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Alexander Trick for Quadratic Poincaré Complexes

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1. Introduction

Let us fix an integer $n\geq 0$, a continuous map $p_{X}:E\rightarrow X$ to a metric space X, a ring R with involution, and a pair of positive numbers $\epsilon\leq\delta$. Then an abelian group $L_{n}^{\delta,\epsilon}(X;p_{X}, R)$ is defined to be the set of equivalence classes of n-dimensional quadratic Poincaré R-module complexes on p_{X} of radius ϵ (= n-dimensional ϵ Poincaré ϵ quadratic R-module complexes on p_{X}), where the equivalence relation is generated by Poincaré cobordisms of radius δ (= δ Poincaré δ cobordisms) ([5][7]). If $\delta\geq\delta'$ and $\epsilon\geq\epsilon'$, there is a natural homomorphism

$$L_{n}^{\delta',\epsilon'}(X;p_{X}, R)\rightarrow L_{n}^{\delta,\epsilon}(X;p_{X}, R)$$

defined by relaxation of control. In general, this map is not surjective or injective. None the less, if X is a finite polyhedron and p_{X} is a fibration, the map above turns out to be an isomorphism for certain values of δ, δ', ϵ, ϵ'. The purpose of this article is to give an outline of a proof of this by E. K. Pedersen and the author. A more detailed account will appear elsewhere ([3]). The precise statement is as follows:

Theorem 1. (Stability Theorem) *Let $n\geq 0$. Suppose X is a finite polyhedron and $p_{X}:E\rightarrow X$ is a fibration. Then there exist constants $\delta_{0}>0$ and $\kappa>1$, which depends on the integer n and X, such that the relax-control map $L_{n}^{\delta',\epsilon'}(X;p_{X}, R)\rightarrow L_{n}^{\delta,\epsilon}(X;p_{X}, R)$ is an isomorphism if $\delta_{0}\geq\delta\geq\kappa\epsilon$, $\delta_{0}\geq\delta'\geq\kappa\epsilon'$, $\delta\geq\delta'$, and $\epsilon\geq\epsilon'$.*

It follows that all the groups $L_{n}^{\delta,\epsilon}(X;p_{X}, R)$ with $\delta_{0}\geq\delta\geq\kappa\epsilon$ are isomorphic and are equal to the controlled L-group $L_{n}^{c}(X;p_{X}, R)$ of p_{X} with coefficient ring R.

Squeezing/stability for controlled K_{0} and K_{1}-groups were known ([1]). 'Splitting' was the key idea there. In general 'splitting' implies 'squeezing'. But splitting in L-theory requires a change of K-theoretic deoration; if you split a free quadratic Poincaré complex, then you get a projective one in the middle. Since the controlled reduced projective class group is known to vanish when the coefficient ring is $\mathbb{Z}$ and the control map is UV^{1}, we do not need to worry about the controlled K-theory and squeezing holds in this case ([2]).

Several years ago E. K. Pedersen proposed an approach to squeezing/stability in controlled L-groups imitating the method of [1]. The idea was to use projective complexes to split and to eventually eliminate the projective pieces using the Eilenberg swindle :

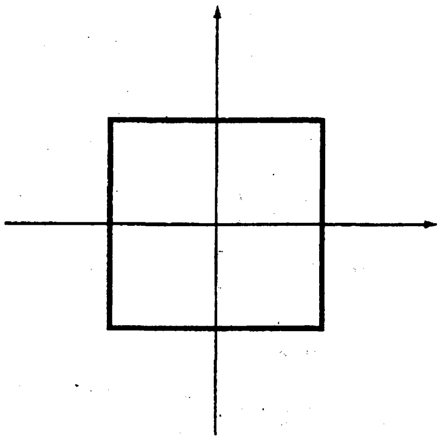
$$\begin{aligned}[P] &= [P] + (-[P] + [P]) + (-[P] + [P]) + (-[P] + [P]) + \cdots \\ &= ([P] - [P]) + ([P] - [P]) + ([P] - [P]) + ([P] - [P]) + \cdots = 0 \ .\end{aligned}$$

This approach works for any R if X is a circle. In the next section we describe the proof slightly modified by the author.

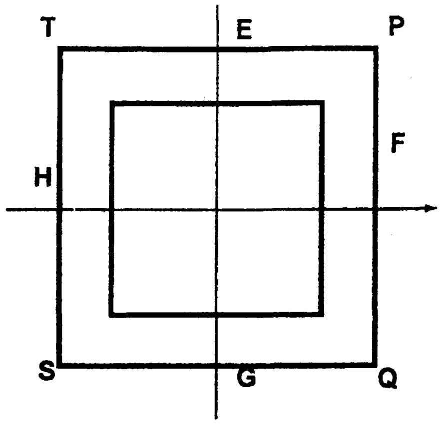
The method in section 2 does not generalize to higher dimensions, because it requires repeated application of splitting but that is impossible for projective complexes. But one construction used in this proof turns out to be very useful: we desribe a construction called Alexander trick in section 3, and use it repeatedly to prove Theorem 1 in section 4.

2. Squeezing over a Circle

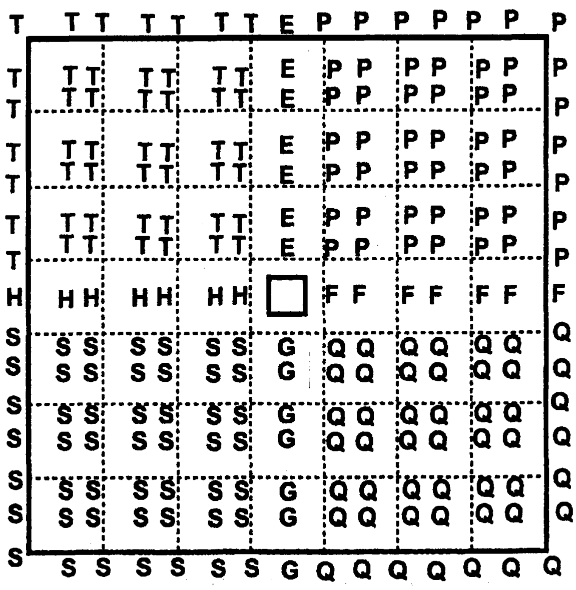
We use the maximum metric of $\mathbb{R}^2$, so the unit circle will look like a square:



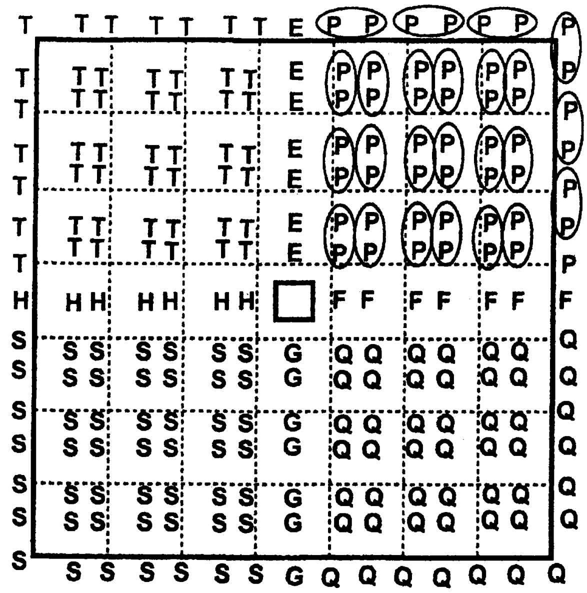
Consider a quadratic Poincaré complex on the unit circle. We assume that its radius is sufficiently small so that it splits into four free pieces E, F, G, H with projective boundary pieces P, Q, S, T as shown in the picture below. The shadowed region is a cobordism between the original complex and the union of E, F, G, H. Although we actually measure the radius using the radial projection to the unit circle (i.e. the square), we pretend that complexes and cobordisms are over the plane.



We extend this cobordism in the following way. On the right vertical edge, we have a quadratic pair $P \oplus Q \to F$. (We are omitting the quadratic structure from notation.) Take the tensor product of this with the symmetric complex of the unit interval $[0, 1]$. Make many copies of such a product and consecutively glue them one after the other to the cobordism. Do the same thing with the other three edges. Then fill in the four quadrants by copies of P, Q, S, T multiplied by the symmetirc complex of $[0, 1]^2$ so that the whole picture looks like a huge square with a square hole at the center:



Although this cobordism is made up of free complexes and projective complexes, the projective complexes sitting over the dotted lines are shifted up 1 dimension, and the projective complexes sitting at the lattice points are shifted up 2 dimension in the union.



We can make pairs of of these (as shown in the picture above for P's) so that each pair contribute the trivial element in the controlled reduced projective class group. Replace each pair by a free complex.

Unlike the real Eilenberg swindle, there are left four projective complexes which do not make pairs. We may assume that they are the boundary pieces of F and H on the outer end. Since the two pairs $P \oplus Q \to F$, $S \oplus T \to H$ are Poincaré, the unions $P \oplus Q$ and $S \oplus T$ are locally chain equivalent to free complexes. Thus we can replace them by free complexes, and now everything is free.

Now recall that we actually measure things by a radial projection to the square. Thus we have a cobordism from the original complex to another complex of very small radius. If we increase the number of layers in the construction, the radius of the outer end becomes arbitrarily small. This is the squeezing in the case of S^1.

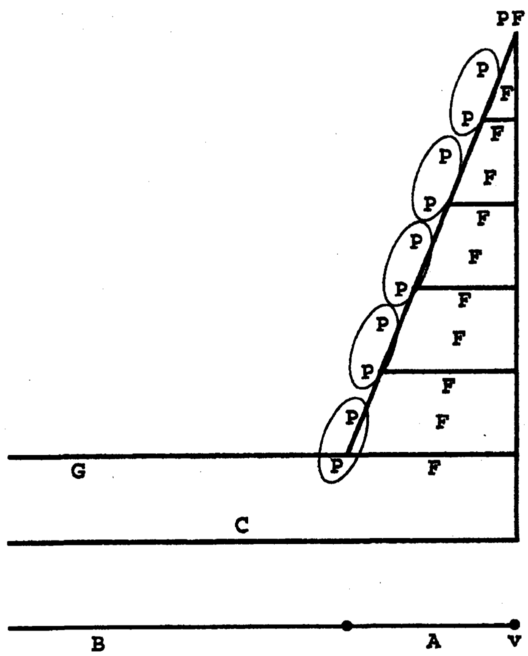
3. Alexander Trick

The method in the previous section does not work for higher dimensions, because we cannot inductively split the projective pieces. But the proof suggests an alternative way toward stability/squeezing. This is the topic of this section. Although we used a radial projection to measure the size in the previous section, we draw pictures of things in their real sizes in this section.

Let X be a polyhedron and $p_X : E \to X$ be a fibration. For a subset Y of X, we denote the restriction $p_X|Y$ of p_X by p_Y. We assume that $n \geq 1$. This is necessary for splitting.

Pick a vertex v of X, and let A be the star neighborhood of v and B be the closure of the complement of A in X. Given a sufficiently small quadratic Poincaré complex $c = (C, \psi)$ on p_X, one can split it according to the splitting of X into A and B : c is cobordant (actually homotopy equivalent if $n \geq 2$) to the union c' of a projective quadratic Poincaré pair $a = (f : P \to F, (\delta\psi', \psi'))$ on p_A and a projective quadratic Poincaré pair $b = (g : P \to G, (\delta\psi'', -\psi'))$ on p_B, where F is an n-dimensional chain complex on p_A, G is an n-dimensional chain complex on p_B, and P is an $(n-1)$-dimensional projective chain complex on $p_{A\cap B}$ ([4][5][7]).

Make many copies of the product cobordism from the pair a to itself, and successively glue them to the cobordism between c and c' as in the picture below. Using the cone structure of A, we shrink these copies toward the central vertex v of A as we go up higher and the copy of a at the top is completely squeezed and lies over v.



This gives us a cobordism from c to a (possibly) projective complex. We will remedy the situation by replacing the projective end by a free complex as follows. The copies of P connecting the layers are actually shifted up 1 dimension in the union, so the marked pairs of P's contribute the trivial element of the controlled $\widetilde{K}_0$ group of (a copy of) $(A \cap B) \times [0,1]$, and we can replace each pair with a free module by adding chain complexes of the form

$$0 \longrightarrow Q_i \xrightarrow{1} Q_i \longrightarrow 0$$

lying over copies of $(A \cap B) \times [0,1]$, where Q_i is a projective module such that $P_i \oplus Q_i$ is free. The last P remaining at the top of the picture can be replaced by a free complex lying over v

as in the following way: Consider the Poincaré duality map for the top copy of the pair a :

$$\mathcal{D}_{(\delta\psi',\psi')} \ : \ F^{n-*} \longrightarrow \mathcal{C}(f) \ .$$

Here $\mathcal{C}(f)$ denotes the algebraic mapping cone of $f : P \to F$. Since this map is a chain equivalence, we have equalities

$$[P] = -([F] - [P]) = -[\mathcal{C}(f)] = -[F^{n-*}] = 0 \in \widetilde{K}_0(R) \ ,$$

in the classical reduced projective class group of R, and hence P is chain equivalent to a free $(n-1)$-dimensional complex F' lying over v. (This is actually obvious if one looks at the construction of P.) Now we can replace the top copy P with F' to finish the construction.

Summary: *Let $n \geq 2$. There exist constants $\delta > 0$ and $\lambda \geq 1$ which depend on n and X such that any n-dimensional quadratic Poincaré complex of radius $\epsilon \leq \delta$ is $\lambda\epsilon$ Poincaré cobordant to another complex which is small in the radial direction of A. The more layers we use, the smaller the result becomes in the radial direction.*

Remarks. (1) We cannot take $\lambda = 1$ in general, since the radius of the complexes gets bigger during the splitting/glueing processes.
(2) This construction will be referred to as the **Alexander trick** at v.

4. Outline of our Approach to the Stability Theorem

We give an outline in the case $n \geq 1$. The stability for $n = 0$ follows from the stability for $n = 4$. We first state the squeezing lemma for quadratic Poincaré complexes:

Lemma 2. (Squeezing of Quadratic Poincaré Complexes) *There exist constants $\delta_0 > 0$ and $\kappa > 1$, which depends on n and X, such that any n-dimensional quadratic Poincaré R-module complex of radius $\epsilon \leq \delta_0$ is $\kappa\epsilon$ Poincaré cobordant to an arbitrarily small quadratic Poincaré complex.*

Sketch of the Argument: Let $v_1, v_2, \ldots, v_m$ be the vertices of X. Let ϵ' be any positive number and $c = (C, \psi)$ be an n-dimensional quadratic Poincaré complex of radius $\epsilon > 0$. We try to construct a cobordism from c to a quadratic Poincaré complex of radius ϵ'. The basic idea is to apply the Alexander trick at every vertex of X.

To keep track of the effects of the Alexander tricks, it is convenient to introduce maps p_1, $\ldots, p_m$ from $X \to [0,1]$ corresponding to the vertices $v_1, \ldots, v_m$: Each point x of X corresponds to its barycentric coordinate $(s_1, \ldots, s_m)$, with $0 \le s_i \le 1$ and $s_1 + \ldots + s_m = 1$. If x does not belong to any simplex of X containing v_i, then $s_i = 0$. The point x is represented as the linear combination $s_1 v_1 + \ldots + s_m v_m$ in a unique way. We define the map $p_i : X \to [0,1]$ by $p_i(x) = s_i$. We are actually normalizing the metric of X by embedding it in the standard $(m-1)$-simplex in $\mathbb{R}^m$ with the maximum metric; X with this metric will be denoted Y.

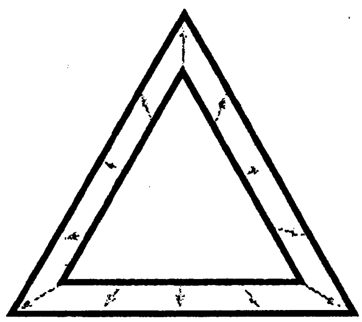
Let λ be the constant appearing in the Alexander trick on Y. Choose a positive number β_0 small enough so that (1) $\lambda^m \beta_0 \le 0.1$, and (2) given a complex of radius $\beta \le \beta_0$ on Y, we can keep on performing Alexander tricks at all the vertices of Y. Find a positive number δ_0 such that objects of radius δ_0 measured in X has radius β_0 measured in Y.

Now suppose $\epsilon \le \delta_0$ and choose $\alpha > 0$ small enough so that objects of radius α measured in Y has radius ϵ' measured in X. The radius β of c measured on Y is smaller than or equal to β_0; perform Alexander tricks at all the vertices using L layers every time to get a new complex c'.

Let r_i denote the radius of a complex with respect to $p_i \circ p_X$ $(i = 1, \ldots, m)$. Then, the radii r_i for c' are all

$$\begin{cases} \lambda^m \beta / L & \text{over } (\lambda^m \beta, 1]\,, \\ \lambda^m \beta & \text{over } [0, \lambda^m \beta]\,. \end{cases}$$

For each simplex of Y which is not a face of other simplices, consider a pseudo-radial projection [6] of a small regular neighborhood of its boundary to the bounday together with the linear stretching of the complement of the regular neighborhood:



Lift this to a map of E and take the functorial image c'' of c'. Then its radii r_i will be

$$\begin{cases} K\lambda^m \beta / L & \text{over } (0, 1]\,, \\ K\lambda^m \beta & \text{at } 0\,. \end{cases}$$

for some constant $K \ge 1$ which comes from the stretching and determined by the integer m.

Now the following lemma implies that all the radii r_i are $mK\lambda^m\beta/L$; therefore, if the number L of the layers used in the Alexander tricks is sufficiently large, the radius of c'' measured on Y is smaller than or equal to α, and hence its radius measured on X is smaller than the given number ϵ'. $\square$

Lemma 3. *Let* $(s_1, \ldots, s_m)$ *and* $(s'_1, \ldots, s'_m)$ *be the barycentric coordinates of* x, $x' \in X$, *and* J *be the subset* $\{j \mid s_j > 0\}$ *of* $\{1, \ldots, m\}$. *If* $|s_i - s'_i| \leq \alpha$ *for every* $i \in J$, *then* $|s_i - s'_i| \leq m\alpha$ *for every* $i = 1, \ldots, m$.

Proof: Let $J' = \{1, \ldots, m\} - J$. From the equalities $\sum s_i = 1$ and $\sum s'_i = 1$, we obtain an equality

$$\sum_{i \in J'} s'_i = \sum_{i \in J} (s'_i - s_i) \ .$$

Therefore

$$\sum_{i \in J'} s'_i \leq \sum_{i \in J} |s'_i - s_i| \leq m\alpha \ .$$

Since all the s'_i's are non-negative, We have $|s'_i - s_i| = s'_i \leq m\alpha$ for $i \in J'$. □

Note that Lemma 2 implies that the relax-control map in Theorem 1 is surjective: Take an element $[c] \in L_n^{\delta,\epsilon}(X; p_X, R)$ with $\delta \leq \delta_0$. Then an inequality $\epsilon \leq \delta_0$ holds and therefore there is a Poincaré cobordism of radius $\kappa\epsilon$ ($\leq \delta_0$) from c to a quadratic Poincaré complex c' of radius ϵ', determining an element $[c'] \in L_n^{\delta',\epsilon'}(X; p_X, R)$ whose image under the relax-control map is $[c]$.

The injectivity can be established in a similar way using a relative version of squeezing.

5. Variations

A. K-theoretic decorations.

There are also controlled analogues of L^p-groups and L^s-groups. $L_n^{p,\delta,\epsilon}(X; p_X, R)$ is defined using ϵ Poincaré ϵ quadratic projective R-module complexes on p_X and δ Poincaré δ projective cobordisms, and $L_n^{s,\delta,\epsilon}(X; p_X, R)$ is defined using ϵ simple Poincaré ϵ quadratic R-module complexes on p_X and δ simple Poincaré δ projective cobordisms. Similar stability results hold for these.

For the L^s-group case, the proof for L-groups work equally well. The chain equivalences used in the proof (including the splitting) are all simple in a controlled fashion.

To get a squeezing result in the L^p-group case, we first take the tensor product of the given projective quadratic Poincaré complex c with the symmetric complex $\sigma(S^1)$ of the circle S^1. We may assume that the radius of $\sigma(S^1)$ is sufficiently small. If the radius of c is also sufficiently small, we can construct a cobordism to a squeezed complex. Split the cobordism along $X \times \{\text{two points}\} \subset X \times S^1$ to get a projective cobordism from the original complex to a squeezed projective complex.

B. UV^1 control maps.

When the control map is UV^1, there is no need to use paths to define morphisms between geometric modules ([2]). This simplifies the definition quite a lot, and we have:

Proposition 4. *Let $p_X : E \to X$ be a UV^1 map to a finite polyhedron. Then for any pair of positive numbers $\delta \geq \epsilon$, there is an isomorohism*

$$L^{\delta,\epsilon}(X; p_X, R) \cong L^{\delta,\epsilon}(X; 1_X, R)$$

for any ring with involution R and any integer $n \geq 0$.

By Theorem 1, the stability holds for $L^{\delta,\epsilon}(X; 1_X, R)$ and hence the stability holds also for $L^{\delta,\epsilon}(X; p_X, R)$.

C. Compact metric ANR's.

Squeezing and stability also hold when X is a compact metric ANR.

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