数理解析研究所講究録 1349 巻 2004 年 1-6

Invex Approaches to Mathematical Programming

大阪大学大学院情報科学研究科情報数理学専攻　齋藤誠慈　石井博昭
Seiji Saito*and Hiroaki Ishii**
Osaka University, Graduate School of Information Science and Technology,
Suita, 565-0871, Japan.
E-mail: *saito@ist.osaka-u.ac.jp, **ishii@ist.osaka-u.ac.jp

Abstract

Invexity was introduced as an extension of differentiable convex functions due to Hanson[6] in 1981. The idea plays an important role in analyzing various types of mathematical programming in which both feasible sets and objective functions are convex. For example, convex functions and affine functions are invex ones. In 1990 Karamardian et al [8] proved that generalized convexity of functions was equivalent to monotonicity of its gradient functions. It is said that the role in generalized monotonicity of the operator in variational inequality problems corresponding to the role in generalized convexity of objective functions in mathematical programming. Variational inequalities arise in models for a wide class of engineering or human sciences, e.g., mathematics, physics, economics, optimization and control., transportation, elasticity and applied sciences, etc. In this article we consider mathematical (optimization) problems and variational inequality problems.

Keywords: invexiety, convexiety, variational inequality problem, monotonicity

1 Introduction.

Consider the following mathematical problem

min f(x) subject to x in C, **(MP)**

where a feasible set C in $\mathbf{R}^n$ and an objective function $f: C \to \mathbf{R}$..

Here $\mathbf{R}$ and $\mathbf{R}^n$ are the set of real numbers, n-dimensional linear space, respectively. Problem (MP) is a particular case of the following variational inequality problems. In this paper we introduce an approach by applying the invex idea and to (MP) and the below problem variational inequality problems to x_0 in C satisfying

$(y - x_0)^T F(x_0) \geqq 0$ for y in C, **(VIP)**

where a function $F : C \to \mathbf{R}^n$. If f is differentiable and $F(x) = \nabla f(x)$, then (VIP) means (MP). According to the similar way as [9] we treat definitions of invexity in Section 2. Our aims are to solve variational-like inequality problems via the invex method (see Section 3) and to discuss invex feasible sets which are extended from the convex sets (see Section 4).

2.Monotonicity and Invexity

In order to find optimal solutions for mathematical problems by finding solutions for variational inequality problems and those for variational-like inequality problems [9] discusses variationals of monotonicity and invexity.

Definition1 A function $F : M \to \mathbf{R}^n$ is said to be *monotone*(M) on C if each x,y in C, then it follows that

$(y - x)^T (F(y) - F(x)) \geqq 0.$

A function F is said to be *pseudo monotone* (PM) on C if each x,y in C such that $(y - x)^T F(x) \geqq 0$, then $(y - x)^T F(y) \geqq 0$.

It follows that (M) means (PM) immediately. In [4] the following theorem is given as follows.

Theorem1 A differential function f on an open set C is convex if and only if ∇f is monotone on C .

Definition2 A function F is said to be *invex monotone*(IM) to a function $\eta : C^2 \to \mathbf{R}^n$ if for each x,y in C it follows that

$\eta(y,x)^T [F(y) - F(x)] \geqq 0.$

F is said to be *pseudo invex monotone* (PIM) to a function $\eta : C^2 \to \mathbf{R}^n$ if for each x, y in C with $\eta(y,x)^T F(x) \geqq 0$, then $\eta(y,x)^T F(y) \geqq 0$.

When F is (IM) to $\eta(y,x) = y - x$, it means that (IM) is (M).It follows that (IM) means (PIM).

The following examples illustrate (IM) and (PIM).

Example 1 Consider the following function $F(x) = x^2$ on $C=\{x \geqq 0\}$. It follows that F is (IM) to $\eta(y,x) = e^y - e^x$ since

$\eta(y,x)[F(y)-F(x)]$

$= (y-x)(1+(y+x)/2 +(y^2+yx+x^2)/3! + \ldots)[(y-x)(y+x)] \geqq 0.$

Example 1 The following function

$$F(x) = -x \quad (x<0); \quad 0 \, (x \geqq 0)$$

defined on $C = \mathbf{R}$ is not (IM) but (PIM) to the same $\eta(y,x) = e^y - e^x$. In case that $y < x \leqq 0$, we get $\eta(y,x)[F(y)- F(x)] = (e^y - e^x)(y^2- x^2)<0$, which means that F is un-(IM). If, however, $\eta(y,x)F(x) \geqq 0$, then $y \geqq x$ together with $\eta(y,x)F(y) \geqq 0$. Therefore F is (PIM) to the $\eta(y,x)$.

Definition3 A Differentiable function f is said to be *invex* (IX) to a function $\eta : C^2 \to \mathbf{R}^n$ if ,for each x,y in C, it follows that $f(y) - f(x) \geqq \eta(y,x)^T f'(x)$. Differentiable f is said to be *pseudo invex* (PIX) to a function $\eta: C^2 \rightarrow \mathbf{R}^n$ if, for each x,y in C with $\eta(y,x)^T f(x) \geqq 0$, it follows that $f(y) - f(x) \geqq 0$.

It follows that (IX) means (PIX). A function $f(x) = x + \sin x$ on $C = \{0 \leqq x < \pi/2\}$ is (IX) to $\eta(y,x) = (y+\sin y - x - \sin x)/(1 + \cos x)$, because

$$f(y) - f(x) = y + \sin y - (x + \sin x) = \eta(y,x) f'(x).$$

3.Variational-like Inequality Problems

In this section we treat variational-like inequality problems to find the following x_0 in C such that

$$\eta(y,x_0)^T F(x_0) \geqq 0 \quad \text{for y in C,} \qquad \text{(VLIP)}$$

which plays an important role in solving optimal solutions for (MP) by utilizing the invex idea. We introduce definitions of hemi-continuity and invex sets. One means the continuity on linear segments and the other is an extension of convexity.

Definition 4 A function F is called *hemi-continuous* on C if for x,y in C, $y^T F(x + ty)$ is continuous on the closed interval [0,1].

Definition 5 The set M in $\mathbf{R}^n$ is an *invex set* to $\eta: C^2 \rightarrow \mathbf{R}^n$ if, for each x, y in C and t in [0,1], it follows that

$x + t\eta(y,x)$ in C.

It can be easily seen that C is convex when C is invex to $y - x$. In the following example we show a different property of invex sets from that of convex sets.

Example 3 Let a subset M in $\mathbf{R}^2$ be invex to $\eta(y,x) = y$ on $C = \mathbf{R}^2 \times \mathbf{R}^2$ .Denote vectors $e_1=(1,0)^T$ and $e_2=(0,1)^T$. Assume that $e_1, e_2 \in M$. Then we get

$M = (\{1 \leqq x < \infty\} \times \mathbf{R}) \cup (\mathbf{R} \times \{1 \leqq y < \infty\})$.

The following definition, lemma and theorem concerning KKM- functions play a significant role in guaranteeing the existence of optimal solutions of (MP).

Definition 6 A function $V: \mathbf{R}^n \rightarrow 2^\wedge\{\mathbf{R}^n\}$, the power set of $\mathbf{R}^n$, is called *KKM-function* if, for every finite set $A = \{x_1, x_2, \ldots, x_m\}$ in $\mathbf{R}^n$, the convex hull conv(A) is contained in $\cup\{V(x_i): I=1,\ldots m\}$.

Lemma1 ([4]) Let a subset A in $\mathbf{R}^n$ be non-empty and $V: A \rightarrow 2^\wedge\{\mathbf{R}^n\}$ a KKM-function. If V(x) is compact for x in A, then $\cap\{V(x): x \text{ in } A\} \neq \phi$.

Theorem2 ([9]) Let C in $\mathbf{R}^n$ be non-empty, compact and convex. Let a function η be continuous, linear in the first argument and $\eta(x,y) + \eta(y,x) = 0$ on C^2. If F is (PIM) to η and hemi-continuous on C, then there exists at least one optimal solution for (VLIP) .

The following relations are essential in proving the existence of optimal solutions of (MP). Let a set of optimal solutions for (VLIP) to y be denoted by

$V_1(y) = \{ x \text{ in } C: \eta (y,x)^T F(x) \geqq 0 \}$ for y in C.

Denote

$$V_2(y) = \{ x \text{ in } C: \eta (y,x)^T F(y) \geqq 0 \} \quad \text{for } y \text{ in } C.$$

In [9] they show that V_1 and V_2 are KKM- functions, respectively, and

$$V_1(y) \subset V_2(y) \quad \text{for } y \text{ in } C.$$

Provided that $\eta (x,y) + \eta (y,x) = 0$ for (x,y) in C^2, then it follows that

$$\cap \{V_1(y): y \text{ in } C\} = \cap \{V_2(y): y \text{ in } C\}.$$

[4] showes the following result.

Theorem 3 It follows that $\cap \{V(x): x \text{ in } C\} \neq \phi$ if C in $\mathbf{R}^n$ is non-empty and the KKM-function $V : C \to 2^\wedge\{\mathbf{R}^n\}$ is compact for x in M.

In [9] authors show the optimal solutions of (VLIP) and (MP) are equivalent each other.

Theorem4 Let $f:C \to \mathbf{R}$ be (IX) to η and C an invex set.

Then x in C is an optimal solution of (VLIP) to the gradient ∇f and η if and only if x is an optimal solution of (MP) .

4. **Invex Feasible Sets**

Theorem 3 and 4 give the following existence criterion Theoerm 5.3 in [9] for (MP) via the idea of invexity provided with compact and convex feasible sets.

Theorem 5 The following conditions (i)-(iii) hold.

(i) Let C in $\mathbf{R}^n$ be non-empty, compact and convex. Let η be continuous, linear in the first argument and $\eta (x,y) + \eta (y,x) = 0$ on C^2.

(ii) Let f be differentiable on C and (IX) to η .

(iii)Let ∇f be (PIM) to η and hemi-continuous on C.

Then there exists an optimal solution x in M for (VLIP) and (MP).

In the following we get an existence criterion for (MP) of invex feasible sets which is non-convex.

Theorem 6 (Extension of Theorem 5.3 in [9]) The following conditions (i)-(iii) hold.

(i)Let η be linear in the first argument on C and $\eta (x,y)+ \eta (y,x) = 0$ on C^2. Let C in $\mathbf{R}^n$ be non-empty, compact and (IX) to η .

(ii) Let f be differentiable on C and (IX) to η .

(iii)Let ∇f be (PIM) to η and $\eta (x,y)^T \nabla f(x)$ be upper semicontinuous in x in C for y in C.

Then there exists an optimal solution x in C for (MP).

In the similar way to [9] invex feasible sets have at least one optimal solutions for (VLIP).

Lemma 2 (Extension of Lemma 5.2 in [9]) The following conditions (i)-(ii) hold.

(i)Let η be linear in the first argument on C and η (x,y)+ η (y,x) = 0 on C^2. Let C in $\mathbf{R}^n$ be nonempty and (IX) to η .

(ii) Let F: C→$\mathbf{R}^n$ be (PIM) to η and $\eta (y,x)^T F(x)$ be upper semicontinuous in x C for y in C.

Then $\cap \{V_1(y)$: y in C } = $\cap \{V_2(y)$: y in C }

For y in C..

Proof. Let x in $\cap \{V_1(y)$: y in C }. From Condition (ii) we have

$\eta (y,x)^T F(y) \geqq 0$ for y in C such that $\eta (y,x)^T F(x) \geqq 0$. Then x in $\cap \{V_2(y)$: y in C }.

Let x in $\cap \{V_2(y)$: y in C }. For y in invex C, denoting w = ty + (1 - t)x in C with $0 < t \leqq 1$, we get $\eta (w,u)^T F(w) \geqq 0$.Conditions (i) leads to that $\eta (x,x)^T F(w) = 0$ and

$\eta (y,x)^T F(ty + (1 - t)x) \geqq 0$,

whcih means that $\limsup_{\xi \to x} \eta (y,x)^T F(\xi) \geqq 0$.Then , by Condition (ii) it follows that $\eta (y,x)^T F(x) \geqq 0$, i.e.,x in $\cap \{V_1(y)$: y in C }.

Q.E.D.

Lemma 3 (Extension of Theorem 5.1 in [9]) Assume that the set C is bounded in addition to conditions of Lemma 2. Then there exists an optimal solution for (VLIP).

Proof. Consider the following function to the above η such that

$V_1(y)$ = { x in C: $\eta (y,x)^T F(x) \geqq 0$ }

for y in C. From Condition (i) it follows that V_1 is a KKM-function. From Condition (ii) the set $V_1(y)$ is closed for y in M. The boundedness of C means that $V_1(y)$ is bounded for y in C. Therefore $V_1(y)$ is compact for y in C, which means that $\cap \{V_1(y)$: y in C } $\neq \phi$ i.e., there exists an optimal solution for (VLIP) in C.

Q.E.D.

Moreover we get the following theorem to ensure the existence of optimal solutions for (MP) under conditions that the feasible sets is invex and compact.

Lemma 4 Assume that f is differentiable with F = ∇f and that C is compact in addition to conditions of Lemma3. Then there exists at least one optimal solution for (MP).

References

[1] Border, K.C. (1985). Fixed Point Theorems with Applications to Economics and Game Theory,

Cambridge Univ. Press, Cambridge.

[2] Clarke, F. H. (1983).Optimization and Nonsmooth Analysis, *Wiely*, New York.

[3] Dafermos S. (1990).Exchange Price Equilibria and Variational Ineqaulities, *Math. Progra.*, **46**, 391-402.

[4] Fan. K.(1961).A generalization of Tychonof's Fixed Point Theorem,*Math. Anall.*, **142**, 305-310 .

[5] Kinderlehrer D. and Stampacchia G. (1980). An Intorduction to Variational Inequalities and Applications, *Academic Press*, New York.

[6] Hanson, M.A. (1981).On Sufficiency of Kuhn-Tucker Condition, Mathematical Analysis and Applications, **80,** 545-550 .

[7] Karamardian, S. (1969)The Nonlinear Complementary Problems with Applications, Part 2, *J. Optimization Theory and Applications*, **4**, 167-181.

[8] Karamardian, S. and Shaible, S. (1990). Seven Kinds of Monotone Maps, J. Optimization Theory and Applications, **66**, 37-46 .

[9] Ruiz-Garzon G., Osuna-Gomez R. and .Rufian-Lizana A. (2003).Generalized Invex Monotonicity,*European J. Operat. Res.*,. **144**, 501-512.