Proceedings of the Commemorative Simion Stoilow (Braşov, 1987) Univ. Braşov, 1987, 43-48.

[5] V. Pescar, *New univalence criteria,* Transilvania University of Braşov, 2002.

[6] Ch. Pommerenke, *Univalent functions*, Vandenhoeck Ruprecht in Gottingen, 1975.

V.Pescar: Department of Mathematics
Faculty of Mathematics and Computer Science
" *Transilvania*" University of Braşov,
2200 Braşov
Romania

S.Owa: Department of Mathematics
Kinki University
Higashi - Osaka, Osaka 577 - 8502
Japan