Bell's results on, and representations of finitely connected planar domains

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1 Ahlfors maps and Bergman kernels

Let $D$ be a domain in $\mathbb{C}$. Consider the subspace $A^2(D)$ of the Hilbert space $L^2(D)$ (of all square integrable functions on $D$ with respect to the Lebesque measure on $\mathbb{C}$) consisting of all elements in $L^2(D)$ holomorphic on $D$. Then there is the natural projection

$$P : L^2(D) \rightarrow A^2(D),$$

which is called the Bergman projection. The corresponding kernel $K(z, w)$ is called the Bergman kernel.

When $D$ is the unit disc,

$$K(z, w) = \frac{1}{\pi(1 - z\overline{w})^2}.$$ 

Hence the Bergman kernel function $K(z, w)$ associated to a simply connected domain $D$ can be written by using the Riemann map $f_a(z)$ (determined uniquely by the conditions $f_a(a) = 0$ and $f'_a(a) > 0$) and its derivative:

$$K(z, w) = \frac{f'_a(z)f'_a(w)}{\pi(1 - f_a(z)f_a(w))^2}.$$ 

Let $D$ be a non-degenerate multiply connected planar domain with smooth boundary. Fix a point $a$ in $D$, and let $f_a$ be the Ahlfors map
associated with the pair $(D, a)$. Among all holomorphic functions $h$ which
map $D$ into the unit disc and satisfy $h(a) = 0$, the Ahlfors map $f_a$ is the
unique function which maximizes $h'(a)$ under the condition $h'(a) > 0$. Such proper holomorphic maps can recover the Bergman projections and
kernels in general.

**Theorem 1** Let $f : D_1 \to D_2$ be a proper holomorphic map between
planar (proper) domains. Let $P_j$ be the Bergman projection for $D_j$. Then

$$P_j(f' \cdot (\phi \circ f)) = f' \cdot ((P_j \phi) \circ f)$$

for all $\phi \in L^2(D_2)$.

But the translation formula for the Bergman kernels is not so simple
in general. For instance, it is hard to write down the following formula
explicitly.

**Proposition 2** Let $f : D_1 \to D_2$ be a proper holomorphic map between
planar (proper) domains. Then the Bergman kernels $K_j(z, w)$ associated
to $D_j$ transform according to

$$f'(z)K_2(f(z), w) = \sum_{k=1}^{m} K_1(z, F_k(w))\overline{F'_k(w)}$$

for $z \in D_1$ and $w \in D_2 - V$ where the multiplicity of the map $f$ is $m$ and
the functions $F_k, k = 1, \cdots, m$, denote the local inverses to $f$ and $V$ is
the set of critical values.

S. Bell obtained several kinds of simpler representations of Bergman
kernel functions.

**Theorem 3** ([1]) For a non-degenerate multiply connected planar do-
main $D$, we can find two points $a, b$ in $D$ such that

$$K(z, w) = f'_a(z)\overline{f'_b(w)}R(z, w)$$

with a rational combination $R(z, w)$ of $f_a$ and $f_b$. 

Here we say that a function $R(z, w)$ is a \textit{rational combination} of $f_a$ and $f_b$ if it is a rational function of

$$f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)}.$$

Such representation as above has the following variant.

\textbf{Theorem 4 ([5])} For a non-degenerate multiply connected planar domain $D$, we can find two points $a, b$ in $D$ such that

$$K(z, w) = \frac{f_a'(z)f_a'(w)}{(1 - f_a(z)f_a(w))^2} \left( \sum_{j,k} H_j(z)K_k(w) \right)$$

where $f_a, f_b$ are the Ahlfors functions, $H$ and $K$ are rational functions of them, and the sum is a finite sum.

Actually, we can use any proper holomorphic maps.

\textbf{Theorem 5 ([2])} Let $D$ be a non-degenerate multiply connected planar domain, and $f$ a proper holomorphic map of $D$ onto the unit disk $U$. Then $K(z, w)$ is an algebraic function of

$$f(z), f'(z), \overline{f(w)}, \overline{f'(w)}.$$

Moreover, we have the following

\textbf{Theorem 6 ([2])} Let $D$ be a non-degenerate multiply connected planar domain. The following conditions are equivalent.

(1) The Bergman kernel $K(z, w)$ associated to $D$ is algebraic, i.e. an algebraic function of $z$ and $\overline{w}$.

(2) The Ahlfors map $f_a(z)$ is an algebraic function of $z$.

(3) There is a proper holomorphic mapping $f : D \to U$ which is an algebraic function.

(4) Every proper holomorphic mapping from $D$ onto the unit disc $U$ is an algebraic function.

Also we have
Theorem 7 ([4]) Let $D$ be a non-degenerate multiply connected planar domain. There are two holomorphic functions $F_1$ and $F_2$ on $D$ such that the Bergman kernel on $D$ is a rational combination of $F_1$ and $F_2$ if and only if there is a proper holomorphic map $f$ of $D$ onto $U$ such that $f$ and $f'$ are algebraically dependent: i.e. there is a polynomial $Q$ such that $Q(f, f') = 0$.

Then, for every proper holomorphic map $f$ of $D$ to $U$, $f$ and $f'$ are algebraically dependent.

Proposition 8 ([4]) Let $D$ be a simply connected planar (proper) domain. The Bergman kernel on $D$ is a rational combination of a function of a complex variable if and only if the Riemann map $f$ of $D$ and $f'$ are algebraically dependent.

Finally, we note the following facts.

Proposition 9 ([2]) If $K(z, w)$ is algebraic, and $f$ be a proper holomorphic map to $U$. Then $K(z, w)$ is an algebraic function of $f(z)$ and $\overline{f(w)}$.

Corollary 1 ([2]) Let $D_1$ and $D_2$ have algebraic Bergman kernels, then every biholomorphic map of $D_1$ onto $D_2$ is algebraic.

2 Bell representations

Now the issue is to find a family of canonical domains which admit a simple proper holomorphic map to $U$. Bell proposed such a family, and actually, they are enough.

Theorem 10 ([6]) Every non-degenerate $n$-connected planar domain with $n > 1$ is mapped biholomorphically onto a domain $W_{a,b}$ defined by

$$W_{a,b} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z-b_k} \right| < 1 \right\}$$

with suitable complex vectors $a = (a_1, a_2, \ldots, a_{n-1})$ and $b = (b_1, b_2, \ldots, b_{n-1})$. 
The above theorem is considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains. The function $f_{a,b}$ defined by

$$f_{a,b}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is a proper holomorphic mapping from $W_{a,b}$ to the unit disc which is rational. Actually, it is a very classical fact that, for such an $f = f_{a,b}$ as above, $f$ and $f'$ are algebraically dependent. Hence the above proposition implies the following corollary.

**Corollary 2** Every non-degenerate $n$-connected planar domain $D$ with $n > 1$ is biholomorphic to a domain with the algebraic Bergman kernel.

**Corollary 3** There are two holomorphic functions $F_1$ and $F_2$ such that the Bergman kernel on $W_{a,b}$ is a rational combination of $F_1$ and $F_2$.

**Definition** The locus $B_n$ in $\mathbb{C}^{2n-2}$ consisting of $(a, b)$ such that the corresponding domain $W_{a,b}$ is a non-degenerate $n$-connected planar domain.

We call this locus $B_n$ the coefficient body for non-degenerate $n$-connected canonical domains.

It is obvious that $B_n$ is contained in the product space

$$(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C},$$

which has the same homotopy type as that of

$$X = (S^1)^{n-1} \times F_{0,n-1}\mathbb{C},$$

where

$$F_{0,n-1}\mathbb{C} = \{(z_1, \ldots, z_{n-1} \in \mathbb{C}^{n-1} \mid z_j \neq z_k \text{ if } j \neq k\}$$

is called a configuration space.

To clarify the topological structure of the coefficient body, it is more convenient to use the following modified representation space.

**Definition** We set

$$B_n^* = \{(a_1, \ldots, a_{n-1}, b) \in (\mathbb{C})^{2n-2} \mid (a_1^2, \ldots, a_{n-1}^2, b) \in B_n\},$$

and call it the modified coefficient body.
Theorem 11  $\mathcal{B}_n^*$ is a circular domain, and has the same homotopy type as that of the product space $X$.

Corollary 4 The homotopy type of $\mathcal{B}_n$ is the same as that of $X$.

Remark  The fundamental group of $F_{0,n-1}\mathbb{C}$ is called the pure braid group, and its structure is well-known.

Problem

1. Determine the Ahlfors locus of $\mathcal{B}_n$ which consists of all $(a, b)$ such that $f_{a,b}$ gives an Ahlfors map ($,$ or more precisely, $e^{i\theta}f_{a,b}$ with a suitable $\theta \in \mathbb{R}$ is an Ahlfors map).

2. Fix a point $(a, b)$ in $\mathcal{B}_n$, and let $W = W_{a,b}$ be the corresponding $n$-connected canonical domain. Determine the leaf $E(W)$ of $\mathcal{B}_n$ for $W$, consisting of all points which correspond to $n$-connected canonical domains biholomorphically equivalent to $W$.

3. Determine the collision locus $C$ of $\mathcal{B}_n$ which consists of all $(a, b)$ such that the corresponding map $f_{a,b}$ has a pair of critical points (counted with multiplicities) whose image is the same. (Note that $\mathcal{B}_n - C$ is a finite-sheeted holomorphic smooth cover of the intersection of $F_{0,2n-2}\mathbb{C}$ and the unit polydisc.)

Example 1

$$\mathcal{B}_2^* = \{(a, b) \in \mathbb{C}^2 : a \neq 0, |b + 2a| < 1, |b - 2a| < 1\},$$

which is biholomorphic to the polydisc deleted the diagonal.

Next, the set

$$\left\{ (a, b) \in \mathcal{B}_2^* : \frac{4a^2}{1 - (b+2a)(b-2a)} = \frac{4r}{4 + r^2} \right\}$$

corresponds to a leaf of $\mathcal{B}_2$ for every given $r > 2$, and the collision locus of $\mathcal{B}_2$ is empty.
参考文献


