

# Asymptotic behavior of spherically symmetric solutions to the compressible Navier-Stokes equations with external forces

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## 1 Introduction

The Navier-Stokes equation with external force for the isentropic motion of compressible viscous gas in the Eulerian coordinate is the system of equations given by

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.1a}$$

$$\rho\{u_t + (u \cdot \nabla)u\} = \mu_1 \Delta u + (\mu_1 + \mu_2) \nabla(\nabla \cdot u) - \nabla P(\rho) + \rho f. \tag{1.1b}$$

We study an asymptotic behavior of a solution  $(\rho, u)$  to (1.1) in an unbounded exterior domain  $\Omega := \{\xi \in \mathbb{R}^n ; |\xi| > 1\}$ , where  $n$  is a space dimension larger than or equal to 2. Here,  $\rho > 0$  is the mass density;  $u = (u_1, \dots, u_n)$  is the velocity of gas;  $P(\rho) = K\rho^\gamma$  ( $K > 0, \gamma \geq 1$ ) is the pressure with the adiabatic exponent  $\gamma$ ;  $f$  is the external force;  $\mu_1$  and  $\mu_2$  are constants called viscosity-coefficients satisfying  $\mu_1 > 0$  and  $2\mu_1 + n\mu_2 > 0$ .

It is assumed that the external force  $f$  is a spherically symmetric potential force and the initial data is also spherically symmetric. Namely, for  $r := |\xi|$

$$[A1] \quad f := -\nabla U = \frac{\xi}{r} U_r(r), \quad U_r \in C^1[1, \infty),$$

$$[A2] \quad \rho_0(x) = \hat{\rho}_0(r), \quad u_0(\xi) = \frac{\xi}{r} \hat{u}_0(r).$$

Under the assumptions [A1] and [A2], it is shown in [5] that the solution  $(\rho, u)$  is spherically symmetric. Here, the spherically symmetric solution means a solution to (1.1) in the form of

$$\rho(\xi, t) = \hat{\rho}(r, t), \quad u(\xi, t) = \frac{\xi}{r} \hat{u}(r, t). \tag{1.2}$$

Substituting (1.2) in (1.1), we reduce the system (1.1) to that of the equations for  $(\hat{\rho}, \hat{u})(r, t)$ . Here and hereafter, we omit the hat “ $\hat{\phantom{x}}$ ” to express a spherically symmetric function without confusion. Hence the spherically symmetric solution  $(\rho, u)(r, t)$  satisfies the system of equations

$$\rho_t + \frac{(r^{n-1} \rho u)_r}{r^{n-1}} = 0, \tag{1.3a}$$

$$\rho(u_t + uu_r) = \mu \left( \frac{(r^{n-1} u)_r}{r^{n-1}} \right)_r - P(\rho)_r - \rho U_r, \tag{1.3b}$$

where  $\mu := 2\mu_1 + \mu_2 > 0$ . The initial data to (1.3) is prescribed to be spatial asymptotically constant:

$$\rho(r, 0) = \rho_0(r) > 0, \quad u(r, 0) = u_0(r), \quad (1.4a)$$

$$\lim_{r \rightarrow \infty} (\rho_0(r), u_0(r)) = (\rho_+, u_+), \quad \rho_+ > 0. \quad (1.4b)$$

As we are interested in the behavior of fluid around a solid sphere, an adhesion boundary condition is adopted:

$$u(1, t) = 0. \quad (1.5)$$

In addition, it is assumed that the initial data (1.4) is compatible with the boundary data (1.5). Since the characteristic speed of (1.3a) is zero on the boundary due to (1.5), one boundary condition (1.5) is necessary and sufficient for the wellposedness of the initial boundary value problem (1.3), (1.4) and (1.5).

This initial boundary value problem is formulated to study the behavior of compressible viscous gas around the solid sphere in a field of external force. We show that the time asymptotic state of the solution to the problem (1.3), (1.4) and (1.5) is the stationary solution, which is a solution to (1.3) independent of time  $t$ , satisfying the same conditions (1.4) and (1.5). Hence the stationary solution  $(\tilde{\rho}(r), \tilde{u}(r))$  satisfies the system of equations

$$\frac{1}{r^{n-1}} (r^{n-1} \tilde{\rho} \tilde{u})_r = 0, \quad (1.6a)$$

$$\tilde{\rho} \tilde{u} \tilde{u}_r = \mu \left( \frac{(r^{n-1} \tilde{u})_r}{r^{n-1}} \right)_r - P(\tilde{\rho})_r - \tilde{\rho} U_r \quad (1.6b)$$

and the boundary and the spatial asymptotic conditions

$$\tilde{u}(1) = 0, \quad \lim_{r \rightarrow \infty} (\tilde{\rho}(r), \tilde{u}(r)) = (\rho_+, u_+). \quad (1.7)$$

Solving (1.6) under the conditions (1.7), we see that  $(\tilde{\rho}(r), \tilde{u}(r))$  is explicitly given by

$$\tilde{\rho}(r) = \begin{cases} \rho_+ \exp \left\{ \frac{1}{K} (U_+ - U(r)) \right\} & \text{for } \gamma = 1, \\ \left\{ \rho_+^{\gamma-1} + \frac{\gamma-1}{K\gamma} (U_+ - U(r)) \right\}^{\frac{1}{\gamma-1}} & \text{for } \gamma > 1, \end{cases} \quad (1.8a)$$

$$\tilde{u}(r) = 0 \quad (1.8b)$$

for  $r \geq 1$ , where  $U_+$  is a constant given by

$$U_+ := \lim_{r \rightarrow \infty} U(r) = \lim_{r \rightarrow \infty} \int_1^r U_r(\eta) d\eta + U(1). \quad (1.9)$$

We see from (1.8b) that the condition

$$u_+ = 0 \quad (1.10)$$

is necessary for the existence of the stationary solution. To avoid occurrence of a vacuum, we assume (1.8a) is positive. Namely, for  $\gamma > 1$ , we assume  $\tilde{\rho}(r) \geq c > 0$ , where  $c$  is a certain constant. We also assume that the external force satisfies

$$-\delta \leq U_r(r) \quad (1.11)$$

for an arbitrary  $r \geq 1$ , where  $\delta$  is a certain positive constant determined suitably small depending only on the initial data. The formula (1.8) implies that the stationary solution is a constant state  $(\rho_+, 0)$  if the external force  $U_r$  is constantly equal to zero.

The stability theorem of the stationary solution (1.8) is summarized in the next theorem, which is the main result in the present paper.

**Theorem 1.1.** *Suppose the initial data satisfies that*

$$r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}), r^{\frac{n-1}{2}}u_0, r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho})_r, r^{\frac{n-1}{2}}u_{0r} \in L^2(1, \infty), \quad (1.12a)$$

$$\rho_0 \in \mathcal{B}^{1+\sigma}[1, \infty), u_0 \in \mathcal{B}^{2+\sigma}[1, \infty) \quad \text{for a certain } \sigma \in (0, 1), \quad (1.12b)$$

(1.10) and the compatibility condition holds. Let the external force  $U_r \in C^1[1, \infty)$  satisfy (1.9). In addition, if the condition (1.11) holds for a positive constant  $\delta$  depending only on the initial data, then the initial boundary value problem (1.3), (1.4) and (1.5) has a unique solution  $(\rho, u)$  satisfying

$$r^{\frac{n-1}{2}}(\rho - \tilde{\rho}), r^{\frac{n-1}{2}}u, r^{\frac{n-1}{2}}(\rho - \tilde{\rho})_r, r^{\frac{n-1}{2}}u_r \in C([0, \infty); L^2(1, \infty)), \quad (1.13a)$$

$$\rho \in \mathcal{B}^{1+\sigma, 1+\sigma/2}([1, \infty) \times [0, T]), u \in \mathcal{B}^{2+\sigma, 1+\sigma/2}([1, \infty) \times [0, T]), \quad (1.13b)$$

for an arbitrary  $T > 0$ . Moreover, the solution  $(\rho, u)$  converges to the corresponding stationary solution  $(\tilde{\rho}, 0)$  given by (1.8) as time  $t$  tends to infinity. Precisely, it holds that

$$\lim_{t \rightarrow \infty} \sup_{r \in [1, \infty)} |(\rho(r, t) - \tilde{\rho}(r), u(r, t))| = 0. \quad (1.14)$$

Notice that any smallness assumptions on the initial data is not necessary for the above stability theorem. Moreover, if the external force is a potential force and  $U_r \geq 0$ , it can be taken arbitrarily large. This condition implies the case that the external force is attractive like the gravitational force. On the other hand, the assumption (1.11) requires that the repulsive part of the external force must be small subject to the initial data.

The Hölder continuity of the initial data (1.12b) is necessary to ensure the validity of the transformation between the Eulerian and the Lagrangian coordinates (see (2.1) below). Actually, we show the asymptotic stability of the stationary solution in the Lagrangian coordinate without the Hölder continuity. In translating this result to that in the Eulerian coordinate, we need the differentiability of the solution. This is the reason why we assume (1.12b), which gives the Hölder continuity of the solution with the aid of the Schauder theory for parabolic equations (see [14]).

**Related results.** The first notable research in the compressible Navier-Stokes equation on the exterior domain is given by A. Matsumura and T. Nishida in [10], where the stability of the stationary solution is proved under the smallness assumptions on

the initial data and the external force. Notice that the research [10] covers the more general solution on more general domain than the present research, which studies the spherically symmetric solution only.

Another pioneering work is given by N. Itaya [5], which establishes the existence of the spherically symmetric solution to the equation for the heat-conductive gas globally in time on a bounded annulus domain without the external force nor the smallness assumption on the initial data. The paper [5] has drawn attentions of researchers to the spherically symmetric solution. For example, T. Nagasawa in [13] shows that the spherically symmetric solution to the heat-conductive fluid without the external force on the annulus domain exists globally in time and it converges to the corresponding stationary solution as time tends to infinity. Moreover, it obtains the exponential convergence rate. For the case of the external force is not zero, A. Matsumura proves in [9] that the solution to the isothermal model tends to the corresponding stationary solution exponentially fast as time tends to infinity. The research by K. Higuchi in [3] extends the results in [9] to the isentropic model. In addition, it considers the equation of heat-conductive ideal gas on the same annulus domain. The present research aims to extend the results in [9] and [3] to those on an unbounded exterior domain.

The study of the spherically symmetric solution over an unbounded exterior domain is started by S. Jiang in [6], where the global existence of the solution is established for the equation of heat-conductive ideal gas. Moreover, the partial result on the asymptotic state is obtained. Precisely, it shows that, for the space dimension  $n = 3$ ,  $\|u(t)\|_{2j} \rightarrow 0$  as  $t \rightarrow \infty$ , where  $j$  is an arbitrarily fixed integer greater than or equal to 2.

**Notation.** For a region  $\Omega \subset \mathbb{R}$  and  $1 \leq p \leq \infty$ ,  $L^p(\Omega)$  denotes the standard Lebesgue space over  $\Omega$  equipped with the norm  $|\cdot|_p$ . For a non-negative integer  $l \geq 0$ ,  $H^l(\Omega)$  denotes the  $l$ -th order Sobolev space over  $\Omega$  in the  $L^2$  sense with the norm  $\|\cdot\|_l$ . We note  $H^0 = L^2$  and  $\|\cdot\| := |\cdot|_2 = \|\cdot\|_0$ . For a non-negative integer  $l$  and  $\sigma \in (0, 1)$ ,  $\mathcal{B}^{l+\sigma}(\Omega)$  denotes the space of Hölder continuous functions over  $\Omega$  which have the  $l$ -th order derivatives of Hölder continuity with exponent  $\sigma$ . For a domain  $Q_T \subseteq [0, \infty) \times [0, T]$ ,  $\mathcal{B}^{\alpha, \beta}(Q_T)$  denotes the Hölder space of continuous functions with the Hölder exponents  $\alpha$  and  $\beta$  with respect to  $x$  and  $t$ , respectively. For integers  $k$  and  $l$ ,  $\mathcal{B}^{k+\alpha, l+\beta}(Q_T)$  denotes the space of the functions satisfying  $\partial_x^i u, \partial_t^j u \in \mathcal{B}^{\alpha, \beta}(Q_T)$  for integers  $0 \leq i \leq k$  and  $0 \leq j \leq l$ .  $c$  and  $C$  denote several generic positive constants.

## 2 Time local solution in Lagrangian coordinate

### 2.1 Problem in Lagrangian coordinate

In the proof of Theorem 1.1, we show the uniform a priori estimate by employing the energy method. For this purpose, it is convenient to adopt the Lagrangian mass coordinate rather than the Eulerian coordinate. The transformation from the Eulerian coordinate  $(r, t)$  to the Lagrangian coordinate  $(x, t)$  is executed by the transformation

$$x = \int_1^r \xi^{n-1} \rho(\xi, t) d\xi, \quad r_x = \frac{v}{r^{n-1}}, \quad r_t = u, \quad (2.1)$$

where  $v := 1/\rho$  is the specific volume. Using (2.1), we deduce the system (1.3) to

$$v_t = (r^{n-1}u)_x, \quad (2.2a)$$

$$u_t = \mu r^{n-1} \left( \frac{(r^{n-1}u)_x}{v} \right) - r^{n-1}p(v)_x - U_r, \quad (2.2b)$$

where  $p(v) = Kv^{-\gamma}$ . The initial and boundary conditions for  $(v, u)$  are derived from (1.4) and (1.5) as

$$v(x, 0) = v_0(x) := \frac{1}{\rho_0(r(x, 0))}, \quad u(x, 0) = u_0(x), \quad \lim_{x \rightarrow \infty} v_0(x) = v_+ := \frac{1}{\rho_+}, \quad (2.3)$$

$$u(0, t) = 0. \quad (2.4)$$

Since the spatial variable  $r$  in the Eulerian coordinate depends on the spatial and time variables  $(x, t)$  in the Lagrangian coordinate, the density  $\bar{\rho}$  in the stationary solution also depends on  $(x, t)$ , that is,  $\bar{\rho}(x, t) := \bar{\rho}(r(x, t))$ . Consequently, the specific volume  $\bar{v}$  in the stationary solution is also a function of  $(x, t)$ . Namely,  $\bar{v}(r) := 1/\bar{\rho}(r) = 1/\bar{\rho}(r(x, t))$ . In addition, let  $r_0(x) := r(x, 0)$ ,  $\bar{\rho}_0(x) := \bar{\rho}(r_0(x))$  and  $\bar{v}_0(x) := 1/\bar{\rho}_0(x)$ .

We consider the initial boundary value problem to the system of equations (2.2) with data (2.3) and (2.4). Here, the coefficients in (2.2) is given by the relation (2.1). The stability theorem of the stationary solution  $(\bar{v}, \bar{u})$  for this problem is stated in the following proposition.

**Theorem 2.1.** *Suppose that the initial data satisfies*

$$v_0 - \bar{v}_0, u_0, r_0^{n-1}(v_0 - \bar{v}_0)_x, r_0^{n-1}u_{0x} \in L^2(0, \infty)$$

and it is compatible with the boundary data (2.4). In addition, the external force  $U_r \in C^1[1, \infty)$  is supposed to satisfy (1.9) and (1.11) for a certain positive constant  $\delta$  depending only on the initial data. Then the initial boundary problem (2.2)–(2.4) has a unique solution  $(v, u)$  satisfying

$$v - \bar{v}, u, r^{n-1}(v - \bar{v})_x, r^{n-1}u_x \in C([0, \infty); L^2(0, \infty)), \quad (2.5a)$$

$$r^{-1}u, r^{n-1}u_x, r^{2n-3}u_{xx} \in L^2(0, \infty; L^2(0, \infty)). \quad (2.5b)$$

Moreover the solution converges to the stationary solution. Precisely, it holds that

$$\lim_{t \rightarrow \infty} \sup_{x \in (0, \infty)} |(v(x, t) - \bar{v}(r(x, t)), u(x, t))| = 0. \quad (2.6)$$

Theorem 2.1 is proved by combining the local existence and the a priori estimate. In order to prove the local existence to the problem (2.2)–(2.4), we solve the approximate problem in bounded domain  $(0, m)$  for  $m = 1, 2, \dots$  to (2.2). This procedure is necessary since some coefficients in (2.2) are unbounded over  $x \in [0, \infty)$ . Following the idea in [1, 6], we employ the “cut-off-function”  $\phi_m(x) \in C^3[0, \infty)$  satisfying

$$\phi_m(x) := \begin{cases} 1, & \text{for } 0 \leq x \leq \frac{m}{2}, \\ 0, & \text{for } m \leq x, \end{cases} \quad (2.7)$$

$$0 \leq \phi_m(x) \leq 1, \quad |\partial_x^i \phi_m(x)| \leq \frac{C}{m^i} \quad (i = 1, 2, 3) \quad \text{for } \frac{m}{2} \leq x \leq m.$$

The initial data  $(v_{m0}, u_{m0})$  on restricted region  $(0, m)$  is derived from (2.3) by using  $\phi_m(x)$  as

$$v_{m0}(x) := (v_0(x) - \tilde{v}_0(x))\phi_m(x) + \tilde{v}_0(x), \quad u_{m0}(x) := u_0(x)\phi_m(x).$$

We consider the initial boundary value problem for the unknown function  $(v_m, u_m)$  in the bounded domain  $(0, m)$

$$v_{mt} = (r_m^{n-1}u_m)_x, \quad (2.8a)$$

$$u_{mt} = \mu r_m^{n-1} \left( \frac{(r_m^{n-1}u_m)_x}{v_m} \right)_x - r_m^{n-1}p(v_m)_x - U_r, \quad (2.8b)$$

with the initial and boundary conditions

$$v_m(x, 0) = v_{m0}(x), \quad u_m(x, 0) = u_{m0}(x), \quad (2.9)$$

$$u_m(0, t) = 0, \quad u_m(m, t) = 0. \quad (2.10)$$

In addition, the compatibility conditions at  $(x, t) = (0, 0)$  and  $(m, 0)$  are supposed to hold. Here, the functions  $r_{m0}$  and  $r_m$  are given by

$$r_{m0}(x) = \left\{ 1 + n \int_0^x v_{m0}(y) dy \right\}^{1/n}, \quad r_m(x, t) = \left\{ 1 + n \int_0^x v_m(y, t) dy \right\}^{1/n}. \quad (2.11)$$

The local existence of the solution to the problem (2.8)–(2.10) in bounded domain is proved by the standard iteration method. See [4] for example. For  $\bar{d} > \underline{d} > 0$ ,  $D > 0$  and positive integer  $m$ , we define the function space as

$$\begin{aligned} X_{\underline{d}, \bar{d}, D}^m(0, T) &:= \{(v, u) \mid (v - \tilde{v}, u) \in C^0([0, T]; H^1(0, m)), u \in L^2(0, T; H^2(0, m)), \\ &\quad \|(v - \tilde{v}, u)(t)\|_{1, r, m} \leq D, \underline{d} \leq v(x, t) \leq \bar{d}\}, \\ \|(v - \tilde{v}, u)(t)\|_{1, r, m} &:= \|(v - \tilde{v}, u, r^{n-1}(v - \tilde{v})_x, r^{n-1}u_x)(t)\|_{L^2(0, m)}, \\ E_0^m &:= \|(v_{m0} - \tilde{v}_0, u_{m0})\|_{1, r, m}^2. \end{aligned}$$

We see that

$$E_0^m \rightarrow E_0 := \|(v_0 - \tilde{v}_0, u_0)\|_{1, r, \infty}^2 \quad \text{as } m \rightarrow \infty. \quad (2.12)$$

**Lemma 2.2.** *If the initial data satisfies  $E_0^m \leq D_0$  and  $\underline{d}_0 \leq v_{m0}(x) \leq \bar{d}_0$  for certain constants  $\underline{d}_0, \bar{d}_0$  and  $D_0$ , then there exists a positive constant  $T = T(\underline{d}_0, \bar{d}_0, D_0)$  such that the problem (2.8)–(2.10) has a unique solution  $(v_m, u_m)$  in the space  $X_{\underline{d}_0/2, 2\bar{d}_0, 2D_0}^m(0, T)$ .*

## 2.2 Energy estimate

In this subsection, we obtain the  $H^1$  a priori estimate for the solution  $(v_m, u_m) \in X_{\underline{d}, \bar{d}, D}^m(0, T)$  uniformly in  $m$  by using the energy method. Then, letting  $m \rightarrow \infty$ , we get the time local solution for the problem (2.2)–(2.4) in unbounded domain  $(0, \infty)$ . Here and hereafter until the end of this subsection, we omit the subscript  $m$  and denote

$(v_m, u_m)$  by  $(v, u)$  for simplicity. To obtain the basic estimate, we employ the energy form  $\mathcal{E}$  defined by

$$\begin{aligned} \mathcal{E} &:= \frac{1}{2}u^2 + \Psi(v, \tilde{v}), \\ \Psi(v, \tilde{v}) &:= p(\tilde{v})(v - \tilde{v}) - \varphi(v, \tilde{v}), \end{aligned} \tag{2.13}$$

where  $\varphi(v, \tilde{v})$  is defined by

$$\varphi(v, \tilde{v}) := \int_{\tilde{v}}^v p(\eta) d\eta. \tag{2.14}$$

The quantity (2.13) is also rewritten as

$$\Psi(v, \tilde{v}) = \tilde{v}p(\tilde{v})\psi\left(\frac{v}{\tilde{v}}\right), \quad \psi(s) := s - 1 - \int_1^s \eta^{-\gamma} d\eta. \tag{2.15}$$

Since the solution  $v$  satisfies

$$\underline{d} \leq v(x, t) \leq \bar{d} \quad \text{for } (x, t) \in (0, m) \times (0, T), \tag{2.16}$$

the quantity  $\Psi(v, \tilde{v})$  is equivalent to  $|v - \tilde{v}|^2$ . Namely,  $c_d|v - \tilde{v}|^2 \leq \Psi(v, \tilde{v}) \leq C_d|v - \tilde{v}|^2$ , where and hereafter  $c_d$  and  $C_d$  are positive constants depending on  $\underline{d}$  or  $\bar{d}$ . Therefore, the energy form  $\mathcal{E}$  is equivalent to  $|u|^2 + |v - \tilde{v}|^2$ , that is,

$$c_d(|u|^2 + |v - \tilde{v}|^2) \leq \mathcal{E} \leq C_d(|u|^2 + |v - \tilde{v}|^2).$$

In this paper, we omit the details of the proof of the following propositions and lemmas. For details, readers are referred to [12].

**Proposition 2.3.** *For the solution  $(v, u) \in X_{\underline{d}, \bar{d}, D}^m(0, T)$  to (2.8)–(2.10), it holds that*

$$\int_0^m \mathcal{E}(t) dx + \mu \int_0^t \int_0^m (n-1) \frac{v}{r^2} u^2 + \frac{r^{2n-2}}{v} u_x^2 dx d\tau = \int_0^m \mathcal{E}(0) dx \tag{2.17}$$

for  $t \in [0, T]$ .

The following lemma is proved by the Sobolev inequality. It is utilized in deriving the pointwise estimate in Subsection 3.1.

**Corollary 2.4.** *For the solution  $(v, u) \in X_{\underline{d}, \bar{d}, D}^m(0, T)$  to (2.8)–(2.10), it holds that*

$$\int_0^t |(r^{n-2}u^2)(\tau)|_\infty d\tau \leq \int_0^m \mathcal{E}(0) dx \tag{2.18}$$

for  $t \in [0, T]$ .

Next, we derive the estimate for the first order derivatives. To this end, we employ  $\varphi(x, t)$  defined in (2.14).

**Proposition 2.5.** For the solution  $(v, u) \in X_{\underline{d}, \bar{d}, D}^m(0, T)$  to (2.8)–(2.10), it holds that

$$\int_0^m r^{2n-2} \varphi_x^2 dx + \int_0^t \int_0^m r^{2n-4} \varphi_x^2 dx d\tau \leq C_d^0 E_0^m, \quad (2.19)$$

$$\int_0^m r^{2n-2} u_x^2 dx + \int_0^t \int_0^m r^{4n-6} u_{xx}^2 dx d\tau \leq C_d^0 E_0^m \quad (2.20)$$

for  $t \in [0, T]$ , where  $C_d^0$  is a positive constant depending on  $\bar{d}$ ,  $\underline{d}$  and the initial data.

Due to (2.14) and the estimate (2.19), we have the estimate for  $(v - \tilde{v})_x$  as

$$\int_0^m r^{2n-2} (v - \tilde{v})_x^2 dx \leq C_d^0 E_0^m. \quad (2.21)$$

Proposition 2.3 and 2.5 give the uniform bound of  $u(x, t)$ .

**Corollary 2.6.** For the solution  $(v, u) \in X_{\underline{d}, \bar{d}, D}^m(0, T)$  to (2.8)–(2.10), it holds that

$$\sup_{x \in (0, m)} |r^{n-1} u^2| \leq C_d^0 E_0^m \quad (2.22)$$

for  $t \in [0, T]$ .

Utilizing the estimates (2.17), (2.19) and (2.20) and letting  $m \rightarrow \infty$ , we have the local solution to the problem (2.2)–(2.4) satisfying

$$v - \tilde{v}, u, r^{n-1} (v - \tilde{v})_x, r^{n-1} u_x \in C^0([0, T]; L^2(0, \infty)), \quad (2.23a)$$

$$r^{-1} u, r^{n-1} u_x, r^{n-2} \varphi_x, r^{2n-3} u_{xx} \in L^2(0, T; L^2(0, \infty)), \quad (2.23b)$$

where  $T = T(\inf_{x \in (0, \infty)} v_0(x), E_0)$ . Moreover, utilizing (2.12), we see that

$$\int_0^\infty \mathcal{E}(t) dx + \int_0^t \int_0^\infty \frac{v}{r^2} u^2 + \frac{r^{2n-2}}{v} u_x^2 dx d\tau \leq C^0 E_0, \quad (2.24a)$$

$$\int_0^\infty r^{2n-2} (v - \tilde{v})_x^2 + r^{2n-2} u_x^2 dx + \int_0^t \int_0^\infty r^{2n-4} \varphi_x^2 + r^{4n-6} u_{xx}^2 dx d\tau \leq C_d^0 E_0, \quad (2.24b)$$

where  $C^0$  is a positive constant depending only on the initial data (2.3) and  $C_d^0$  is a positive constant depending only on  $\underline{d}, \bar{d}$  and the initial data (2.3).

### 3 Large time behavior of solutions in Lagrangian coordinate

#### 3.1 Pointwise estimate of the specific volume

In this subsection, we show the outline of the proof of the pointwise positive bounds of the specific volume  $v(x, t)$  uniformly in time. Combining this pointwise estimate with (2.24b) yields the  $H^1$  estimate uniformly in time. It immediately gives the time global solution to the initial boundary value problem (2.2)–(2.4) by the standard continuation argument with the local existence.

Here, let us note that in proving Proposition 3.1, we use the estimate (2.24a), but not (2.24b).



**Proposition 3.1.** *Suppose that  $(v, u)$  is the solution to the problem (2.2)–(2.4) in the space  $X_{\underline{d}, \bar{d}, D}^\infty(0, T)$ . Moreover let the condition (1.11) holds. Then the specific volume  $v$  satisfies*

$$\underline{v} \leq v(x, t) \leq \bar{v} \quad \text{for } x \geq 0 \text{ and } t \in [0, T], \quad (3.1)$$

where  $\underline{v}$  and  $\bar{v}$  are positive constants depending only on the initial data.

The proof of Proposition 3.1 is divided into the several lemmas. The next lemma follows from (2.24a).

**Lemma 3.2.** *Suppose that the same assumptions as in Proposition 3.1 hold. Let  $\varepsilon$  be a positive constant. Then, there exist positive constants  $c_\varepsilon$  and  $C_\varepsilon$  depending only on  $\varepsilon$  and the initial data such that*

$$0 < c_\varepsilon \leq \int_a^{a+\varepsilon} v(x, t) dx \leq C_\varepsilon \quad (3.2)$$

for  $t \in [0, T]$  and an arbitrary  $a \geq 0$ .

To obtain the pointwise estimate of the specific volume  $v(x, t)$ , we use the representation formula of the specific volume  $v(x, t)$ . To derive this formula, we employ the “cut-off-function”, which is defined by

$$\eta(x) := \begin{cases} 1, & \text{for } 0 \leq x \leq k\varepsilon, \\ k + 1 - \frac{x}{\varepsilon}, & \text{for } k\varepsilon \leq x \leq (k+1)\varepsilon, \\ 0, & \text{for } (k+1)\varepsilon \leq x \end{cases} \quad (3.3)$$

for  $\varepsilon > 0$  and a positive integer  $k$ .

**Lemma 3.3.** *Suppose that the same assumptions as in Proposition 3.1 hold. Then the specific volume  $v(x, t)$  is given by the formula*

$$v(x, t)^\gamma = \frac{v_0(x)^\gamma + \frac{K\gamma}{\mu} \int_0^t A_\varepsilon(x, \tau) B_\varepsilon(x, \tau) d\tau}{A_\varepsilon(x, t) B_\varepsilon(x, t)}, \quad (3.4)$$

for  $x \in [(k-1)\varepsilon, k\varepsilon)$  and  $t \in [0, T]$ , where

$$A_\varepsilon(x, t) := \exp \left( \frac{K\gamma}{\mu\varepsilon} \int_0^t \int_{k\varepsilon}^{(k+1)\varepsilon} v^{-\gamma} dx d\tau + \frac{\gamma}{\mu} \int_0^t \int_x^\infty \frac{U_r}{r^{n-1}} \eta dx d\tau \right), \quad (3.5)$$

$$B_\varepsilon(x, t) := \exp \left( \frac{\gamma}{\mu} \int_x^\infty \left( \frac{u}{r^{n-1}} - \frac{u_0}{r_0^{n-1}} \right) \eta dx + \frac{\gamma}{\mu} \int_0^t \int_x^\infty (n-1) \frac{u^2}{r^n} \eta dx d\tau - \frac{\gamma}{\varepsilon} \int_{k\varepsilon}^{(k+1)\varepsilon} \log \frac{v}{v_0} dx \right). \quad (3.6)$$

Utilizing (3.2) and the estimate (2.24a), we have the estimates for  $A_\varepsilon(x, t)$  and  $B_\varepsilon(x, t)$ .

**Lemma 3.4.** *Suppose that the same assumptions as in Proposition 3.1 hold. Then we have*

$$e^{(c_\varepsilon - \bar{c}\delta\varepsilon)(t-\tau)} \leq \frac{A_\varepsilon(x, t)}{A_\varepsilon(x, \tau)}, \quad 0 < c_\varepsilon \leq B_\varepsilon(x, t) \leq C_\varepsilon \quad (3.7)$$

for  $x \in [(k-1)\varepsilon, k\varepsilon)$  and  $0 \leq \tau \leq t \leq T$ , where  $c_\varepsilon$ ,  $C_\varepsilon$  and  $\bar{c}$  are positive constants independent of  $T$ ,  $t$ ,  $\tau$  and  $k$ .

Applying the estimates (3.7) to the representation formula (3.4) and taking  $\varepsilon$  suitably small, we can prove Proposition 3.1. The combination of the estimate (2.24) and Proposition 3.1 gives the uniform  $H^1$ -estimate

$$\begin{aligned} & \int_0^\infty (v - \tilde{v})^2 + u^2 + r^{2n-2}(v - \tilde{v})_x^2 + r^{2n-2}u_x^2 dx \\ & + \int_0^t \int_0^\infty \frac{u^2}{r^2} + r^{2n-2}u_x^2 + r^{2n-4}\varphi_x^2 + r^{4n-6}u_{xx}^2 dx d\tau \leq CE_0, \end{aligned} \quad (3.8)$$

where  $C$  is a positive constant depending only on the initial data. This uniform estimate immediately gives the time global solution to the problem (2.2)–(2.4) by the standard continuation argument.

By virtue of the uniform estimate (3.8), we show the convergence (2.6). Then, utilizing the Schauder theory for the parabolic equations, we prove the Hölder continuity of the solution, which immediately yields the global existence in the Eulerian coordinate and the convergence (1.14).

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