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ENTROPY FOR AUTOMORPHISMS OF THE CROSSED PRODUCTS
エルゴード理論におけるエントロピーから非可換力学系でのエントロピーへ

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Abstract. Let $G \subset \mathrm{Aut}(A)$ be a discrete group which is exact, that is, admits an amenable action on some compact space. Then the entropy of an automorphism of the algebra A does not change by the canonical extension to the crossed product $A \times G$. This is shown for the topological entropy of an exact C^*-algebra A and for the dynamical entropy of an AFD von Neumann algebra A. These have applications to the case of transformations on Lebesgue spaces.

1. INTRODUCTION

The notion of the Kolmogoroff-Sinai entropy in the ergodic theory was brought into the theory of finite von Neumann algebras by Connes-Størmer ([10]), as a non-commutative extension. Replacing the finite trace to a state ϕ, it was extended to general von Neumann algebras and to C^*-algebras by Connes-Narnhofer-Thirring ([9]). In this paper we call the Connes-Størmer entropy the CS-entropy and the Connes-Narnhofer-Thirring entropy the CNT-entropy. We denote by $H(\cdot)$ the CS-entropy and the by $h_\phi(\cdot)$ the CNT-entropy.

In the ergodic theory, we are given a probability space (X, μ) together with a measure preserving nonsingular transformation T of X. Then we have the abelian von Neumann algebra $L^\infty(X, \mu)$ with the trace τ_μ induced by μ and the automorphism α_T of $L^\infty(X, \mu)$ induced by T. In this setting, the Connes-Størmer entropy $H(\alpha_T)$ with respect to the trace τ_μ is nothing but the Kolmogoroff-Sinai entropy $h(T)$.

The noncommutative algebra M is given from this dynamical ststem (X, μ, T) by taking the crossed product $M = L^\infty(X, \mu) \times_\alpha \mathbf{Z}$. The automorphism α_T is extended naturally to the automorphism $\overline{\alpha_T}$ of M, and it preserves the natural extension $\overline{\tau_\mu}$ of τ_μ. As a logical consequence, the following question was suggested by Størmer in ([17]) : Do we have $H(\overline{\alpha_T}) = h(T)$?

The first positive answer is due to Voiculescu. He showed that

$$H(\overline{\alpha_T}) = h(T) = \log n$$

for the ergodic measure preserving Bernoulli transformation T on the space (X, μ), where X is the product space $\{1, \cdots, n\}^{\mathbf{Z}}$ and the measure μ is the product measure $\mu_n^{\otimes \mathbf{Z}}$. Here μ_n is the equal weights probability measure on the set $\{1, \cdots, n\}$.

It was an application of the result on his topological entropy $ht(\cdot)$ introduced in the paper ([20]) for automorphisms of nuclear C^*-algebras. After then, Brown [3] extended the notion to automorphisms of more large class of C^*-algebras, that is, exact C^*-algebras.

Let us replace the integer group $\mathbf{Z}$ to a discrete group G, and let us replace the abelian von Neumann algebra $L^\infty(X, \mu)$ to a general von Neumann algebra M with a state μ, or a C^*-algebra A. Then we have the von Neumann crossed product $M \times_\alpha G$ with respect to an action α of G on M with

$$\mu \circ \alpha_g = \mu, \quad \text{for all} \quad g \in G$$

and also we have the C^*- crossed product $A \times_\alpha G$ with respect to an action α of G on A.

In the case of von Neumann algebras, the state μ has the natural extension $\bar{\mu}$ to $M \times_\alpha G$ which is $\overline{\theta}$-invariant. If an automorphism θ of M satisfies that

$$\alpha_g \theta = \theta \alpha_g, \quad \text{for all } g \in G$$

and

$$\mu \circ \theta = \mu,$$

then θ can be canonically extended to the automorphism $\overline{\theta}$ of $M \times_\alpha G$, and the following problem naturally arises :

$$h_\mu(\theta) = ht_{\bar{\mu}}(\overline{\theta}) \quad ?$$

Similarly in the case of C^*-algebras, if an automorphism θ of A satisfies $\alpha_g\theta = \theta\alpha_g$ for all $g \in G$, then it is canonically extended to the automorphism $\overline{\theta}$ of $A \times_\alpha G$, and the following problem naturally arises :

$$ht(\theta) = ht(\overline{\theta}) \quad ?$$

When G is amenable, known results for these two problems are as follows :

Theorem. [6, 11, 13]. *Assume that G is an amenable discrete countable group.*

(1) [6, 11]. *Let A be a unital exact C^*-algebra and α an action of G on A. If θ is an automorphism of A such that $\alpha_g\theta = \theta\alpha_g$ for all $g \in G$, then*

$$h_\phi(\theta) = h_{\bar{\phi}}(\bar{\theta}).$$

(2) [13]. *Let M be an approximately finite-dimensional von Neumann algebra with a normal state ϕ and α an action of G on M with $\phi \cdot \alpha_g = \phi$ for all $g \in G$.*

If θ is an automorphism of M such that $\phi \circ \theta = \phi$ and $\alpha_g\theta = \theta\alpha_g$ for all $g \in G$, then

$$h_\phi(\theta) = h_{\bar{\phi}}(\bar{\theta}),$$

where $\bar{\phi}$ is the canonical extension of ϕ to $M \times_\alpha G$.

There are a large class of interesting non amenable discrete groups such as free groups $F_n, n \geq 2$ and discrete subgroups of connected Lie groups, etc.. . However each of these nonamenable groups has an amenable action on some compact space ([1, 2, 15]).

A discrete group G has an amenable action on some compact space if and only if G is *exact* in the sense of Kirchberg and Wassermann ([14]), that is, its reduced group C^*-algebra $C_r^*(G)$ is exact. This is first proved by Ozawa in [15].

Here, we report our results which show that the amenability of G is not always necessary and it is replaced to more large class of groups, that is, exact groups.

2. BASIC NOTATIONS AND TERMINOLOGIES.

Proofs of the main results are given using partly some methods in [4, 5, 6, 7, 13]. Here we only denote some basic notations and terminologies.

2.1. **Approximation property and exactness.** Here our C^*−algebras are all separable. Let M be a von Neumann algebra (resp. unital C^*-algebra). Then M is called *approximately finite dimensional* if there exists an increasing sequence $(N_k)_k$ of finite dimensional subalgebras such that $\cup_k N_k$ is weakly (resp. norm) dense in M.

This approximation property is extended in the case of C^*-algebras in [14, 21] as exactness.

A C^*-algebra is *exact* if there exists a representation π of A on a Hilbert space H and triplets $(\varphi_\iota, B_\iota, \psi_\iota)_\iota$ of finite dimensional algebras B_ι, completely positive maps

$$\varphi_\iota : A \to B_\iota, \quad \psi_\iota : B_\iota \mapsto B(H)$$

such that

$$\| \pi(a) - \psi_\iota \cdot \varphi_\iota(a) \| \to 0$$

for all $a \in A$.

A discrete group G is called *exact* if the C^*-algebra $C_r^*(G)$ generated by the left regular representation is exact.

2.2. **Entropy.** Topological entropy $ht(\theta)$ is defined for an automorphism θ of an exact C^*-algebra. CS-entropy $H(\alpha)$ is defined for an automorphism α of a finite von Neumann algebra M with a finite trace τ such that $\tau \cdot \alpha = \tau$ and CNT-entropy $h_\phi(\theta)$ is defined for an automorphism θ of a unital C^*-algebra A with a state ϕ such that $\phi \cdot \theta = \phi$. They are both called dynamical entropies. CS-entropy $H(\alpha)$ depends on a finite trace τ such that $\tau \cdot \alpha = \tau$ and CNT-entropy $h_\phi(\theta)$ also depends on a state ϕ such that $\phi \cdot \theta = \phi$. Let M be the von Neumann algebra generated by the GNS representation $\pi_\phi(A)$. Then such a θ as $\phi \cdot \theta = \phi$ is extended to the automorphism $\bar{\theta}$ of M naturally . If ϕ is a tracial state, then the natural extension $\bar{\phi}$ is a trace of a finite von Neumann algebra M and

$$H(\bar{\theta}) = h_\phi(\theta).$$

If $\phi \cdot \theta = \phi$, then topological entropy $ht(\theta)$ and CNT-entropy $h_\phi(\theta)$ has the following relation :

$$h_\phi(\theta) \leq ht(\theta).$$

We refer these [3, 10, 9, 20]

2.2.1. *Topological entropy.* We refer [3, 20] for definitions and notations about the topological entropy.

2.2.2. *Dynamical entropy.* We refer [10, 9] for definitions and notations about the topological entropy.

2.3. **Crossed product.** Let A be a unutal C^*-algebra (resp. von Neumann algebra), G a discrete countable group and α be an action of G on A, that is a homomorphism from G to the automorphism group $\mathrm{Aut}(A)$ of A. We may assume that A is acting on a Hilbert space H faithfully. The crossed product $A \times_\alpha G$ is the C^*-subalgebra (resp. von Neumann subalgebra) of

$$B(l^2(G,H)) \cong B(l^2(G)) \otimes B(H)$$

generated by $\pi_\alpha(A)$ and λ_G, where

$$\pi_\alpha(a)\xi(g) = \alpha_{g^{-1}}(a)\xi(g), \qquad (a \in A, g \in G, \xi \in l^2(G,H))$$

and

$$\lambda_g\xi(h) = \xi(g^{-1}h), \quad (a \in A, g \in G, \xi \in l^2(G,H)).$$

Essentially, we use the following representation as in [3, 4, 6, 7, 13, 19] :

$$\pi_\alpha(a)\lambda_g = \sum_{t\in G} e_{t,g^{-1}t} \otimes \alpha_t^{-1}(a), \quad (a \in A, g \in G),$$

where $\{e_{s,t}\}_{s,t\in G}$ is the standard matrix units in $B(l^2(G))$.

Since G is discrete, there exists always the conditional expectation E of $A \times_\alpha G$ onto $\pi_\alpha(A)$ such that

$$E(\lambda_g) = 0$$

for all $g \in G$ except the unit. If ϕ is a state of A with $\phi \circ \alpha_g = \phi$ for all $g \in G$, we denote the state $\phi \circ E$ by $\bar{\phi}$ and call it the canonical extension of ϕ to $A \times_\alpha G$.

If $\theta \in \mathrm{Aut}(A)$ commutes with α_g for all $g \in G$, then there exists always an automorphism $\bar{\theta} \in \mathrm{Aut}(A \times_\alpha G)$ such that

$$\bar{\theta}(\pi_\alpha(a)\lambda_g) = \pi_\alpha(\theta(a))\lambda_g, \qquad (a \in A, g \in G).$$

We call the $\bar{\theta}$ the canonical extension of θ.

2.4. **Amenability.** The notion of amenablity for groups is generalized to amenability of actions of groups, that is, a group admits an amenable action on some compact space (cf. [1, 2, 5, 14]).

For example, the discriptions in [1] and [5] are as follows :

2.4.1. *Amenable action.* ([5]) :

Let G be a countable discrete group, and let α^G be the action : $G \to \mathrm{Aut}(l^\infty(G))$ given by

$$\alpha_g^G(x)(h) = x(g^{-1}h), \quad (x \in l^\infty(G),\ g, h \in G).$$

Let $l^1(G, l^\infty(G))$ be the closure of the linear space of finitely supported functions $T : G \to l^\infty(G)$ with respect to the norm

$$||T||_1 = ||\sum_g |T(g)|\ ||_{l^\infty(G)}.$$

Let us put

$$s.T(g) = \alpha_s^G(T(s^{-1}g)), \quad (s, g \in G).$$

The action α^G is *amenable* if there exist functions $T_n \in l^1(G, l^\infty(G))$ such that

(1) T_n is nonnegative (i.e. $T_n(g) \geq 0, (g \in G)$),

(2) finitely supported,

(3) $\sum_g T_n(g) = 1_{l^\infty(G)}$ and

(4) $||s.T_n - T_n||_1 \to 0$ for all $s \in G$.

2.4.2. *Amenable at infinity.* ([1]) :

A group G is amenable at infinity if and only if there exists a sequence $(g_n)_{n\geq 1}$ of nonnegative functions on $G \times G$ with support in a tube such that

a) for each n and each s,

$$\sum_t g_n(s,t) = 1,$$

b) uniformly on tubes,

$$\lim_n \sum_{u\in G} |g_n(s,u) - g_n(t,u)| = 0.$$

Here, a tube means the set $\{(s,t) : s^{-1}t \in F\}$ for some finite suset F of G.

2.4.3. *Equivalence.* These two notions of 3.4.1 and 3.4.2 are equivalent. In fact, let

$$(T_n(t))(s) = g_n(s^{-1}, s^{-1}t)$$

for all $s, t \in G$, then conditions in one side are implied by the other side.

A group G is exact if G admits an amenable action on a compact space ([15]) which is equivalent to that G is amenable at infinity ([1]) and also it is equivalent to that α^G is amenable.

2.4.4. *Typical examples of exact groups.*

(1) Amenable groups.

(2) Free groups.

(3) Discrete subgroups of connected Lie groups.

(4) Subgroups, extensions, free products of the above groups.

(5) Quotients by classical amenable groups

3. MAIN RESULTS

Our results are followings :

3.1. **Theorem.** *Let A be a unital exact C^*-algebra, G an exact discrete countable group, and α an action of G on A.*

If $\beta \in Aut(A \times_\alpha G)$ satisfies $\beta(\lambda_g) = \lambda_g$ for all $g \in G$ and $\beta(\pi_\alpha(A)) = \pi_\alpha(A)$, then

$$ht(\beta) = ht(\beta|_{\pi_\alpha(A)}).$$

Here π_α is the representation of A and λ is the unitary representation of G such that $A \times_\alpha G$ is generated by $\{\pi(A), \lambda_G\}$.

3.2. **Remark.** We have more general result on the topological entropy. In fact, by replacing the condition that

$$\beta(\lambda_g) = \lambda_g \quad \text{for all} \quad g \in G$$

to the condition that

$$\beta(\lambda_G) = \lambda_G$$

we have a similar result in [7]. This gives an application to the proof of the main theorem in [12].

3.3. **Theorem.** *Let M be an approximately finite-dimensional von Neumann algebra with a normal state ϕ, G an exact discrete countable group, and α an action of G on M with*

$$\phi \cdot \alpha_g = \phi \quad \textit{for all} \quad g \in G.$$

If θ is an automorphism of M such that $\phi \circ \theta = \phi$ and

$$\alpha_g \theta = \theta \alpha_g \quad \textit{for all} \quad g \in G,$$

then

$$h_\phi(\theta) = h_{\bar{\phi}}(\bar{\theta}),$$

where $\bar{\phi}$ is the canonical extension of ϕ to $M \times_\alpha G$.

3.4. *Proof.* Proofs for these results are in [8]. In [8], we adopt as exactness of the group G the amenability of the canonical action α^G in [5]. □

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