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On a bounding problem of Calabi-Yau threefolds

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Abstract

In this report, we shall give a tentative argument on bounding probelm of Calabi Yau's the study of which was started by B.Hunt and M.Gross.

1 Introduction

Many families of projective smooth Calabi-Yau threefolds defined over the complex number field have been found, but as far as I know, they are all bounded (i.e., parametrized by quasi projective schemes over the complex number field). It is natural to ask how many families of projective smooth Calabi-Yau threefolds exitst. So far, specialists seem to think that their deformation types are maybe finite or maybe not. Reid's fantasy asserts that there are essentially just one family which corresponds to the fact(conjectured by A.Weil-A.Andreotti and solved affirmatively by K.Kodaira) that every $K3$ surface is a deformation of a non-singular quartic surface in a projective 3-spaces. If Reid's fatasy is true, even if there are non finitely many families of projective smooth Calabi-Yau threefolds, they are "transformed" to a bounded family and one may get feedback. For example, let us recall that K.Kodaira showed that any $K3$ surface can be deformed to an elliptic $K3$ surface in a one family which implies the conjecture of A.Weil-A.Andreotti is true. In this direction, B.Hunt ([9]) asserted the Euler numbers of projective smooth Calabi-Yau threefolds with a fiber structure are bounded but unfortunately his proof based on the theory of variation of Hodge structures contains many crucial gaps. Obviously more geometric information seems to be needed. Later, M.Gross ([7]) showed, extending the Ogg-Shafarevich theory, the surprizingly strong result that elliptic Calabi-Yau threefolds with a rational base is birationally bounded. In this report, we shall consider the other cases of Calabi-Yau threefolds with a fiber structure, that is, the cases where a general fibre is a surface with trivial canonical bundle and the base is a projective line.

2 Bimeromorphic invariants of degenerations of surfaces of Kodaira dimension zero

In this section, we define bimeromorphic invariants of degenerations of surfaces of Kodaira dimension zero.

Definition 2.1 Let (X, Δ) be a normal log variety and let Δ' be a boundary on X such that $\Delta' \leq \Delta$. Assume that (X, Δ) is log canonical and that $K_X + \Delta'$ is $\mathbf{Q}$-Cartier. (X, Δ) is said to be *moderately log canonical with respect* $K_X + \Delta'$ if for any exceptional prime divisor E of the function field of X with $a_l(E; X, \Delta) = 0$, the inequality $a_l(E; X, \Delta') > 1$ holds.

Let $f : X \to B$ be a proper connected morphism from a normal $\mathbf{Q}$-factorial variety defined over the complex number field $\boldsymbol{C}$ (resp. a normal $\mathbf{Q}$-factorial complex analytic space) X onto a smooth projective curve (resp. a unit disk $\mathcal{D} := \{z \in \boldsymbol{C}; |z| < 1\}$) B such that a general fibre $f^*(p)$ (resp. any fibre $f^*(p)$ where p is not the origin) is a normal algebraic variety with only terminal singularity. Let Σ be a set of points in B (resp. the origin 0) such that the fibre $f^*(p)$ is not a normal algebraic variety with only terminal singularity. Put $\Theta_p := f^*(p)_{\mathrm{red}}$ and $\Theta := \sum_{p \in \Sigma} \Theta_p$. We shall consider the following three birational (or bimeromorphic) models.

Definition 2.2 We define the following three birational (bimeromorphic) models.

(1) $f : X \to B$ is called a *minimal fibration* (*resp. degeneration*) if X has only terminal singularity and K_X is f-nef (i.e., the intersection number of K_X and any complete curve contained in a fibre of f is non-negative).

(2) $f : X \to B$ is called a *logarithmic minimal* (*or abbreviated, log minimal*) *fibration* (*resp. degeneration*) if (X, Θ) is divisorially log terminal and $K_X + \Theta$ is f-nef.

(3) $f : X \to B$ is called a *strictly logarithmic minimal* (*or abbreviated, strictly log minimal*) *fibration* (*resp. degeneration*) if (X, Θ) is log canonical with K_X, Θ being both f-nef.

Given a fibrations (resp. degenerations) of algebraic surfaces with Kodaira dimension zero over a smooth projective curve (resp. 1-dimensional unit disk) B, one can get the model (1), (2) by using MMP and LMMP. The model (3) can be constructed also by LMMP starting from the model (2). The model (3) constructed as above enjoyies also the following property; (X, Θ) is moderately log canonical with respect to K_X. Moreover, the model (3) is good in the following sense.

Proposition 2.1 (c.f. [11], Theorem 4.9) *Let $f_i^s : X_i^s \to B$ $(i = 1, 2)$ be two strictly log minimal fibration (or degeneration) of surfaces with Kodaira dimension zero projective over B which are birationally equivalent to each other over B. Assume that $(X_i^s, \Theta_{i,p}^s)$ is moderately log canonical with respect to $K_{X_i^s}$ for $i = 1, 2$. Then $f_1^s : X_1^s \to B$ and $f_2^s : X_2^s \to B$ are connected by a sequence of log flips over a neighbourhood of $p \in B$, that is, there exist birational morphisms between normal threefolds over a neighbourhood of $p \in B$ which are isomorphic in codimension one:*

$$X_1^s := X^{(0)} \to Z^{(0)} \leftarrow X^{(1)} \to Z^{(1)} \cdots \leftarrow X^{(n)} =: X_2^s,$$

where $X^{(k)}$ is $\mathbf{Q}$-factorial for $k = 0, 1, \dots n$.

Let $f^s : X^s \to B$ be a strictly log minimal fibration (or degeneration) of surfaces with Kodaira dimension zero projective over B such that (X^s, Θ^s) is moderately log canonical with respect to K_{X^s}. Since K_{X^s} is numerically trivial over B, there exists a positive integer $\ell_p^* \in \boldsymbol{N}$ such that $f^{s*}(p) = \ell_p^* \Theta_p^s$ for any $p \in B$, where $\Theta_p^s := f^{s*}(p)_{\mathrm{red}}$. Let $\mu : Y \to X^s$ be a minimal model over X^s, that is, μ is a projective birational morphism from a normal $\mathbf{Q}$-factorial Y with only terminal singularity to X^s such that K_Y is μ-nef. By running the minimal model program over B starting from the induced morphism $g := f^s \circ \mu : Y \to B$, we obtain a minimal fibration $h : Z \to B$ and a dominating rational map $\lambda : Y - \to Z$ over B. Since X^s has only log terminal singularity, there exists an effective $\mathbf{Q}$-divisor Δ with $\lfloor \Delta \rfloor = 0$ on Y such that

$$K_Y + \Delta = \mu^* K_{X^s}.$$

Since $K_Y + \Delta$, $K_Z + \lambda_* \Delta$ and K_Z are all numerically trivial over B, there exists a non-negative rational number $\mu_p^* \in \boldsymbol{Q}$ such that

$$\lambda_* \Delta_p = \mu_p^* h^*(p), \tag{2.1}$$

where Δ_p denotes the restriction of Δ in a neighbourhood of the fibre over $p \in B$. Put

$$s_p^* := b(\frac{\ell_p^* - 1}{\ell_p^*} - \mu_p^*), \quad c_p^* := \mu^* \ell_p^*.$$

Proposition 2.1 gives the following:

Corollary 2.1 *$\ell_p^* \in \boldsymbol{N}$ and $\mu_p^* \in \boldsymbol{Q}$ are birational (or bimeromorphic) invariants of germs of singular fibres over $p \in B$ and hence so are s_p^* and c_p^*.*

Let $f : X \to B$ be a minimal fibration of surfaces with Kodaira dimension zero projective over a projective smooth connected curve B defined over the complex number field. The above invariants fit into the canonical bundle formula as follows;

$$K_X = f^*(K_B + \frac{1}{b}L_{X/B}^{ss} + \sum_{p\in B}(\frac{\ell_p^* - 1}{\ell_p^*} - \mu_p^*)p). \qquad (2.2)$$

where $\frac{1}{b}L_{X/B}^{ss}$ is a $\boldsymbol{Q}$-divisor on B defined in [4].

Example 2.1 For degenerations of elliptic curves, one can define invariants ℓ_p^*, μ_p^* and s_p^* in the same way and it can be checked that ℓ_p^* coincides with the multiplicty if the singular fibre is of type ${}_m\mathrm{I}_b$ or otherwise, with the order of the semisimple part of the monodromy group around the singular fibre. We can also obtain the following table:

Table V

	${}_m\mathrm{I}_b$	I_b^*	II	II*	III	III*	IV	IV*
ℓ_p^*	m	2	6	6	4	4	3	3
μ_p^*	0	0	2/3	0	1/2	0	1/3	0
s_p^*	(m-1)/m	1/2	1/6	5/6	1/4	3/4	1/3	2/3
c_p^*	0	0	4	0	2	0	1	0

Here we are using the Kodaira's notation ([10]). See also [6].

Seeing the above table, naively one may expect that the invariatnt c_p^* is the one determined by the variation of Hodge structure around p, but c_p^* involves the index of K_{X^s} which is usually a minimal model theoretic invariant not the Hodge theoretic one. So, in higher dimensional cases, c_p^* seems to be extremely complicated, but still one may hope;

Conjecture 2.1 For any degeneration of algebraic surfaces with Kodaira dimension zero over a one-dimensional complex disk, the number of possible values of c_p^* is finite .

For the above conjecture, we have a partial affirmative answer.

Theorem 2.1 *For any degeneration of abelian surfaces over a one-dimensional complex disk, we have*

$$c_p^* \in \{0, 1/5, 1/4, 1/3, 2/5, 1/2, 2/3, 1, 3/2, 2, 3, 4, 5, 6\}.$$

3 Bounding the number of singular fibres of Abelian Fibred Calabi-Yau threefolds over a projective line

In [9], there is no argument on the number of singular fibres which is important to get the certain finiteness result. For, recalling the Arakelov-Faltings theory which was developed for solving Shafarevich conjecture, smooth families over fixed base tends to have a certain finiteness property. The study of the number of singular fibres was started by Oguiso ([12]).

Let $f : X \to B$ is a projective connected morphism from a normal $\mathbf{Q}$-factorial projective variety X with only canonical singularity onto a B. We define the subset of closed points of B, Σ_f by

$$\Sigma_f := \{p \in B |\ f \text{ is not smooth over a neighbourhood of } p\ \}.$$

Definition 3.1 Let $\mathcal{CY}^3_{B,\mathrm{ab}}$ be the set of all the triple (X, f, B) where X is a normal projective threefold X with only canonical singularity whose canonical divisor K_X is numerically trivial and $f : X \to B$ is a projective connected morphism onto $B \simeq \boldsymbol{P}^1$ whose geometric generic fibre is an abelian surface.

From Theorem 2.1 , we can deduce the following theorem using G.Faltings theory and Zarhin's trick.

Theorem 3.1 *There exists $s \in \boldsymbol{N}$, such that for any triple $(X, f, B) \in \mathcal{CY}^3_{B,\mathrm{ab}}$,*

$$s_f := \text{Card } \Sigma_f \leq s,$$

that is, the number of singular fibres of Abelian Fibred Calabi-Yau threefolds over a projective line is bounded from above by a universal constant.

4 Further tentative argument

To get more results on Abelian Fibred Calabi-Yau threefolds over a projective line, we should study albanese fibration, which is defined from $\mathrm{Alb}^0(X_\eta)$

(see, [8] and also [3]). Albanese fibration plays a role of jacobian fibration for elliptic fibrations. Since jacobian fibrations of eliiptic fibered Calabi-Yau threefolds are again a birationally Calabi-Yau threefolds([7]), It is natural to hope;

Conjecture 4.1 Albanese fibration of Abelian Fibred Calabi-Yau are again Calabi-Yau.

For the above conjecture, we have:

Proposition 4.1 *Let $f : X \to B$ be a projective connected morphism from a normal $\mathbf{Q}$-factorial projective variety X with only canonical singularity onto B whose geometric generic fiber is an abelian variety and let $\varphi : \mathcal{A} \to B$ be the Albanese group scheme associated to f. Let $\bar{\varphi} : \bar{\mathcal{A}} \to B$ be a projective fibration from a smooth $\bar{\mathcal{A}}$ which is birational to $\mathcal{A}$ over B.*

(i) *we have $L^{ss}_{X/B} \sim_{\mathbf{Q}} L^{ss}_{\bar{\mathcal{A}}/B}$ and*

(ii) *moreover, if we assume that $\dim X = 3$ and that $f : X \to B$ admits an analytic local section in a neighbourhood of any closed points $p \in B$, then we have $\Sigma_f = \Sigma_{\bar{\varphi}}$ and $s^*_p(f) = s^*_p(\bar{\varphi})$, where $s^*_p(f)$ and $s^*_p(\bar{\varphi})$ are the analytic local bimeromorphic invariants s^*_p of the fibres of f and φ over $p \in B$ respectively. In particular, the Kodaira dimensions of X and $\bar{\mathcal{A}}$ are the same.*

One may also hope;

Conjecture 4.2 Abelian Fibred Calabi-Yau has local sections everywhere.

For further investigation, R-R will be useful.

Proposition 4.2 *Let $f : X \to B$ be a projective fibration, where X is a $\mathbf{Q}$-factorial normal variety with only canonical singularity and B is a connected smooth projective curve defined over the complex number field. Then,*

$$\sum_q (-1)^q \deg R^q f_* \mathcal{O}(K_{X/B}) = (-1)^{\dim X}\{\chi(\mathcal{O}) - \chi(\mathcal{O}_{X_\eta})\chi(\mathcal{O}_B)\}$$

In particular, If $\dim X = 3$, we have

$$\deg f_*\mathcal{O}(K_{X/B}) - \deg R^1 f_*\mathcal{O}(K_{X/B}) = \frac{c_2(X_{\bar{\eta}})}{24}\deg(L^{ss}_{X/B} + \sum_{p\in B} s^*_p) - \sum_{P_\alpha} \frac{r_\alpha^2 - 1}{24 r_\alpha}$$

(see [14] for notation).

If $q(X_{\bar{\eta}}) = 0$ (for example, $K3$ fibred case), we have

$$\deg f_*\mathcal{O}(K_{X/B}) = \frac{c_2(X_{\bar{\eta}})}{24}\deg(L^{ss}_{X/B} + \sum_{p\in B} s_p^*) - \sum_{P_\alpha}\frac{r_\alpha^2 - 1}{24r_\alpha}.$$

[9] asserts that boundedness results follows if we fix $\deg f_*\mathcal{O}(K_{X/B})$ (for example, $\deg f_*\mathcal{O}(K_{X/B}) = 2$ if X is Calabi-Yau) not using any other property of Calabi-Yau. But seeing the above formula, it seems that fixing $\deg f_*\mathcal{O}(K_{X/B})$ is geomertrically nonsense.

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