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Automorphic forms on type IV symmetric domains

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Introduction

Here is a short introduction to the type IV symmetric domains and to the moduli theory of K3 surfaces. Because of the lack of time, I cannot write enough details at various points. The referencens given below are far from complete to fill the missing details. But the readers might meet the necessary papers using them as the starting points. Also the articles of the speakers of the workshop should contain related more references.

1 Hermitian symmetric domain of type IV

For this section, we refer to Helgason [1], Satake [2].

1.1 Symmetric spaces

A Riemannian symmetric manifold X with metric form $\mu = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ is called a *symmetric space*, if for any point $x \in X$ there is an involutive isometry s_x of X whose fixed points set $\{y \in X | s_x(y) = y\}$ is $\{x\}$. In particular at the tangent space T_x of X at x, s_x induces (-1) multiplication.

Let $Iso(X)$ be the group of all the isometries of X with compact-open topology. Then the subgroup of $Iso(X)$ generated by all the symmetries acts transitively on X, because any two points x, y in X is connected by a finite number of geodesic arcs C_i $(1 \leq i \leq n)$ such that the terminal points are finite number of points $x = x_0, x_1, \cdots, x_n = y$ with $End(C_i) = \{x_{i-1}, x_i\}$. All the more $Iso(X)$ acts on the symmetric space X transitively.

The stabilzer $Stab(x)$ of x in $Iso(X)$ is a closed subgroup, which is known to be compact (*cf.* Theroem 2.5 of [1]). The derivation induces a natural continuous homomorphism i_x : $Stab(x) \ni g \mapsto dg \in O(T_x, \mu_x)$. Here $O(T_x, \mu_x)$ is the orthogonal group on the linear space with definite inner product μ_x, hence it is the orthoganal group $O(n)$ with $n = \dim_{\mathbf{R}} X$.

Given an element h in $O(T_x, \mu_x)$, then by the uniqueness of the solution of the geodesic equation with inital value $t \in T_x$, it is uniquely extended to an element of $Stab(x)$ (i.e., we use the exponential map $\exp : T_x \to X$. Therefore i_x is a bijective continuous homomorphism from a compact group, hence an isomorphism. $Stab(x)$ is a compact Lie group and the quotient $Iso(X)/Stab(x) \cong X$ is a manifold. We can show that $Iso(X)$ is also a Lie group with compatible smooth structure on $Iso(X)/Stab(x) \cong X$ *cf.* Theorem 3.3 of [1]).

1.2 Decomposition

There are symmetric spaces of compact type which is isomorphic to a homogeneous space G/K with G a compact Lie group and K a closed subgroup. There are symmetric spaces of non-compact type which is isomorphic to G/K with G a non-compact semisimple Lie group and K is a maximal compact subgroup of G. There are symmetric space of Euclidean type, which is a flat manifold, i.e., locally an Euclidean space.

In general a simply connected (globally) symmetric space X decomposes as a product $X^0 \times X^+ \times X^-$ of Euclidean type X^0, compact type X^+ and non-compact type X^- (*cf.* Proposition 4.2 of [1]).

A symmetric space of non-compact type (resp. compact type) decomposes into irreducible factors corresponding to the decomposition of G into simple factors. An irreducible symmetric space X of non-compact type (resp. compact type) is a quotient of a simple Lie group G.

1.3 Cartan decomposition

If $X = G/K$ is a non-compact symmetric space with G a semisimple Lie group of non-compact type, then the symmetry s_{x_0} at $x_0 = 1 \cdot K \in G/K$ induces an isomorphoism $g \in G \to s_{x_0} g s_{x_0}^{-1} \in G$ of G. Passing to the Lie algebra we have $Ad(s_{x_0}) : \mathfrak{g} \to \mathfrak{g}$. The eigenspace decompostion $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ with respect to this involution is the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the non-compact semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The space $\mathfrak{p}$ which is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form is canonically identified with the tangent space $T_{x_0} \cong \mathfrak{g}/\mathfrak{k}$ of X at x_0. Moreover the invaraiant Riemannian metric on T_{x_0} is proportional to the restriction of the Killing form to $\mathfrak{p}$, if X is irreducible.

1.4 Classification

Irreducible symmetric spaces of compact type and non-comapct type are classified by Élie Cartan. Among them, Type BD I

$$SO_0(p+q)/SO(p) \times SO(q)$$

is our concern (*cf.* Chapter X of [1], p.453 for BD I) .

1.5 A direct description of BD I type symmetric spaces

Assume that $p, q \geq 1, p+q \geq 3$. Let $Q : \mathbf{R}^{p+q} \to \mathbf{R}$ be a real quadratic form of signature $(p+, q-)$. Let G be the identity component of the orthogonal group $O(Q)$, which is identified with the identity component $SO_0(p,q)$ of $SO(p,q)$ if $p+q$ even, and with the group $SO(p,q)$ itself if $p+q$ is odd.

There is a natural description of the symmeytric space

$$X = G/K = SO_0(p,q)/SO(p) \times SO(q),$$

in terms of the minimal majorants of Q, which appears in the reduction theory of indefinite quadratic forms (*cf.* A.Borel [3]).

Proposition For the quadratic form Q given above, there is canonical bijections between the following 3 data:
(i) $R : \mathbf{R}^{p+q} \to \mathbf{R}$ is a positive-definite quadratic forms such that for any v in $\mathbf{R}^{p+q}$ we have $|Q(v)| \leq |R(v)|$ and R is minimal among such majorating positive-definite quadratic forms (minimal majorant);
(ii) a decompostion of $V = \mathbf{R}^{p+q}$ into two subspaces $V = V_+ \oplus V_-$ such that

$$Q|V_+ \text{ is positive-definite, } Q|V_- \text{ is negative-definite, and } S_Q|V_+ \times V_- \equiv 0.$$

Here S_Q is the symmetric bilinear form on $V \times V$ associated with Q;
(iii) a positive-definite matrix R such that $(QR^{-1})^2 = 1_n$, or equivalently $QR^{-1}Q = R$;
(iv) a choice of maximal compact subgroup in $G = SO_0(p,q)$.

Proof) Probably it is not necessary to give a detailed proof. Similutaneous diagonalization of Q and R shows that both Q and R are written in diagonal forms: $Q(v) = \sum_{i=1}^{p+q} a_i v_i^2$, $R(v) = \sum_{i=1}^{p+q} b_i v_i^2$. Here among a_i, p elements are positive and q elements are negative by Sylvester's law of inertia. For R to be a minimal majorant, we have to set $b_i = |a_i|$ for each i.

The correspondence are given as follows:

(i) $\Rightarrow$ (ii): Given a minimal majorant R, we set

$$V_\pm = \{v \in V| \text{ for any } w \in V, S_Q(v,w) = \pm S_R(v,w)\}.$$

(ii) $\Rightarrow$ (i): Given a decomposition in the statement (ii), we define R by

$$R(v) = Q(v_+) - Q(v_-) \text{ for } v = v_+ + v_- \ (\pm \in V_\pm).$$

(ii) $\Rightarrow$ (iii): The decomposition in (ii) gives an involutive automorphism

$$P : v = v_+ + v_- \mapsto v_+ - v_- \ (v_\pm \in V_\pm).$$

Let us denote by the same sybol P the matrix corresponding to P. Then $P^2 = 1_{p+q}$ and QP ($= R$) is a positive definite matrix which is obviously minimal majorant by the first part of this proof.

(iii) $\Rightarrow$ (ii): $V = V_+ + V_-$ is the eigenspace decomposition with respect to the involutive automorphism QR^{-1} ($= P$), i.e.,

$$V_\pm = \{v \in V|Pv = \pm v\}.$$

(i), (ii), (iii) $\Rightarrow$ (iv): Let K be the subgroup of G defined by $K = G \cap O(R) = \{g \in G|g(V_\pm) \subset V_\pm\}$. Then this is isomorphic to $SO(p) \times SO(q)$, a maximal compact subroup. Conversely if a maximal compact subgroup K is given. Then the the integral

$$R(v) = \int_K |Q(k \cdot v)| dk$$

Here dk is the normalized Haar measure on K.

We refer to Proposition (5.2) of Borel [3] here.

2 Hermitian symmetric spaces of type IV

A symmetric space $X = G/K$ with a complex structure and the given Riemannian metric is Hermitian is called a *Hermitian symmetric space*, if the symmetry s_x at each point $x \in X$ is also holomorphic with respect this complex structure. In particular the multiplication of $U(1) = \{z \in \mathbf{C}||z| = 1\}$ on the tangent space T_x of each point $x \in X$ is induced by elements in the stabilizer $Stab(x)$, the (connected) group K have a subgroup isomorphic to $U(1)$ which is central in K.

We can check those symmterci spaces $X = G/K$ with connected G and non-trivial center $Z(K)$ which contains $U(1)$. For BDI type symmetric spaces $SO_0(p,q)/SO(p \times SO(q)$, this happens only when $p = 2$ or $q = 2$.

2.1 A description by real Hodge structure

This section is a reproduction of Appendix of the book of Satake [2].

The type IV classical domains have various important realizations. We review those briefly here.

2.2 Poincare model (Harish-Chandra realiztion)

This is the unit disk model. Our domain is written as

$$\mathcal{D}_{IV} = \{z = (z_1, \cdots, z_q) \in \mathbf{C}^q \mid |{}^t z \cdot z|^2 + 1 - 2{}^t\bar{z} \cdot z > 0 \text{ and } |{}^t z \cdot z| < 1\}$$

$$= \{z \in \mathbf{C}^q \mid 1 - {}^t\bar{z}z > \sqrt{({}^t\bar{z} \cdot z)^2 - |{}^t z \cdot z|^2}, 1 - \sum_{i=1}^{q} |z_i|^2 > \sqrt{(\sum_i |z_i|^2)^2) - |\sum z_i^2|}\}.$$

We may refer to [4].

The Borel embedding of this realization is given by

$$(z_1, \cdots, z_q) \to (1 : z_1 : \cdots : z_q : \sum_{i=1}^{q} z_i^2) \in \mathbf{P}^{q+1}.$$

2.3 Realization as a tube domain

A domain in $\mathbf{C}^q$ of the form $\mathbf{R}^q + \sqrt{-1}V$ with a (positive) cone V in $\mathbf{R}^q$ is called a *tube domain.* The symmetric domains of type IV are isomorphic to tube domains. The description of this realization as tube domain is given as follows.

Set

$$\mathcal{D}_{tube} = \{(\zeta_1, \cdots, \zeta_q) \in \mathbf{C}^q \mid \mathrm{Im}\zeta_1 > \sqrt{\sum_{i=2}^{q} (\mathrm{Im}\zeta_i)^2}\}.$$

Then the Borel embedding is given by the mapping

$$(\zeta_1, \cdots, \zeta_q) \in \mathcal{D}_{tube} \mapsto (1 : \zeta_1 : \cdots : \zeta_q : \zeta_1^2 - \sum_{i=2}^{q} \zeta_i^2) \in \mathbf{P}^{q+1}.$$

Any point $(\xi_0 : \xi_1 : \cdots : \xi_{q+1})$ in the image satisfies a quadratic relation:

$$Q(\xi) := -\xi_0\xi_{q+1} + \xi_1^2 - \sum_{i=2}^{q} \xi_i^2 = 0.$$

Moreover for the symmetric bilinear form S_Q associated with Q, we have

$$\begin{aligned}
S_Q(\xi, \bar{\xi}) &= -\bar{\xi}_0\xi_{q+1} - \xi_0\bar{\xi}_{q+1} + 2\xi_1\bar{\xi}_1 - 2\sum_{i=2}^{q} \xi_i\bar{\xi}_i \\
&= -(\zeta_1^2 - \sum_{i=2}^{q} \zeta_i^2) - \overline{\zeta_1^2 - \sum_{i=2}^{q} \zeta_i^2} + 2\zeta_1\bar{\zeta}_1 - 2\sum_{i=2}^{q} \zeta_i\bar{\zeta}_i \\
&= 4(\mathrm{Im}\zeta_1^2 - \sum_{i=2}^{q} \mathrm{Im}\zeta_i^2) \\
&> 0.
\end{aligned}$$

2.4 Real parabolic subgroups

In general the Witt index r of Q with signature $(p+, q-)$ over $\mathbf{R}$ is $\min(p, q)$. The split component A of a minimal parabolic subgroup of $G = SO(Q)$ is of rank r.

When $r = 2$, the restricted root system $\Phi(\mathfrak{g}, \mathfrak{a})$ is of BC_2-type: there are two (types of) maximal standard parabolic subgroups P_J and P_S containing the minimal paraboric subgroup P_{min}. One has non-abelian unipotent radical, the other abelian unipotent radical which is the translation of the real directions for the tube domain model of G/K. The semisimple non-compact part of the Levi componet of P_J is $SL(2, \mathbf{R})$. The semisimple part of the Levi part of P_S, which is sometimes refered as the Siegel parabolic subgroup, is isomorphic to $SO(1, q-1)$.

The parabolic subgroups defined over $\mathbf{Q}$ is discussed later.

3 Arithmetic discrete subgroups

Here we recall the typical ways to construct arithmetic discrete subgroups Γ in $G = SO_0(2, q)$, and review the basic facts related them.

3.1 Definition

The simplest way to obtain such group in $SO_0(p, q)$ for general p is to consider a quadratic form $Q : \mathbf{Q}^{p+q} \to \mathbf{Q}$ of signature $(p+, q-)$ defined over the rational number field $\mathbf{Q}$. Then we can consider the orthogonal group $SO(Q)$ (or $O(Q)$ depending on one's purpose) which is a semisimple algebraic group defined over $\mathbf{Q}$ if $p + q \geq 3$.

Choose a lattice L in $\mathbf{Q}^{p+q}$, then there is a rational number r such that rQ becomes an integral-valued function on L (or even-integral valued if you like). Then in the group of $\mathbf{Q}$-rational points $SO(Q)(\mathbf{Q})$ of the algebraic group $SO(Q)$ or in the real semisimple Lie group of the real points of $SO(Q)$, we can consider the intersection

$$\Gamma := \mathrm{Aut}(L) \cap SO(Q)(\mathbf{Q}) \cap SO(Q)_0(\mathbf{R}) = \mathrm{Aut}(L) \cap SO(Q)_0(\mathbf{R}).$$

Then Γ is a discrete subgroup of $G = SO(Q)_0(\mathbf{R})$ with finite covolume by the reduction theory (*cf.* Borel, and Harish-Chandra []).

The Witt index of the quadratic form Q over $\mathbf{Q}$ is equal to the dimension of the maximal $\mathbf{Q}$-split torus in $SO(Q)$, i.e., the $\mathbf{Q}$-rank of $SO(Q)$.

More general way to have an arithmetic subgroup Γ in $SO_0(p, q)$ is to consider a totally real number field F of finite degree d and a quadratic form

$$Q : F^{p+q} \to F$$

over F, which is of signature $(p+, q-)$ with respect a real embedding $v_1 : F \subset \mathbf{R}$ and definite with repect to the remaining $d-1$ embeddings $v_i : F \subset \mathbf{R}$ $(2 \leq i \leq d)$.

Now consider the diagonal map

$$SO(Q)(F) \to \prod_{i=1}^{d} SO(Q \otimes_{(F, v_i)} \mathbf{R})$$

from the F-rational points $SO(Q)(F)$ of the special orthogonal group $SO(Q)$ over F to the product of real groups. Compose this with the first projection to $SO(Q \otimes_{(F, v_1)} \mathbf{R})$. Then the image Γ of the integral part $\mathrm{Aut}(\mathcal{O}_F^{p+q}) \cap SO(Q)(F)$ of $SO(Q)(F)$ is the requited arithmetic subgroup. When $d \geq 2$, this group is of $\mathbf{Q}$-rank 0.

3.2 Parabolic subgroups (global)

Let V be a finite dimensional vector space of dimension n with a non-degenerate $\mathbf{Q}$-valued quadratic form ψ on V. We consider the algebraic group $G = SO(V, \psi)$.

Now assume that either of the following equivalent condistions:
(i) $\mathrm{rank}_{\mathbf{Q}} G = 2$;
(ii) the Witt index of (V, ψ) is equal to 2.

Under this assumption, we can find a maximally totally isotropic subsapce of $\dim_{\mathbf{Q}} W_{-1}(V) = 2$. We set

$$W_0(V) := \{v \in V | \psi(v, w) = 0, \text{ for any } w \in W_{-1}(V)\}.$$

Further choose a subspace $W_{-2}(V) \subset W_{-1}(V)$, $\dim_{\mathbf{Q}} W_{-2](V)} = 1$ and the assocaited subsapce

$$W_1(V) := \{v \in V | \psi(v, w) = 0, \text{ for any } w \in W_{-2}(V)\}.$$

Then we obtain a flag

$$\mathcal{F} := \{W_{-3}(V) = \{0\} \subset W_{-2}(V) \subset W_{-1}(V) \subset W_0(V) \subset W_1(V) \subset W_2(V) = V\}$$

and the associated minimal parabolic subgroup

$$P_{\mathcal{F}} = Stab(\mathcal{F}) := \{g \in G | g(W_i(V)) \subset W_i(V)\}.$$

and its unipotent radical

$$N_{\mathcal{F}} := \{g \in P_{\mathcal{F}} | gr(g)|_{gr_{W_i(V)}} \equiv 1 \text{ for any } i\}.$$

We have the natural isomorphism of algebraic groups

$$P_{\mathcal{F}}/N_{\mathcal{F}} \cong \mathbf{G}_m \times \mathbf{G}_m \times SO(Gr_{W_0}(V), \psi').$$

The reduction theory implies that the set of double cosets: $\Gamma \backslash G/P_{\mathcal{F}}$ is finite.

We have two standard maximal parabolic subgroups containing the above minimal parabolic subgroup, by forgetting the part of the data of the falg:
(A): Siegel parabolic subgroup P_S assocaited with the partial flag:

$$W_{-2}(V) \subset W_0(V) \subset W_2(V) = V.$$

In this case, $P_S/N_S \cong \mathbf{G}_m \times SO(Gr_{W_0}, \psi')$. Here ψ' is the naturally induced metric from ψ.
(B): 'Jacobi' parabolic subgroup P_J associated with the partial flag:

$$W_{-1}(V) \subset W_0(V) \subset W_1(V) = V.$$

In this case, the Levi part of P_J is isomorphic to the quotient $P_J/N_J \cong GL(Gr_{W_{-1}(V)}) \times SO(gr_{W_0}(V), \psi'')$

3.3 Compactification

The Baily-Borel-Satake compactification of the aritmetic quotient $\Gamma \backslash \mathcal{D}_{IV}$ is obtained by attaching a finite number of points (=zero-dimensional boundaries) parametrized by the double cosets $\Gamma \backslash G/P_S$ and a finite number of elliptic modular curves (= one dimensional boundaries) numbered by the finite set of double cosets $\Gamma \backslash G/P_J$. The latter boundaries are associated with the semisimple part $SL(GrW_{-1}) \cong SL(2, \mathbf{Q})$ of the Levi subgroup of P_J. Hence these are elliptic modular curves.

The topology and the analytic structure on this enlargement of the quotient $\Gamma \backslash \mathcal{D}_{IV}$ requires some more space and time. The readers should consult with the original papers.

4 Fundamentals on K3 surfaces

4.1 Definition of K3 surfaces

Definition A connected complex analytic manifold of dimesnion 2 is called an analytic surface. A compact analytic surface S with the conditions:
(i) $q(S) = \dim_{\mathbf{C}} H^1(S, \mathcal{O}_S) = 0$;
(ii) $c_1(S) = 0$
is called *K3* surface.

The short exact sequence of sheaves on S:

$$0 \to \mathbf{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 1$$

derives a long cohomological sequence:

$$0 \to H^1(S, \mathbf{Z}) \to H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^*) \to H^2(S.\mathbf{Z}) \to H^2(S, \mathcal{O}_S) \to \cdots .$$

Then the first condition $q(S) = 0$ implies that

$$H^1(S, \mathcal{O}_S) = \{0\}, \qquad H^1(S, \mathbf{Z}) = \{0\},$$

and the Picard variety

$$\mathrm{Pic}^0(S) := H^1(S, \mathbf{Z}) \backslash H^1(S, \mathcal{O}_S)$$

vanishes. Therefore the Picard group $\mathrm{Pic}(S) := H^1(S, \mathcal{O}_S^*)$ is isomorphic to the Néron-Severi group

$$NS(S) := \mathrm{Im}(c_{1,B} = \delta : H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbf{Z})) = \mathrm{Ker}(H^2(i) : H^2(S, \mathbf{Z}) \to H^2(S, \mathcal{O}_S)).$$

The vanishing of the first Chern class $c_1(S)$ means that the image of the class of $\wedge^2 \Theta_S$ or its dual $\Omega_S^2 = \wedge^2 \Omega_S^1$ in $\mathrm{Pic}(\mathcal{O}_S)$ via $\delta = c_{1,B}$ vanishes in $NS(S)$. Here Θ_S is the sheaf of holomorphic tangent on S and Ω_S^1 the sheaf of holomorphic cotangent on S, and Ω_S^2 the canonical sheaf on S, respectively. Thus we have an isomorphism of sheaves

$$\Omega_S^2 \cong \mathcal{O}_S.$$

Therefore $\Gamma(S, \Omega_S^2)$ has non-zero section ω which is unique up to constant multiple, that is nowhere vanishing on S. Moreover Serre duality implies that $H^2(S, \mathcal{O}_S)$ is also of one dimension. Hence

$$p_g(S) = \dim_{\mathbf{C}} H^2(S, \mathcal{O}_S) = 1,$$

and

$$\chi(\mathcal{O}_S) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbf{C}} H^i(S, \mathcal{O}_S) = 1 - q(S) + p_g(S) = 2.$$

As a part of Riemann-Roch theorem, we have Max Noether's formula:

$$\chi(\mathcal{O}_S) = \frac{1}{12}\{c_1^2(S) + c_2(S)\}$$

with $c_2(S) = e(S)$ the Euler number of S, for any compact complex analytic surface S. For K3 surfaces tis means that

$$2 = \frac{1}{12}(0 + c_2(S)), \text{i.e., } c_2(S) = e(S) = 24.$$

We know already that $H^1(S,\mathbf{Z})=\{0\}, i.e., b_1(S)=0$. Therefore by Poincaré duality $b_3(S)=0$. Hence

$$24=e(S)=b_0(S)-b_1(S)+b_2(S)-b_3(S)+b_4(S)=1-0+b_2(S)-0+1=b_2(S)+2, \text{i.e., } b_2(S)=22.$$

Since S has a Kähler metric by assumption, we have Hodge decompostion of the cohomology groups with real coefficients of S. The unique non-trivial Hodge structure on these cohomology groups is at the degree 2:

$$H^2(S,\mathbf{R})\otimes_{\mathbf{R}}\mathbf{C}=H^2(S,\mathbf{C})=H^{(2,0)}\oplus H^{(1,1)}\oplus H^{(0,2)}$$

with

$$H^{(2,0)}=\mathrm{Im}(\Gamma(S,\Omega_S^2)\to H^2(S,\mathbf{C}))\cong\Gamma(S,\Omega_S^2)$$
$$H^{(1,1)}=H^1(S,\Omega_S^1)),\quad H^{(0,2)}\cong H^2(S,\mathcal{O}_S).$$

The Hodge symmetry implies $\bar{H^{(2,0)}}=H^{(0,2)}$ has dimension 1 for K3 surfaces, again.

4.2 H_2 and H^2 are torsion-free

4.3 Examples of K3 surfaces

(0):*Kummer surfaces.* Let $[-1]$ be the isomorphism (-1) multiplication on an abelian variety A of dimension 2, which has $16=2^4$ isolated fixed points corresponding to the 2−divison ppoints $2\cdot P=0$. Then the quotient variety $A/\{id_A,[-1]\}$ by order 2 cyclic group generated bo $[-1]$ has 16 normal singularities whose local chart is given by Spec$\mathbf{C}[z^2,w^2,zw]$. Here $\mathbf{C}[z^2,w^2,zw]$ is the subring in the polynomial ring $\mathbf{C}[z,w]$ of 2 variables. Since it is isomorphic to the quotient ring $\mathbf{C}[u,v,t]/(uv-t^2)$, these singularities are conical. By blowing-up these 16 singularities, we have a smooth algebraic surface $Kum(A)$, which is a K3 surface.

Firstly $H^1(A/<[-1]>,\mathbf{Z})=H^1(A,\mathbf{Z})^{<[-1]>}=\{0\}$ implies $H^1(Kum(A),\mathbf{Z})=\{0\}$, this means $b_1(S)=2q(S)=0$. Secondly the fact that the canonical bundle Ω_A^2 is trivial implies that there is a unique nowhere vanishing 2-form ω_A unique up to constant multiple. Direct computation using local coordinates shows that this is extendable to $Kum(A)$ uniquely without zeros. This means $\Omega^2_{Kum(A)}\cong\mathcal{O}_S$.

Polarization.

(1):*Double covering of* $\mathbf{P}^2$ Some K3 surfaces are obtained as double coverings of $\mathbf{P}^2$ branched along degree 6 curves in $\mathbf{P}^2$. We consider weighted variables (x,y,z,w) of weight $(1,1,1,3)$ respectively. And we can define the associated weighted projective space $\mathbf{P}^{(1,1,1,3)}$ obtained as the quotient of $\mathbf{A}^4-\{(0,0,0,0)\}$ by the relation $(x,y,z,w)\tilde{(}tx,ty,tz,t^3w)\quad(t\in\mathbf{C}^*)$.

An equation $w^2=F_6(x,y,z)$ with $F_6(x,y,z)$ a homogeneous polynomial of degree 6 in this 3-dimensional weighted projective space defines a $K3$ surface if it has no singularities. The projection to $\mathbf{P}^2$ corresponding to the 3 coordinates (x,y,z) defines a double covering.

The pull-back of the tautological line bundle $O(1)$ of $\mathbf{P}^2$ gives an ample line bundle of degree 2 on this $K3$ surface.

(2):*Quartic surfaces in* $\mathbf{P}^3$ A non-zero homogeneous polynomial $F_4(x,y,z,w)$ of degree 4 in 4 variables $(x:y:z:w)$ defines an algebraic surface. If this quartic surface has mild singuarities, it is a $K3$ surface. In particular, a smooth quartic surface is a $K3$ surface. This is because the irregulairty $q(S)$ of this surface S vanishes by the Lefschetz hypersurface (section) theroem (i.e., $q(S)=q(\mathbf{P}^3)=0$) on one hand. On the other hand, the adjunction formula implies that the canonical sheaf Ω_S^2 of S is isomorphic to

$$(\Omega^3_{\mathbf{P}^3}|S)\otimes N^*_{S/\mathbf{P}^3}\cong(\mathcal{O}_{\mathbf{P}^3}(4)|S)\otimes\mathcal{O}_S(-4)\cong\mathcal{O}_S(4)\otimes\mathcal{O}_S(-4)\cong\mathcal{O}_S,$$

i.e., the trivial invertible sheaf.

The possible number of coefficeints of F_4 is ${}_4H_4 = {}_7C_4 = 35$ and the dimension of the automorphism of $\mathbf{P}^3$ is $16 - 1 = 15$. Therefore the heuristic 'Anzhal de Modul' is $35 - 1 - 15 = 19$.

The polariztion is the hyperplane section in $\mathbf{P}^3$, hence it is the degree of the surface S, 4.

(3):*Complete intersection of a quadric and a cubic in* $\mathbf{P}^4$ By the same theorems as the case (2), a smooth intesersection gives a $K4$ surface. The polarization is the hyperplane section, hence its degree is $2 \cdot\cdot = 6$. For a fixed non-degenerate quadric, the dimension of the projective orthogonal group stabilzeing this quadric is 10. For a fixed quadric Q, the choice of cubics should be counted modulo Q times some linear form L. Thus the heuristic number of moduli is ${}_5H_3 - 10 - 5 - 1 = 35 - 16 = 19$!

(4):*Complete intersection of type* $(2,2,2)$ *in* $\mathbf{P}^5$ By the same theorems as in the case (2), (3), the smooth intersections are $K3$. The polarization, the hyperplane section is of degree $2^3 = 8$.

Exercise Confirm that in this case also the heuristic number of moduli is 19. Try the case (1) also.

4.4 Simply connectedness of $K3$ surfaces

It is an easy exercise to show that a $K3$ surface S has no non-trivial finite etale covering, using Noether's formula etc. But the fact that a complex analytic surafce has trivial (topological) fundamental is proved by much deeper result.

The Lefchetz hyperplane section theorem implies that any smooth quartic in the 3 dimensional projecitve space is simply connected.

Theorem (Kodaira) Any two $K3$ surfaces S_1, S_2 are included in some analytic family of (analytic) $K3$ surfaces, i.e., they are connected by deformation of complex structures. In particular, all the $K3$ surfaces are diffeomorphic as C^∞-manifold.

Because a complex quartic surface is simply connected, all other $K3$ surafces are also simply connected.

5 Moduli spaces of $K3$ surfaces

Unfortunately we do not yet have purely algebraic construction of the global moduli spaces of $K3$ surfaces by using Geometric Invariant Theory. There seems to be satisfactory local theory. The remaining problem is the problem of 'stablity' to apply the method of G.I.T.

The current construction uses the transcendental method via periods firstly, after that the existence of moduli space over $\mathbf{C}$ implies the stability. Thus we have moduli spaces over subfield of $\mathbf{C}$, say, over $\mathbf{Q}$. And by the fact that almost all p is good, we have models over such large p. But we have no model over $\mathbf{Z}$ or no effective control of bad primes p.

We recall this transcendental method to construct moduli spaces. This is directly related type IV symmetric domain. And accordingly automorphic forms on this domain, similarly as elliptic modular forms are invoved in the moduli space of elliptic curves.

5.1 The Hodge structure of a K3 surrface

The non-trivial homology or cohomology groups of a K3 surfaces S is the second homology (cohomology) group $H_2(S, \mathbf{Z})$ (resp. $H^2(S, \mathbf{Z})$). This is a free $\mathbf{Z}$ module of rank 22. The Hodge decomposition is given by

$$H^2(S, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = H^{(2,0)} \oplus H^{(1,1)} \oplus H^{(0,2)}$$

$$H^{(2,0)} = \Gamma(S, \Omega_S^2) \cong \mathbf{C}, \quad H^{(0,2)} = \overline{H^{(0,2)}} \cong H^2(S, \mathcal{O}_S) \cong \mathbf{C},$$

and

$$H^{(1,1)} \cong H^1(S, \Omega_S^1) \cong \mathbf{C}^2 2.$$

If S is algebraic and a poralization class $c_1(L) \in NS(S)$ of an ample invertible sheaf L of degree $2d$ is given, then the orthogonal complement of L in $H^2(S, \mathbf{Z})$ with respect to the intersection form

$$H^2_{prim}(S, \mathbf{Z}) = \{\eta \in H^2(S, \mathbf{Z}) | \mathrm{tr}(\eta \cup c_1(L)) = 0\}$$

is a Hodge structure of weight 2 with a polarization form ψ which is the restriction of the intersection form.

The restriction of $\psi_{\mathbf{R}} = \psi \otimes_{\mathbf{Z}} \mathbf{R}$ to $H^2_{prim}(S, \mathbf{R}) \cap \{H^{(2,0)} \oplus H^{(0,2)}\}$ is positive definite, and the restriction to $H^2_{prim}(S, \mathbf{R}) \cap H^{(1,1)}$ is negative definite by Hodge index theorem. Hence the signature of $\psi_{\mathbf{R}}$ on $H^2_{prim}(S, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R}$ is $(2+, 19-)$.

Returning to the original lattice $(H^2(S, \mathbf{Z}), \psi_S)$ with intersectionform ψ_S, we find that this satisied the following 3 properties:
(i) ψ_S is unimodular, and even.
(ii) it is of signature $(3+, 19-)$ over $\mathbf{R}$.
(iii) $\psi_S \cong (-E_8)^{\oplus 2} \oplus H^{\oplus 3}$.
The last result is a conclusion of the theory of quadratic forms. And we find the ismorphism class of such lattice is unique.

Choose such an abstract lattice (Λ, ψ_Λ) of signature $(3+, 19-)$, integral even unimodular. Then by an analogue of Witt theorem for any two vectors λ, $\lambda' \in \Lambda$ of the same length $\psi_\Lambda(\lambda) = \psi_\Lambda(\lambda') = 2d$, there is an isometry γ of (Λ, ψ_Λ) such that $\lambda' = \gamma(\lambda)$.

From now on, we identify $H^2(S, \mathbf{Z})$ with $H_2(S, \mathbf{Z})$ by Poincaré duality.

5.2 Periods of marked K3 surfaces and the moduli map

We fix a lattice (Λ, ψ_Λ) of the type given above. Also we fix an element $\lambda_0 \in \Lambda$ with postive length $\psi_\Lambda(\lambda_0) = 2d$.

Definition A marked K3 surface with polarization is a pair (S, L) of a K3 surafce and an ample invertible sheaf L, with added strutures:
(i) an isomorphism

$$\alpha : \{H_2(S, \mathbf{Z}), \psi_S; c_1(L)\} \cong \{\Lambda, \psi_\Lambda; \lambda_0\}$$

and
(ii) an isomrphism

$$\beta : \Gamma(S, \Omega_S) \cong \mathbf{C}.$$

Then for the above data $(S, L; \alpha, \beta)$, we can associate
(a): a free $\mathbf{Z}$ module

$$\Lambda(\lambda_0) = \{l \in \Lambda | \psi_\Lambda(\lambda_0, l) = 0\}$$

of rank 21.
(b): an element $p(S; \alpha, \beta)$ in

$$\Lambda^*(\lambda_0)_{\mathbf{C}} = \mathrm{Hom}_{\mathbf{Z}}(\Lambda(\lambda_0), \mathbf{C})$$

is defined by

$$l \in \Lambda \to \int_{\alpha^{-1}(l)} \omega$$

. Here $\omega \in \Gamma(S, \Omega_S)$ which is mapped to $1 \in \mathbf{C}$ by β. Then the (dual) of the intesection form ψ^* gives two period relations:
(i): $\psi_\Lambda^*(p(S;\alpha,\beta), p(S;\alpha,\beta)) = 0$
(ii): $\psi_\Lambda^*(p(S;\alpha,\beta), \overline{p(S;\alpha,\beta)}) > 0$.

This implies that the point $p(S;\alpha,\beta)$ modulo $\mathbf{C}^\times$ belongs to the Borel embedding of the type IV symmteric domain $\mathcal{D}$ of complex dimension 19 belonging to the real orthogonal group $SO(\Lambda_\mathbf{R}^*, \psi_{\Lambda,\mathbf{R}})$. Here note that to consider the homogenous coordinates $p(S;\alpha,\beta)$ modulo $\mathbf{C}^\times$ is equivalent to forget the second marking β.

We can consider a complex analytic family $\mathcal{S} \to X$ of complex analytic surfaces of K3 type with relative ample invertible sheal on $\mathcal{S}$ relative to X, with continuous family of markings α_x, β_x for eaxh point $x \in X$. Then we can define a period map $x \in X \to p(S_x;\alpha_x,\beta_x)$. Forget the second marking β_x to get a holomorphic map form X to the type IV symmetric domain $\mathcal{D}_{IV}$.

Finally we forget the first marking α_x. This is equivalent to the division by the action of the discrete subgroup $\Gamma := \mathrm{Aut}((\Lambda, \psi_\lambda, \lambda_0))$ on $\mathcal{D}$.

There remains the problem to show the bijectivity of this moduli mapp defined by the periods. The local injectivity comes from the local deformation theory of K3 surfaces. The 'Anzahl der Modul' is 19 etc., etc. The surjectivity is proved by compactification and by investigation of degeneration of K3 surfaces. For global injectivity we refer to the original papers.

5.3 Degeneration of K3 surfaces

A degeneration of K3 surfaces is a proper flat analytic morphism $\varphi : \mathcal{S} \to D = \{z \in \mathbf{C} | |z| < \varepsilon\}$ from a complex anaytic 3-fold $\mathcal{S}$ to the open disk D such that for $z \in D, z \neq 0$ the fibers $\varphi^{-1}(z) = S_z$ is a K3 surface and the fiber S_0 at the center $z = 0$ has some singularities in general, which is of semistable type.

Different from the case of degeneration of curves, the 3-fold $\mathcal{S}$ has possiblity of alternations which preserve the singular fiber S_0 and the local monodromy aroud it. To get only a unique denegeration with prescribed local monodromy aroud a given singular fiber, Kulikov imposed the following condition for φ:
(*) the relative dualizing complex of φ is a single sheaf $\omega_\varphi = \omega_{\mathcal{S}/D}$ (the relative canical sheaf)m and this is trivial, i.e., $\omega_\varphi \cong \mathcal{O}_\mathcal{S}$ (not only over $\varphi^{-1}(D - \{0\})$) over the whole $\mathcal{S}$. Under this Kulikov [5] proved the following:

Theorem There are 3 following possibilities of degenerations of K3 surafces:
(0): φ is a smooth morphism, i.e., in particular S_0 is a non-singular K3 surface. Hence this case is not a real degeneration.
(1): $S_0 = \sum_{i=1}^n V_i$, where V_1, V_n are rational surfaces, $V_2, \cdots, V_{n-1}$ are ruled surfaces with irregularity 1. plus the graph of $\{V_i\}$is of type A_n.
(2): $S_0 = \sum_{i=1}^n V_i$, where all the V_i are rational surfaces with nonsingular double curves $C_{ij} = V_i \cap V_j$ $(i \neq j)$ which rational. There are some more conditions on the dual graph....

The last two types of degenerations correponds to two types of maximal parabolic subgroups P_J and P_S discussed in the section of arithemtic subgroups.

I am sorry for not giving enough references.

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