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On discrete Morse semi-flow

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1 Introduction.

Set d and D be positive integers greater than 1. Let $\mathbb{B}^d$ and $\mathbb{S}^D$ be the unit ball centered at the origin in $\mathbb{R}^d$, the unit sphere $\mathbb{S}^D$ in $\mathbb{R}^{D+1}$ and T a positive number. Give Q by $(0,T)\times\mathbb{B}^d$. This article studies a certain time-difference space-differential system; We call the solution to it "Discrete Morse Semiflow", which is abbreviated to "DMS". This system enables us discuss at least two important problems in Geometric evolutional problems: Heat flows for harmonic mappings and mean curvature motion. To explain DMS, we introduce a several notation: Let h be a positive number and N_T be $[T/h]+1$. We put $t_n := nh$ $(n=0,\ldots,N_T)$ and set $k_0 = \big(1-h/(16T)\log(1/h)\big)$. $\chi(t)\in C^\infty$ with

$$\chi(t) := \begin{cases} t & t\le 2, \\ 3 & t>4, \end{cases} \tag{1.1}$$

Give a mapping $u_0\in H^{1,2}(\mathbb{B}^d;\mathbb{S}^D)$. Then DMS is designated by a sequence of mappings $\{u_n\}(n=1,\ldots,N_T)\subset\{u\in H^{1,2}(\mathbb{B}^d;\mathbb{S}^D)\,;\,u-u_0\in \overset{\circ}{H}{}^{1,2}(\mathbb{B}^d;\mathbb{R}^{D+1})\}$ of the solution of the following difference-differential systems:

$$\frac{u_n-u_{n-1}}{h} - \triangle u_n + \frac{k_n}{\sqrt[4]{h}}\dot\chi\big((|u_n|^2-1)^2\big)\big(|u_n|^2-1\big)u_n = 0 \quad \text{in } \mathbb{B}^d, \tag{1.2}$$

$$u_n = u_0 \quad \text{on} \quad \partial\mathbb{B}^d. \tag{1.3}$$

An interpolational convention $u_{\bar{h}}(t,x)$ and $u_h(t,x)$ $(t>0)$ respectively indicates

$$
\begin{aligned}
u_{\bar{h}}(t,x) &:= u_n(x) && \text{for} \quad t_{n-1} < t \le t_n, \\
u_h(t,x) &:= \frac{t-t_{n-1}}{h}u_n(x) + \frac{t_n - t}{h}u_{n-1}(x) && \text{for} \quad t_{n-1} < t \le t_n.
\end{aligned}
$$

Note $\partial u_h/\partial t(t,x) = (u_n(x) - u_{n-1}(x))/h$ for $t_{n-1} < t < t_n$. When no confusion may arise, we say a pair of functions $u_{\bar{h}}$ and u_h to be DMS; $u_{\bar{h}}$ and u_h satisfy

$$
u_{\bar{h}} \in L^\infty(0,T;H^{1,2}(\mathbb{B}^d;\mathbb{S}^D)), \tag{1.4}
$$

$$
\int\limits_{\mathbb{B}^d} \left(\langle \frac{\partial u_h}{\partial t}, \phi\rangle + \langle \nabla u_{\bar{h}}, \nabla\phi\rangle\right) dx = -\frac{k_{\bar{h}}}{\sqrt[4]{h}} \int\limits_{\mathbb{B}^d} \left(|u_{\bar{h}}|^2 - 1\right)\langle u_{\bar{h}}, \phi\rangle\, dx \tag{1.5}
$$

$$\text{for all } \phi \in C_0^\infty(\mathbb{B}^d;\mathbb{R}^{D+1}),$$

$$
u_{\bar{h}}(t,x) - u_0(x) \in \overset{\circ}{H}{}^{1,2}(\mathbb{B}^d;\mathbb{R}^{D+1}) \quad \text{for every} \quad t \ (0 \le t \le N_T h),
$$

$$
\lim_{h\searrow 0} \|u_h(t,\circ) - u_0(\circ)\|_{L^2(\mathbb{B}^d)} = 0. \tag{1.6}
$$

I addict to DMS: We show that DMS satisfies a maximal principle, a few global energy inequalities, a monotonicity inequality for the scaled energy and finally a reverse Poincaré inequality. By using the inequalities above, we prove that DMS converges to a heat flow for harmonic mappings and discuss a partial regularity result on it. Here for any given mapping $u_0 \in H^{1,2}(\mathbb{B}^d;\mathbb{S}^D)$, we call $u \in L^\infty(0,T;H^{1,2}(\mathbb{B}^d;\mathbb{S}^D)) \cap H^{1,2}(0,T;L^2(\mathbb{B}^d;\mathbb{S}^D))$ a heat flow for harmonic mappings provided

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + |\nabla u|^2 u && \text{in } Q, \qquad (1.7)\\
u(0,x) &= u_0(x) && \text{in} \quad \{0\}\times\mathbb{B}^d, \\
u(t,x) &= u_0(x) && \text{in} \quad (0,T)\times\partial\mathbb{B}^d.
\end{aligned}
$$

The following fact is well-known

Remark 1 (1.7) *is equivalent to*

$$
\frac{\partial u}{\partial t}\wedge u - \Delta u \wedge u = 0 \quad \text{in} \quad \left(C_0^\infty(Q;\mathbb{R}^{D+1})\right)^*, \tag{1.8}
$$

$$
|u| = 1 \qquad \text{in} \quad \text{a.e} \quad z \in Q. \tag{1.9}
$$

The parabolic system holds in the following weak sense:

$$\int_Q \left(\langle \frac{\partial u}{\partial t}, \phi \rangle + \langle \nabla u, \nabla \phi \rangle - \langle u, \phi \rangle |\nabla u|^2 \right) dz = 0 \text{ for any } \quad \phi \in C_0^\infty(Q; \mathbb{R}^{D+1}), \tag{1.10}$$

$$u(t,x) - u_0(x) \in \overset{\circ}{H}{}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) \qquad \text{for almost every} \quad t \in (0,T), \tag{1.11}$$

$$\lim_{t \to +0} u(t, \circ) = u_0(\circ) \qquad \text{in} \quad L^2(\mathbb{B}^d; \mathbb{R}^{D+1}). \tag{1.12}$$

My main result of this article is

Theorem 1 (Partial Regularity) *There exists a heat flow for harmonic mappings and it is smooth on a relative open set in Q whose compliment has* 0 *d-dimensional Hausdorff measure with respect to the parabolic metric.*

The proof of Theorem 1 can be performed by combining Theorem 8 with Theorem 9.

2 DMS.

In this chapter, we state a discrete maximal principle and a few global energy-estimates. Thereafter we establish a monotonicity inequality for the scaled energies and a reverse Poincaré inequality, which are the main techinical tools of this sort of study. The first is

Theorem 2 (Discrete maximal Principle) *Each of DMS $\{u_n\}$ $(n = 1, \ldots, N_T)$ implies*

$$|u_n| \le 1 \quad \text{for all point } x \in \mathbb{B}^d.$$

Theorem 3 (Energy Estimate). *For any given mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$, DMS $\{u_n\}$ $(n = 1, 2, \ldots, N_T)$ satisfies*

$$\int_{\mathbb{B}^d} \left(\frac{1}{2} |\nabla u_n|^2 + \frac{k_n}{4\sqrt[4]{h}} \left(|u_n|^2 - 1 \right)^2 \right) dx$$

$$\leq \frac{1}{2}\int\limits_{\mathbb{B}^d} |\nabla u_0|^2\, dx \quad \textit{for any integer } n \quad (n = 1, \ldots, N_T), \tag{2.1}$$

$$\frac{h}{2}\sum_{n=1}^{N_T}\int\limits_{\mathbb{B}^d}\left|\frac{u_n - u_{n-1}}{h}\right|^2 dx$$
$$+ \frac{\log(1/h)}{16T} h \sum_{n=1}^{N_T} \frac{k_{n-1}}{4\sqrt[4]{h}} \int\limits_{\mathbb{B}^d} \left(|u_{n-1}|^2 - 1\right)^2 dx \leq \frac{1}{2}\int\limits_{\mathbb{B}^d} |\nabla u_0|^2 dx. \tag{2.2}$$

Lemma 1 (Global Pokhojaev Identity). *DMS u_n $(n = 1, 2, \ldots, N_T)$ have the following property:*

$$\frac{1}{2}\int\limits_{\partial\mathbb{B}^d}\left|\frac{\partial u_n}{\partial |x|}\right|^2 d\mathcal{H}_x^{d-1}$$
$$+ \frac{d-2}{2}\int\limits_{\mathbb{B}^d} |\nabla u_n|\, dx + \frac{dk_n}{4\sqrt[4]{h}}\int\limits_{\mathbb{B}^d}\left(|u_n|^2 - 1\right)^2 dx$$
$$= \frac{1}{2}\int\limits_{\partial\mathbb{B}^d} |\nabla_{\tan} u_0|^2\, d\mathcal{H}_x^{d-1} + \int\limits_{\mathbb{B}^d} \langle \frac{u_n - u_{n-1}}{h}, \langle x, \nabla\rangle u_n\rangle\, dx. \tag{2.3}$$

Corollary 1 (The first derivatives estimates at $\partial\mathbb{B}^d$).

$$\frac{1}{2}\int\limits_{h}^{T} dt \int\limits_{\mathbb{B}^d}\left|\frac{\partial u_{\bar{h}}}{\partial |x|}\right|^2 d\mathcal{H}_x^{d-1}$$
$$\leq 2T\int\limits_{\mathbb{B}^d} |\nabla_{\tan} u_0|^2\, d\mathcal{H}_x^{d-1} + 2(T+1)\int\limits_{\mathbb{B}^d} |\nabla u_0|^2\, dx. \tag{2.4}$$

Corollary 2 (The rate of the convergence). *If $\triangle u_0 \in L^{p_0}(\mathbb{B}^d; \mathbb{R}^{D+1})$ for some $p_0 > 1$,*

$$\int\limits_{\mathbb{B}^d} |\nabla(u_1 - u_0)|^2\, dx \leq 2^{1-2/p_0'} ||\nabla u_0||_{L^2(\mathbb{B}^d)}^{2/p_0'} ||\triangle u_0||_{L^{p_0}(\mathbb{B}^d)} \cdot h^{1-1/p_0}, \tag{2.5}$$

holds with $1/p_0 + 1/p_0' = 1$.

Remark 2 *The typical example of map from $\mathbb{B}^d$ to $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ may be the equator map given by $x/|x|$. If $u_0(x) = x/|x|$, then $\Delta u_0 \in L^{q_0}(\mathbb{B}^d)$ as long as $1 < q_0 < d/2$. We refer to F.Bethuel and X.Zheng [1]. Namely the assumption on L^{p_0}-integrability about Δu_0 is just peril.*

Lemma 2 (Higher Order Differential Estimates). *DMS u_n $(n = 1, 2, \ldots, N_T)$ satisifies*

$$h\sum_{n=2}^{N_T}\int_{\mathbb{B}^d} |\Delta(u_n - u_{n-1})|^2\,dx \leq Ch^{3/2}\int_{\mathbb{B}^d} |\nabla u_0|^2\,dx + \frac{1}{2}\int_{\mathbb{B}^d} |\nabla(u_1 - u_0)|^2\,dx. \tag{2.6}$$

Now, we are in the position to state a monotonicity inequality for the scaled energy; For $z_0 = (t_{n_0}, x_0) \in Q$ and a positive number R, the scaled energy is denoted by

$$E_h(R; z_0) := \frac{1}{2R^d}\int_{t_{n_0}-\theta_0(2R)^2}^{t_{n_0}-\theta_0R^2} dt\int_{\mathbb{B}^d}\Big(|\nabla u_h|^2 + \frac{k_{\bar{h}}}{2\sqrt[4]{h}}\big(|u_h|^2 - 1\big)^2\Big) \times \exp\Big(\frac{|x - x_0|^2}{4(t - t_{n_0})}\Big)\,dx. \tag{2.7}$$

Lemma 3 (Monotonicity for the Scaled Energy). *For any point $z_0 = (t_{n_0}, x_0)$ and any positive number R,*

$$\begin{aligned}\frac{dE_h}{dR}(R; z_0) \geq& -\frac{1}{R^{d-1}}\int_{t_{n_0}-\theta_0(2R)^2}^{t_{n_0}-\theta_0R^2} \frac{t - t_{n_0}}{R^2}\,dt\int_{\mathbb{B}^d} \\ &\times \Big|\frac{\partial u_h}{\partial t} + \langle\frac{x - x_0}{2(t - t_{n_0})}, \nabla\rangle u_h\Big|^2 \exp\Big(\frac{|x - x_0|^2}{4(t - t_{n_0})}\Big)\,dx \\ &+ \frac{1}{2R^{d+1}}\int_{t_{n_0}-\theta_0(2R)^2}^{t_{n_0}-\theta_0R^2} \frac{k_{\bar{h}}dt}{\sqrt[4]{h}}\int_{\mathbb{B}^d}\big(|u_h|^2 - 1\big)^2\exp\Big(\frac{|x - x_0|}{4(t - t_{n_0})}\Big)\,dx \\ &- C_{\mathrm{M}}(R, R_0; h).\end{aligned} \tag{2.8}$$

where

$$C_{\mathrm{M}}(R, R_0; h). := C_{\mathrm{M},1} + C_{\mathrm{M},2}$$

$$C_{\mathrm{M},1} := \frac{CR}{\mathrm{dist}^{d+1}(x_0, \partial\mathbb{B}^d)}\left(\int_{\mathbb{B}^d} |\nabla u_0|^2\, dx + \int_{\partial\mathbb{B}^d} |\nabla_{\tan} u_0|^2\, d\mathcal{H}_x^{d-1}\right),$$

$$C_{\mathrm{M},2} := -C||\nabla(u_1 - u_0)||_{L^2(\mathbb{B}^d)}||\nabla u_0||_{L^2(\mathbb{B}^d)} + \frac{C\sqrt[4]{h}}{R^{d+1}}||\nabla u_0||_{L^2(\mathbb{B}^d)}.$$

Hereafter we state a point-wise estimate and the inequality of "a hybrid type". The latter part of the chapter will be devoted to saying these inequalities. We supposed θ_0, θ_1 and R be positive numbers with

$$0 < \theta_0 < 1, \;\; 1 < \theta_1, \;\; R > 0, \;\; \max\left(\frac{2\theta_1}{3}, 2\right) < \frac{\theta_0 R^2}{h}, \tag{2.9}$$

and we set

$$N_1 := \left[\frac{\theta_0 R^2}{h}\right], \;\; N_2 := \left[\frac{\theta_0 (2R)^2}{h}\right].$$

We must remark that all N_i $(i = 1, 2, 3, 4)$ are positive integers by assumption (2.9).

Theorem 4 (A Point-wise Estimate) *There exists a positive number ϵ_0 depending only on d, such that if $w_{\bar{h}}$ satisfies*

$$\fint_{t_{n_1}-2N_1}^{t_{n_1}} dt \fint_{B_{2R}(x_0)} (1 - \langle u_{\bar{h}}, K\rangle)\, dx < \epsilon_0 \tag{2.10}$$

for any cylinder $Q_{2R,2N_1h}(t_{n_1}, x_0)$ $(:= (t_{n_1-2N_1}, t_{n_1}) \times B_{2R}(x_0)) \subset\subset Q$, then

$$|\{z \in Q_{2R,N_1h}(t_{n_1}, x_0)\; ;\; \langle u_{\bar{h}}, K\rangle \le 1 - \delta_0\}| \le C\frac{h\log(1/h)}{\delta_0^3}\int_{\mathbb{B}^d} |\nabla u_0|^2\, dx \tag{2.11}$$

with $n_1 = n_0 + N_2 - N_1$ and K is any vector in $\mathbb{R}^{D+1}$.

We must remark that all N_i $(i = 1, 2, 3, 4)$ are positive integers by assumption (2.9).

Theorem 5 (Discrete Hybrid Inequality) *DMS $u_{\bar{h}}$ and u_h have the following inequality: There exists positive constant C_H depending only on d such that for any numbers θ_0, θ_1, R satisfying the condition* (2.9), *for any cylinders $(t_{n_0}, t_{n_0+N_2}) \times B_{2R}(x_0) \subset\subset Q$,*

$$\int_{t_{n_0}+N_2-N_1}^{t_{n_0}+N_2} dt \int_{B_R(x_0)} \left(\frac{1}{2}|\nabla u_{\bar{h}}|^2 + \frac{k_{\bar{h}}}{4\sqrt[4]{h}}\left(|u_{\bar{h}}|^2 - 1\right)^2 + \frac{\theta_0 R^2}{2}\left|\frac{\partial u_h}{\partial t}\right|^2 \right) dx$$
$$+ \frac{\theta_0 R^2}{10} \int_{B_R(x_0)} \left(\frac{1}{2}|\nabla u_{\bar{h}}|^2 + \frac{k_{\bar{h}}}{4\sqrt[4]{h}}\left(|u_{\bar{h}}|^2 - 1\right)^2 \right) dx \Bigg|_{t=t_{n_0+N_2}}$$
$$\leq C_H \max\left(\left(1 - \frac{\theta_1}{N_1}\right)^{N_1}, \theta_0, \delta(\epsilon_0) \right) \tag{2.12}$$
$$\times \int_{t_{n_0}+N_2-3N_1-1}^{t_{n_0}+N_2} dt \int_{B_{2R}(x_0)} \left(\frac{1}{2}|\nabla u_{\bar{h}}|^2 + \frac{k_{\bar{h}}}{4\sqrt[4]{h}}\left(|u_{\bar{h}}|^2 - 1\right)^2 + \frac{\theta_0 R^2}{2}\left|\frac{\partial u_h}{\partial t}\right|^2 \right) dx$$
$$+ \left(1 + \theta_0 + \frac{1}{\theta_0}\right) \frac{C_H}{\log(1/\theta_1)^2 R^2} \int_{t_{n_0}+N_2-3N_1}^{t_{n_0}+N_2} dt \int_{B_{3R/2}(x_0)} |u_{\bar{h}} - K|^2 \, dx$$
$$+ O(h).$$

where $R_0 = \min(\sqrt{t_{n_0}}/2\theta_0, \operatorname{dist}(x_0, \partial\mathbb{B}^d))$, ϵ_0 is a certain positive constant appeared in Theorem 4, respectively and $\delta_0(\epsilon_0) = \epsilon_0^{1/d \cdot 1/(1+2/d)(1+4/d)}$.

Remark 3 *If one takes θ_1 being sufficiently large and θ_0, ϵ_0 being sufficiently small, then the coefficient of the first term on the right-hand side above is small.*

3 Heat Flows for Harmonic Mapping.

This chapter establishes the existence and a partial regularity on a heat flow for harmonic mappings that are obtained as the limit of DMS. The existence theorem is a slight modification of Y.Chen [3] and see also L.C.Evans [5, p.48, 5.A.1] and J.Shatah [9]. On the other hand the regularity result will be established by means of a blow-up technique. For the blow-up technique used here, we refer to R.Hardt, D.Kinderlehler and F.H.Lin [7] and R.Schöen and K.Uhlenbeck [11]. First of all we mention two convergence theorems directly derived from Theorem 2 and Theorem 3:

Theorem 6 (Convergence) *There exists a subsequence* $\{u_{\bar{h}_k}\}, \{u_{h_k}\}$ $(k = 1, 2, \ldots)$ *of* $\{u_{\bar{h}}\}, \{u_h\}$ $(h > 0)$ *respectively and a mapping* $u \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D))$ $\cap\, H^{1,2}\,(0, T; L^2(\mathbb{B}^d; \mathbb{S}^D))$ *such that* $u_{\bar{h}_k}$ *and* u_h *respectively converges weakly-* $*$ *and weakly to* u *in* $L^\infty(0, T; H^{1,2}(\mathbb{B}^d\,; \mathbb{R}^{D+1}))$ *and* $H^{1,2}\,(0, T; L^2(\mathbb{B}^d\,; \mathbb{R}^{D+1}))$, *so does* $u_{\bar{h}_k}$ *strongly to* u *in* $L^2(Q)$ *and* $u_{\bar{h}_k}$ *point-wisely to* u *as* $k \nearrow \infty$.

Theorem 6 enables us state the following existence theorem:

Theorem 7 (Existence) *Each of DMS:* $u_{\bar{h}}$ *and* u_h *respectively converges to a heat flow for harmonic mappings* u *in* $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ *and* $H^{1,2}\,(0, T\,; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ *as* $h \searrow 0$(modulo a subsequence of h).

Proof of Theorem 7. Since $\nabla u_{\bar{h}}$ and $\partial_t u_h$ is uniform bounded in $L^\infty(0, T; L^2(\mathbb{B}^d\,; \mathbb{R}^{D+1}))$ and $L^2(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ respectively, and a subsequence of $u_{\bar{h}}$ and u_h also converges weakly-$*$ and weakly to a map u in $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $H^{1,2}(0, T; L^2\,(\mathbb{B}^d; \mathbb{R}^{D+1}))$ respectively, strongly in $L^2(\mathbb{B}^d; \mathbb{R}^{D+1})$, $|u_{\bar{h}}| \leq 1$, almost everywhere in Q as $h \searrow 0$; We show that the map u is indeed a heat flow for harmonic mappings. Since $u_{\bar{h}}$ and u_h satisfy

$$\frac{\partial u_h}{\partial t} - \triangle u_{\bar{h}} = \frac{k_{\bar{h}}}{\sqrt[4]{h}}(1 - |u_{\bar{h}}|^2)u_{\bar{h}},$$

by taking a wedge product, we have

$$\left(\frac{\partial u_h}{\partial t} \wedge u_{\bar{h}} - \triangle u_{\bar{h}}\right) \wedge u_{\bar{h}} = 0 \quad \text{in} \quad \left(C_0^\infty(Q; \mathbb{R}^{D+1})\right)^*. \tag{3.1}$$

Thus by virtue of $u \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d\,; \mathbb{S}^D)) \cap H^{1,2}\,(0, T; L^2(\mathbb{B}^d\,; \mathbb{S}^D))$, Remark 1, Theorem 6, we observe that u satisfies (1.10), (1.11) and (1.12), i.e. u is a heat flow for harmonic mappings $\square$

Remark 4 *In the following, we fix a subsequences* $\{h_k\}$ $(k = 1, 2, 3, \ldots)$ *of* $\{h\}$ $(h > 0)$ *that makes DMS converge to a heat flow for harmonic mappings* u.

Definition 1 *Fix a point $z_0 = (t_0, x_0) \in Q$. We indicate $\mathcal{M}$ by the following rescaled Radon measure:*

$$\mathcal{M}\,(Q_R(z_0)) := \frac{\liminf_{h_k \searrow 0}}{2\theta_0 R^d} \int\limits_{Q_R(z_0)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}}(|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0 R^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz,$$

for any positive number θ_0 and any cylinder $Q_R(z_0) \subset\subset Q$.

Remark 5 (Measured Hybrid Inequality) *Assume that $u_{\bar{h}_k}$ and u_{h_k} respectively converges weakly-$*$ and weakly in $L^\infty(0,T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $H^{1,2}(0,T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ to a heat flow for harmonic mappings $u \in L^\infty(0,T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1})) \cap H^{1,2}\,(0,T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ as $h_k \searrow 0$. Then take the pass to the limit $h_k \searrow 0$ in (2.12) to infer the following: For any positive θ_2, there exists a positive constant C_{HM} depending only on d, θ_2 such that*

$$\mathcal{M}\,\big(Q_R(z_0)\big) \le \theta_2 \mathcal{M}\,\big(Q_{2R}(z_0)\big) + C_{HM} \fint\limits_{Q_{2R}(z_0)} |u - K|^2\,dz \qquad (3.2)$$

holds for any vector $K \in \mathbb{R}^{D+1}$ and $Q_R(z_0) \subset Q_{2R}(z_0) \subset\subset Q$ with $z_0 = (t_0, x_0)$ and $Q_{2R}(z_0) = (t_0 - \theta_0(2R)^2, t_0) \times B_{2R}(x_0)$.

In the similar way as in L.Simon [10, Lemma 2, p31], we can assert the following reverse Ponicaré inequality:

Corollary 3 (Reverse Poincaré inequality). *The rescaled Radon measure implies the reverse Poincaré inequality: whenever $Q_{4R} \subset\subset Q$,*

$$R^{d+2}\mathcal{M}\,(Q_R(z_0)) \le C_{\mathrm{PO}} \fint\limits_{Q_{4R}(z_0)} |u - K|^2\,dz \qquad (3.3)$$

holds, where C_{PO} is a certain positive constant depending only on θ_2 and d.

Proof of Corollary 3. Let

$$M = \sup_{\{Q_\sigma(z); Q_\sigma(z) \subset Q_{2R}(z_0)\}} \sigma^{d+2}\mathcal{M}\,(Q_\sigma(z))$$

and then take any cylinder $Q_\sigma(z)$ with $Q_\sigma(z) \subset Q_{2R}(z_0)$. Notice that such a cylinder can be covered by cylinders $Q_{\sigma/4}(z_i)$ $(i = 1, 2, 3, \ldots, S)$

with $z_i \in Q_\sigma(z)$ and $Q_\sigma(z_i) \subset Q_{2R}(z_0)$. We can evidently bound the number S by a fixed constant depending only on d. Recall (3.2); Then

$$\begin{aligned} \sigma^{d+2}\mathcal{M}(Q_\sigma(z)) &\leq 4^{d+2}\sum_{i=1}^{S}\left(\frac{\sigma}{4}\right)^{d+2}\mathcal{M}(Q_{\sigma/4}(z_i)) \\ &\leq 4^{d+2}\theta_2\left(\frac{\sigma}{2}\right)^{d+2}\mathcal{M}(Q_{\sigma/2}(z_i)) \\ &+ 4^{d+2}C_{\mathrm{MH}}\sum_{i=1}^{S}\int_{Q_{2\sigma}(z_i)} |u - K|^2\,dz \\ &\leq 4^{d+2}S\theta_2 M + 4^{d+2}C_{\mathrm{MH}}S\int_{Q_{4R}(z_0)} |u - K|^2\,dz. \end{aligned}$$

Taking "sup"on the right-hand side above, we have

$$M \leq 4^{d+2}S\theta_2 M + 4^{d+2}SC_{\mathrm{MH}}S\int_{Q_{4R}(z_0)} |u - K|^2\,dz,$$

whereupon $\theta_2 = 1/(24^{d+2}S)$, we infer

$$R^{d+2}\mathcal{M}(Q_R(z_0)) \leq 24^{d+2}SC_{\mathrm{MH}}\int_{Q_{4R}(z_0)} |u - K|^2\,dz. \tag{3.4}$$

We can state one of the main assertions:

Theorem 8 (Energy Improvement) *For some positive numbers ϵ_0, θ_0 and θ_1, the following holds: for any positive number R and point $z_0 = (t_0, x_0)$ and any measure $\mathcal{M}$, for any cylinder $Q_R(z_0)$ $(=: (t_0 - \theta_0 R^2, t_0) \times B_R(x_0)) \subset\subset Q$,*

$$\begin{aligned} &\mathcal{M}\Big(Q_R(z_0)\Big) < \epsilon_0^2 \quad \text{implies} \\ &\mathcal{M}\left(Q_{\theta_1 R}(z_0)\right) < \frac{1}{2}\mathcal{M}\Big(Q_R(z_0)\Big). \end{aligned} \tag{3.5}$$

<u>Proof of Theorem 8.</u> The proof can be proceeded by a contradiction: If the statement would be false, then for any positive number θ_1 less than $1/2$, there would exist sequences of positive numbers R_i, of points $z_i =$

$(t_i, x_i) \subset Q$, of measures $\mathcal{M}_i^t$ and of heat flow for harmonic mappings u_i $(i = 1, 2, \ldots)$ such that for any θ_0 with $Q_{R_i}(z_i) \subset\subset Q$,

$$\mathcal{M}_i^t\left(Q_{R_i}(z_i)\right) =: \epsilon_i^2 < \frac{1}{i}, \tag{3.6}$$

$$\text{but} \quad \mathcal{M}_i^t\left(Q_{\theta_1 R_i}(z_i)\right) \geq \frac{\epsilon_i^2}{2}. \tag{3.7}$$

By rescaling

$$z = (t, x) \to \bar{z} = (\bar{t}, \bar{x}) = \left(\frac{t - t_i}{\theta_0 R_i^2}, \frac{x - x_i}{R_i}\right),$$

without a loss of generality, we can rewrite (3.6) and (3.7) as

$$\mathcal{M}_i^t\left(Q_1(0)\right) = \epsilon_i^2 < \frac{1}{i}, \tag{3.8}$$

$$\text{but} \quad \mathcal{M}_i^t\left(Q_{\theta_1}(0)\right) > \frac{\epsilon_i^2}{2}. \tag{3.9}$$

By using the rescaling $\bar{z} = ((t - t_i)/\theta_0 R_i^2, (x - x_i)/R_i)$ and a positive number r, (3.2) becomes

$$\mathcal{M}_i^t\left(Q_{\theta_1}(0)\right) \leq C \fint_{Q_{2\theta_1}(0)} |u_i - u_{i,Q_{2\theta_1}}|^2 \, d\bar{z}. \tag{3.10}$$

Set $v_i(\bar{z}) := \dfrac{1}{\epsilon_i}(u_i(\bar{z}) - u_{i,Q_1})$ for any r with $\theta_1 \leq r \leq \dfrac{1}{2}$.

By assumption (3.6), a subsequence of v_i converges weakly to a mapping $v_\infty \in L^2(0, T; H^{1,2}(B_1(0); \mathbb{R}^{D+1})) \cap H^{1,2}(0, T; L^2(B_1(0); \mathbb{R}^{D+1}))$ as $i \nearrow \infty$ (modulo a subsequence of i). In addition, since v_i satisfies the systems:

$$\frac{1}{\theta_0}\frac{\partial v_i}{\partial \bar{t}} - \Delta v_i = \epsilon_i |\nabla v_i|^2 u_i$$

in the sense of $(C_0^\infty(Q_1(0); \mathbb{R}^{D+1}))^*$, by using L.C.Evans [E1, p.39, Theorem 3] and noting (3.6) again, we find that v_∞ is the solution of

$$\frac{1}{\theta_0}\frac{\partial v_\infty}{\partial \bar{t}} - \Delta v_\infty = 0, \tag{3.11}$$

in the classical sense. From the gradient estimate on the solution of the equation (3.11) by S.Campanato [2] and the Rellich-Kondrachev theorem, it follows that $\operatorname{ess}\cdot\sup_{\bar z\in Q_{\theta_1}(0)} (|\nabla v_\infty| + |\partial v_\infty/\partial t|) \le C\|\nabla v_\infty\|_{L^2(Q_1(0))}$ and v_i converges strongly to v_∞ in $L^2(Q_1(0))$ as $i \nearrow \infty$. Thus

$$\fint_{Q_{2\theta_1}} |v_i|^2\, d\bar z \le 2\fint_{Q_{2\theta_1}} |v_\infty|^2\, d\bar z + 2\fint_{Q_{2\theta_1}} |v_i - v_\infty|^2\, d\bar z \le Cr^2,$$

$$\fint_{Q_{2\theta_1}} |u_i - u_{i,Q_{2\theta_1}}|^2\, d\bar z \le \epsilon_i^2 \fint_{Q_{2\theta_1}} |v_i|^2\, d\bar z \le 2C\theta_1^2\epsilon_i^2,$$

holds if i is sufficiently large possibly depending on θ_1. Consequently we infer

$$\mathcal{M}_i^{\flat} (Q_{\theta_1}(0,0)) \le C\theta_1\epsilon_i^2. \tag{3.12}$$

If we choose $C\theta_1 < 1/2$, which is $\mathcal{M}^{\flat} (Q_{\theta_1}(0)) < \epsilon_i^2/2$, then we find that this is a contradiction of our choice.

Theorem 9 (Singular Set) *Let ϵ_0 be the positive number appeared in Theorem 8. Define*

$$\mathbf{sing} := \bigcap_{R>0}\{z_0 \in Q\, ;\, \mathcal{M}^{\flat}(P_R(z_0)) \ge \epsilon_0\}, \tag{3.13}$$

with $P_R(z_0) = (t_0 - \theta_0R^2, t_0 + \theta_0R^2) \times B_R(x_0)$. Then **sing** *is a relatively closed set and*

$$\mathcal{H}^{(d)}(\mathbf{sing}) = 0. \tag{3.14}$$

Proof of Theorem 9. **sing** is a relatively closed set. Indeed, if $z_0 \in \overline{\mathbf{sing}} \cap Q$, some sequence $z_\nu = (t_\nu, x_\nu) \in \mathbf{sing} \cap Q$ $(\nu = 1, 2, \ldots)$ implies $z_\nu \to z_0$ as $\nu \nearrow \infty$, i.e. for any positive δ, there exists a positive number ν_δ such that $\operatorname{dist}(z_\nu, z_0) \le \delta$ holds for any positive integer $\nu \ge \nu_\delta$. From definition on **sing**, for any $R > \delta$ and any points z_ν $(\nu = \nu_\delta, \nu_\delta + 1, \ldots)$, we obtain

$$\epsilon_0 \le \frac{\liminf_{h_k\searrow 0}}{2\theta_2(R-\delta)^d} \times \int_{P_{R-\delta}(t_\nu,x_\nu)} \left(|\nabla u_{\bar h_k}|^2 + \frac{k_{\bar h_k}}{2\sqrt[4]{h_k}}(|u_{\bar h_k}|^2 - 1)^2 + \theta_0(R-\delta)^2\left|\frac{\partial u_{h_k}}{\partial t}\right|^2\right) dz$$

$$\leq \frac{\liminf_{h \searrow 0}}{2\theta_2(R-\delta)^d} \int\limits_{P_R(z_0)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{\bar{h}_k}} \left(|u_{\bar{h}_k}|^2 - 1\right)^2 + \theta_0 R^2 \left|\frac{\partial u_{h_k}}{\partial t}\right|^2 \right) dz. \tag{3.15}$$

By the arbitrariness of δ, passing to the limit $\delta \searrow 0$, we can say $\overline{\mathbf{sing}} \cap Q \subset \mathbf{sing} \cap Q$, which provides us with our first assertion. Next we estimate the size of **sing** in the d-dimensional Hausdorff measure with respect to the parabolic metric. Fix a positive $R < 1$ and set a compact set **comp** in Q. Let $\{P_{2R_k}(z_k)\}$ $(2R_k < R)$, be a cover of **sing**. Since $\overline{\mathbf{sing}} \cap \mathbf{comp}$ is compact set in Q, we can assume that the cover is finite. Moreover the parabolic version of Vitali covering theorem shows that there is a disjoint finite sub-family $\{P_{R_k}(z_k)\}$, $k \in \mathcal{K}$ with

$$\mathbf{sing} \cap \mathbf{comp} \subset \bigcup_{k \in \mathcal{K}} P_{10R_k}(z_k),$$

$$2\epsilon_0 R_k^d \leq \liminf_{h_k \searrow 0} \int\limits_{P_{R_k}(z_k)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{\bar{h}_k}} \left(|u_{\bar{h}_k}|^2 - 1\right)^2 + \theta_0 R_k^2 \left|\frac{\partial u_{h_k}}{\partial t}\right|^2 \right) dz.$$

From Corollary 3, we have

$$\begin{aligned}
\epsilon_0 R_k^d &\leq \liminf_{h_k \searrow 0} \int\limits_{P_{R_k}(z_k)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{\bar{h}_k}} \left(|u_{\bar{h}_k}|^2 - 1\right)^2 + \theta_0 R_k^2 \left|\frac{\partial u_{h_k}}{\partial t}\right|^2 \right) dz \\
&\leq \frac{C_{\mathrm{PO}}}{R_k^2} \int\limits_{P_{2R_k}(z_k)} |u - u_{P_{2R_k}(z_k)}|^2 \, dz \\
&\leq C \int\limits_{P_{2R_k}(z_k)} \left(|\nabla u|^2 + \theta_0 R_k^2 \left|\frac{\partial u}{\partial t}\right|^2 \right) dz.
\end{aligned} \tag{3.16}$$

Thus we obtain

$$\sum_{k=1}^{K} (10R_k)^d \leq C \bigcup_{k=1}^{K} \int\limits_{P_{2R_k}(z_k)} \left(|\nabla u|^2 + \theta_0 R_k^2 \left|\frac{\partial u}{\partial t}\right|^2 \right) dz.$$

From

$$\sum_{k=1}^{K} (10R_k)^{d+2} \leq CR^2 \int\limits_{Q} \left(|\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \right) dz$$

and the absolute continuity of the Lebesgue integral, we conclude

$$\mathcal{H}^{(d)}(\mathbf{sing} \cap \mathbf{comp}) \leq C \lim_{R \searrow 0} \sum_{k=1}^{K} (10R_k)^d = 0. \tag{3.17}$$

If we set $\mathbf{comp}_n := \{z \in Q;\ \mathrm{dist}(z, \complement Q) \geq 1/n\}$ $(n = 1, 2, \ldots)$, by $\lim_{n\to\infty} \mathcal{H}^{(d)}$ $(\mathbf{sing} \cap \mathbf{comp}_n) = \mathcal{H}^{(d)}(\mathbf{sing})$, we cane deduce our assertion.

Theorem 10 (Recursive Inequality) *The heat flow for harmonic mapping u is Hölder continuous on $Q \setminus \mathbf{sing}$.*

Proof of Theorem 10. Fix a point $z_0 = (t_0, x_0) \in Q \setminus \mathbf{sing}$ and choose R so that $\mathcal{M}(P_R(z_0)) < \epsilon_0$ with some θ_0 possibly depending on z_0 and R. Because $Q/\mathbf{sing}$ is an open set, there exists some $P_{R_0}(z_0)$ so that

$$\mathcal{M}\left(P_R(\bar{z}_0)\right) < \epsilon_0$$

for all point $\bar{z}_0 \in P_{R_0}(z_0)$. Then by Theorem 8, we obtain

$$\mathcal{M}\left(P_r(\bar{z}_0)\right) \leq C\left(\frac{r}{R}\right)^{\alpha_0} \mathcal{M}\left(P_R(\bar{z}_0)\right) \tag{3.18}$$

for any positive number $r > 0$ with $\alpha_0 = \log 2/\log(1/\theta_1)$. This leads to our claim. □

We next collect a few properties of the heat flow for harmonic mappings obtained by the perturbation of DMS:

Corollary 4 *From* (2.2) *in Theorem 3, we obtain*

$$\limsup_{h_k \searrow 0} \int_Q \frac{k_{\bar{h}_k}}{\sqrt[4]{h_k}} \left(|u_{\bar{h}_k}|^2 - 1\right)^2 dz = \limsup_{h_k \searrow 0} \frac{1}{\log 1/h_k} = 0. \tag{3.19}$$

From Lemma 4, we infer that there is a positive number ϵ_0 such that for any positive number ϵ less than ϵ_0 if the heat flow for harmonic mappings u satisfies

$$⨍_{Q_{2R}} |u - u_{Q_{4R}}|^2 \, dz < \epsilon,$$

for any cylinder $Q_{2R}(z_0) \subset\subset Q$, then we infer

$$\limsup_{h_k \searrow 0} \sup_{Q_R(z_0)} |u_{\bar{h}_k} - u_{\bar{h}_k, Q_{2R}}|^2 < C(\epsilon),$$

where $C(\epsilon)$ is a positive number satisfying $C(\epsilon) \searrow 0$ as $\epsilon_0 \searrow 0$.

Finally, we close this section by showing the strong convergence of $u_{\bar{h}_k}$ to a heat flow for harmonic mappings u in $H^{1,2}_{\mathrm{loc}}$ -topology as $h_k \searrow 0$;

Theorem 11 (Strong Convergencity of Gradients) *The gradients of* $u_{\bar{h}_k}$ *converges strongly to the gradients of* u *in* $L^2_{\mathrm{loc}}(Q)$.

Proof of Theorem 11. Fix two compact sets $\mathbf{comp} \subset \mathbf{comp}_1 \subset Q$, which are compactly contained each other. Take the difference between (1.10) and (1.5), for a map $\phi \in C_0^\infty(\mathbf{comp}_1; \mathbb{R}^{D+1})$, then we obtain

$$\int_{\mathbf{comp}_1} \langle \frac{\partial}{\partial t}(u_{h_k} - u), \phi\rangle \, dz + \int_{\mathbf{comp}_1} \langle \nabla(u_{\bar{h}_k} - u), \nabla\phi\rangle \, dz$$
$$= - \int_{\mathbf{comp}_1} |\nabla u|^2 \langle u, \phi\rangle \, dz + \int_{\mathbf{comp}_1} \frac{k_{\bar{h}_k}}{\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\langle u_{\bar{h}_k}, \phi\rangle \, dz. \quad (3.20)$$

Substituting ϕ for $(u_{h_k} - u)\eta_1$, we obtain

$$\int_{\mathbf{comp}_1} |\nabla(u_{\bar{h}_k} - u)|^2 \eta_1 \, dz \le \int_{\mathbf{comp}_1} \left(|\frac{\partial u_{h_k}}{\partial t}| + |\frac{\partial u}{\partial t}|\right) |u_{h_k} - u| \, dz$$
$$+ \int_{\mathbf{comp}_1} (|\nabla u_{h_k}| + |\nabla u|)\, |u_{h_k} - u||\nabla \eta| \, dz$$
$$+ \int_{\mathbf{comp}_1} |\nabla u|^2 |u_{h_k} - u| \, dz$$
$$+ \int_{\mathbf{comp}_1} \frac{k_{\bar{h}_k}}{\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u| \, dz, \quad (3.21)$$

where η_1 is a smooth function with the support of $\mathbf{comp}_1$ and $\eta_1 = 1$ in $\mathbf{comp}$.

By using Schwarz's inequality and recalling the energy inequality (2.1) and (2.2) in Theorem 3 and the strong convergencity of $u_{\bar{h}_k}$: Theorem 6, we can easily estimate the 1st, the 2nd and the 3rd terms on the right-hand side in (3.21). We estimate the last term of the right-hand side in (3.21). Since $\mathcal{H}^{(d)}(\mathbf{sing}) = 0$ and $\mathbf{sing} \cap \mathbf{comp}$ is compact, from definition of Hausdorff measure, for any positive number ϵ, there exists a positive number R_ϵ less than $R_\epsilon < R_0$ ($=$ dist$(\mathbf{comp}, \partial\mathbf{comp}_1)$) and a finite cover of **sing**: $\{Q_{R_i}(z_i)\}$ $(i = 1, 2, \ldots, K_1)$ with $R_i < R_\epsilon$ such that

$$\mathbf{sing} \subset \bigcup_{i=1}^{K_1} Q_{R_i}(z_i), \quad \mathcal{H}^{(d)}(\mathbf{sing}) \leq \sum_{i=1}^{K_1} R_i^d + \epsilon,$$
$$\mathcal{M}^{+}(Q_{R_i}(z_i)) \leq CR^{\alpha_0}.$$

We decompose the last term as follows:

$$\begin{aligned}
&\int\limits_{\mathbf{comp}_1} \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&\leq \int\limits_{\cup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&+ \int\limits_{\mathbf{comp}_1 \setminus \cup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz.
\end{aligned} \tag{3.22}$$

Moreover there exists a finite cover $\{Q_{R_j}\}$ $(j = 1, 2, \ldots K_2)$ with $z_i \in$ $\mathbf{comp}_1 \setminus \cup_{i=1}^{K_1} Q_{R_i}(z_i)$, because it is compact; we can proceed to estimate (3.22) as follows:

$$\begin{aligned}
&\int\limits_{\mathbf{comp}_1} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&\leq \int\limits_{\cup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&+ \int\limits_{\cup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&\leq 2 \int\limits_{\cup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&+ \int\limits_{\cup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\left|u_{h_k} - u_{h_k Q_{R_j}(z_j)}\right|\,dz \\
&+ \int\limits_{\cup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\left|u_{Q_{R_j}(z_j)} - u_{h_k Q_{R_j}(z_j)}\right|\,dz
\end{aligned}$$

$$+ \int\limits_{\cup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u - u_{Q_{R_j}(z_j)}|\,dz. \tag{3.23}$$

From now on, we estimate the each term of (3.23). First, we majorize the 1st term as follows: Recall (1.5) as $h = h_k$ and substitute $u_{\bar{h}_k}\eta_i$ for ϕ in (1.5) where η_i is smooth function having only x -variable with the compact support in $B_{2R_i}(x_i)$ satisfying

$$\eta_i = \begin{cases} 1 & \text{in } \ B_{R_i}(x_i), \\ 0 & \text{outside } \ B_{2R_i}(x_i) \end{cases}$$

to obtain

$$\begin{aligned} &\int\limits_{Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz \\ &\le \int\limits_{Q_{2R_i}(z_i)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(|u_{\bar{h}_k}|^2 - 1)^2\right) dz \\ &+ \left| \int\limits_{Q_{2R_i}(z_i)} \left(\langle\frac{\partial u_{h_k}}{\partial t}, u_{h_k}\rangle + \frac{1}{2}\langle\nabla|u_{\bar{h}_k}|^2, \nabla\eta_1\rangle\right) dz \right|. \end{aligned} \tag{3.24}$$

From Lemma 3, for $R_0 = 1/2\,\mathrm{dist}\,(\mathbf{comp}, \partial\mathbf{comp}_1)$, we obtain

$$\begin{aligned} \int\limits_{Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz \le CR_i^d \int\limits_{Q} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(|u_{\bar{h}_k}|^2 - 1)^2\right) dz \\ + \left| \int\limits_{Q_{2R_i}(z_i)} \left(\langle\frac{\partial u_{h_k}}{\partial t}, u_{h_k}\rangle\eta_1 - \frac{1}{2}\langle|u_{\bar{h}_k}|^2, \Delta\eta_1\rangle\right) dz \right|. \end{aligned} \tag{3.25}$$

That is, noting that $\partial u_{h_k}/\partial t$ and $u_{\bar{h}_k}$ converges weakly to $\partial u/\partial t$ and strongly to u as $k \nearrow \infty$, respectively and $|u| = 1$, a.e,

$$\begin{aligned} &\limsup_{h_k \searrow 0} \int\limits_{\cup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz \le CR_i^d \\ &+ \sum_{i=1}^{K_1} \limsup_{h_k \searrow 0} \left| \int\limits_{Q_{2R_i}(z_i)} \left(\langle\frac{\partial u_{h_k}}{\partial t}, u_{h_k}\rangle\eta_1 - \frac{1}{2}\langle|u_{\bar{h}_k}|^2, \Delta\eta_1\rangle\right) dz \right| \end{aligned}$$

$$\leq C\mathcal{H}^{(d)}(\mathbf{sing}) + \epsilon = \epsilon. \tag{3.26}$$

Next we estimate the 2nd and the 4th term on the right-hand side: First recall that since $z_i \in Q \setminus \mathbf{sing}$, by using Corollary 4 and Theorem 10, we obtain

$$\begin{aligned}
&\limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u|\,dz \\
&\leq \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} \widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{h_k} - u_{h_k Q_{R_i}(z_i)}|\,dz \\
&\leq \sum_{j=1}^{K_2} \limsup_{h_k \searrow 0} \int_{\widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz \sup_{z \in \widehat{Q}_{R_j}(z_j)} |u_{h_k} - u_{h_k Q_{R_i}(z_i)}| \\
&\leq C(R^{\alpha_0}) \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} \widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz, \qquad (3.27)
\end{aligned}$$

$$\begin{aligned}
&\limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} (1 - |u_{\bar{h}_k}|^2)|u - u_{Q_{R_j}(z_j)}|\,dz \\
&\leq \sum_{i=1}^{K_2} \limsup_{h_k \searrow 0} \int_{\widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz \sup_{z \in \widehat{Q}_{R_j}(z_j)} |u - u_{Q_{R_j}(z_i)}| \\
&\leq C(R^{\alpha_0}) \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} \widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\,dz, \qquad (3.28)
\end{aligned}$$

where we set $\widehat{Q}_{R_1}(z_1) = Q_{R_1}(z_1)$, $\widehat{Q}_{R_j}(z_j) = Q_{R_j}(z_j) \setminus \cup_{i=1}^{j-1} Q_{R_i}(z_i)$ $(j = 2, 3, \ldots K_2)$. If we now estimate $\int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} k_{\bar{h}_k}/2\sqrt[4]{h_k}\,(1 - |u_{\bar{h}_k}|^2)\,dz$ in the same way as before, we find

$$\begin{aligned}
&\limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u - u_{Q_{R_j}(z_j)}|\,dz \\
&\leq C \int_Q \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h}}\left(|u_{\bar{h}_k}|^2 - 1\right)^2 \right) dz \leq C. \qquad (3.29)
\end{aligned}$$

Finally, we estimate the 3rd term on the right-hand side:

$$\limsup_{h_k \searrow 0} \int\limits_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)|u_{\bar{h}_k} - u_{\bar{h}_k Q_{R_j}(z_j)}|\, dz$$

$$\leq \limsup_{h_k \searrow 0} \int\limits_{\text{comp}} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\, dz$$

$$\times \max_j \limsup_{h_k \searrow 0} \frac{1}{|Q_{R_j}|} \int\limits_{Q_{R_j}(z_j)} |u_{\bar{h}_k} - u|\, dz = 0. \tag{3.30}$$

By applying (3.29) into (3.27) and (3.28) and gathering the estimates of (3.26), (3.27), (3.28) and (3.30), we arrive at

$$\limsup_{h_k \searrow 0} \int\limits_{\text{comp}} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\, dz \leq C\epsilon + C(R^{\alpha_0}).$$

Let $\epsilon \searrow 0$ and recall $C(R^{\alpha_0}) \searrow 0$ as $\epsilon \searrow 0$ to deduce our claim:

$$\limsup_{h_k \searrow 0} \int\limits_{\text{comp}} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}}(1 - |u_{\bar{h}_k}|^2)\, dz = 0,$$

which implies

$$\limsup_{h_k \searrow 0} \int\limits_{\text{comp}} |\nabla(u_{\bar{h}_k} - u)|^2\, dz = 0.$$

4 Final Remarks.

We will mention a few open problems that we should research from now: The 1st is a so-called Federer's dimension reduction argument (For the Federer's dimension reduction argument, we refer to R.Schöen and K.Uhlenbeck [11])

Conjecture 1 $\mathcal{H}^{(d-1)}(\mathbf{sing}) < \infty$.

Note that the estimate above is sharp in the sense that for any fixed time $t > 0$, any mapping $u(t, \cdot) : \mathbb{B}^3 \to \mathbb{S}^2$ i.e $d = 3$, $D = 2$ must have as least one point singularity and "the time"has two-dimension in the parabolic metric.

Next we are drawn into the regularity on the singular set:

Problem 1 **sing** *is rectifiable set.*

Author believes that once we can show the above, we also arrive

Problem 2 *Let $(s, a) \in Q$ be a singular point of the heat flow for harmonic mappings; There exists a matrix $R \in O(3)$ (may be independent of (t, x), but possibly depending on a) such that the heat flow for harmonic mappings u behaves $R(x-a)/|x-a|$ around the singular points a at each time slice s.*

These will be proved by establishing the parabolic analogue of L.Simon [10]. We do emphasis that a monotonicity for scaled energy is crucially made the best of his theory.

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