数理解析研究所講究録 1355 巻 2004 年 24-32

On the solution to nonlinear Schrödinger equation with superposed δ-function as initial data

九州大学大学院・数理学研究院 北 直泰 (Naoyasu Kita)
Faculty of Mathematics, Kyushu University

1 Introduction

We consider the Cauchy problem for the nonlinear Schrödinger equation with very singular initial data described as the superposition of point mass measures:

$$\begin{cases} i\partial_t u = -\Delta u + \mathcal{N}(u), \\ u(0,x) = \mu_0\delta_0(x) + \mu_1\delta_a(x). \end{cases} \tag{1.1}$$

In the above equation, u is a complex valued unknown function of $(t,x) \in \mathbf{R}\times\mathbf{R}^n$ $(n \geq 1)$. The nonlinearity $\mathcal{N}(u)$ is of gauge invariant power type, i.e.,

$$\mathcal{N}(u) \;=\; \lambda|u|^{p-1}u,$$

where $\lambda \in \mathbf{C}$ and $1 < p < 1 + 2/n$. The functional $\delta_b(x)$ denotes Dirac's δ function supported at $x = b$ and the coefficiet μ_j $(j = 0, 1)$ belongs to $\mathbf{C}$. Some generalization of the initial data will be given as the remark later.

The nonlinear evolution equations with measures as initial data are extensively sutudied. For nonlinear parabolic equations, Brezis-Friedman [2] gives the critical power of nonlinearity concerning the solvability and unsoluvability of the equation. For the KdV equation, Tsutsumi [5] constructs a solution by making use of Miura transformation. Recently, Abe-Okazawa [1] have studied this problem for the complex Ginzburg-Landau equation. The idea of the proof for these known results is based on the strong smoothing effect of linear part or the nonlinear transformation of unknown functions into the suitably handled equation. In the present case, however, the nonlinear Schrödinger equation does not posesse the useful smoothing properties and the transformation into easily handeled equation. Therefore, it is still open whether we can construct a solution when the initial data is arbitrary measure.

The author considered the case in which the initial data is single delta function supported at the origin. In this case, the solution is explicitly described as

$$\begin{aligned} u(t,x) &= A(t)\exp(it\Delta)\delta_0 \\ &= A(t)(4\pi it)^{-n/2}\exp(ix^2/4t), \end{aligned} \tag{1.2}$$

where the modified amplitude $A(t)$ is

$$A(t) = \begin{cases} \exp\left(\frac{\lambda}{i}\frac{(4\pi)^{-n(p-1)/2}}{1-n(p-1)/2}|t|^{-n(p-1)/2}t\right) & \text{if } \mathrm{Im}\lambda = 0, \\ \exp\left(\frac{i\lambda}{(p-1)\mathrm{Im}\lambda}\log(1 - C_{n,p}\mathrm{Im}\lambda|t|^{-n(p-1)/2}t)\right) & \text{if } \mathrm{Im}\lambda \neq 0, \end{cases} \tag{1.3}$$

where $C_{n,p} = (p-1)(4\pi)^{-n(p-1)/2}(1-n(p-1)/2)^{-1}$. We note that (1.3) gives the global solution if $\mathrm{Im}\lambda = 0$ and the blow-up solution at positive (resp. negative) finite time if $\mathrm{Im}\lambda > 0$ (resp. < 0). In fact, by substitute the expression (1.2) into (1.1), we have the ordinary differential equation of $A(t)$ such that

$$i\frac{dA}{dt} = |4\pi t|^{-n(p-1)/2}\mathcal{N}(A),$$

with the initial data $A(0) = 1$. This ODE is easily solved as we obtain (1.3). In [4], we also study the case in which the initial data cosists of the superposition of δ_0 and $L^2(\mathbf{R}^n)$-perturbation. In this case, the global existence in time follows if $\lambda \in \mathbf{R}$ and sme additional conditions on the power of nonlinearity are imposed.

Our concern in this proceeding is to construct a solution to (1.1) with the formar $L^2(\mathbf{R}^n)$-perturbation replaced by δ functions supported away from the origin. Before stating our main theorems we introduce the space of sequences:

$$\ell^2_\alpha = \{(A_k)_{k\in\mathbf{Z}};\ \|(A_k)_{k\in\mathbf{Z}}\|_{\ell^2_\alpha} < \infty\},$$

where $\|(A_k)_{k\in\mathbf{Z}}\|^2_{\ell^2_\alpha} = \sum_{k\in\mathbf{Z}}|(1+k^2)^{\alpha/2}A_k|^2$. The first main theorem is concerning the local existence of the solution.

Theorem 1.1 *For some $T = T(\mu) > 0$, there exists a unique solution $u(t,x)$ to (1.1) described like*

$$u(t,x) = \sum_{k\in\mathbf{Z}} A_k(t)\exp(it\Delta)\delta_{ka}, \tag{1.4}$$

where $(A_k(t))_{k\in\mathbf{Z}} \in C([-T,T];\ell^2_1)\cap C^1([-T,T]\backslash\{0\};\ell^2_1)$ with $A_0(0) = \mu_0$, $A_1(0) = \mu_1$ and $A_k(0) = 0$ $(k \neq 0,1)$.

The idea of the proof is based on the reduction of (1.1) into the system of ordinary differential equations (see section 2).

Remark1.1. Let us call $A_k(t)\exp(it\Delta)\delta_{ka}$ the k-th mode. Then, (1.4) suggests that new modes away from 0-th and first ones appear in the solution while the initial data contains only the two modes. This special property is visible only in the nonlinear case. We can not expect this kind of phenomena in the linear case ($\lambda = 0$). The representation (1.4) is deduced by the following rough consideration. Since the nonlinear solution is first well-approximated by the linear solution $u_1(t,x) = \exp(it\Delta)(\mu_0\delta_0 + \mu_1\delta_a)$, the second approximation $u_2(t,x)$ is given by solving

$$\begin{aligned}(i\partial_t + \Delta)u_2 &= \mathcal{N}(u_1)\\ &= \mathcal{N}((2\pi)^{-n/2}e^{ix^2/4t}D(\mu_0 + \mu_1 e^{-ia\cdot x}e^{ia^2/4t}))\\ &= |4\pi t|^{-n(p-1)/2}(2\pi)^{-n/2}e^{ix^2/4t}D\mathcal{N}(1 + e^{-ia\cdot x}e^{ia^2/4t}),\end{aligned}\tag{1.5}$$

where we have used $u_1 = e^{ix^2/4t}D\mathcal{F}e^{ix^2/4t}u(0,x)$, $Df(t,x) = (2it)^{-n/2}f(t,x/2t)$ and $\mathcal{F}$ denotes the Fourier transform. Let us replace $a\cdot x$ by θ. Then, the nonlinearity in (1.5) is regarded as a 2π-periodic function of θ, and hence the Fourier series expansion yields

$$\begin{aligned}(\text{the right hand side of (1.5)}) &= |4\pi t|^{n(p-1)/2}(2\pi)^{-n/2}e^{ix^2/4t}D\sum_{k\in\mathbf{Z}}\tilde{B}_k(t)e^{i(ka)^2/4t}e^{-ik\theta}\\ &= |4\pi t|^{-n(p-1)/2}\sum_{k\in\mathbf{Z}}B_k(t)\exp(it\Delta)\delta_{ka},\end{aligned}$$

where $B_k(t)e^{i(ka)^2/4t}$ is the Fourier coefficient. By the Duhamel principle, we can imagine that the solution to (1.1) has the description as in (1.4).

Remark1.2. Reading the proof of Theorem 1.1, we see that it is possible to generalize the initial data. Namely, we can construct a solution even when point masses are distributed on a line at equal intervals – more precisely, the initial data is given like

$$u(0,x) = \sum_{k\in\mathbf{Z}}\mu_k\delta_{ka}(x),$$

where $(\mu_k)_{k\in\mathbf{Z}} \in \ell_1^2$. In this case, the solution has the description similar to (1.4) but $\{A_k(0)\} = \{\mu_k\}$. The decay condition on the coefficients described in terms of ℓ_1^2 is required to estimate the nonlinearity. This is because we will use the inequality like $\|\mathcal{N}(g)\|_{L^2_\theta} \le C\|g\|_{L^\infty_\theta}^{p-1}\|g\|_{L^2_\theta}$ where $g = g(t,\theta) = \sum_k A_k e^{-ik\theta}e^{i(ka)^2/4t}$ and $\theta \in [0, 2\pi]$. Accordingly, to estimate $\|g\|_{L^\infty_\theta}$, we require the decay condition of $\{A_k\}$.

Remark1.3. Actually, we can construct a solution in more general situation on the initial data. There is no need for the point masses to be distributed on a line at the equal interval. For instance, even when the initial data is given as

$$u(0,x) = \mu_{00}\delta_0(x) + \mu_{10}\delta_a(x) + \mu_{01}\delta_b(x),$$

where a and b are linearly independent on the quotient number field, i.e., $a \neq qb$ for any $q \in \mathbf{Q}$, we can construct a solution to (1.1). This solution is described as

$$u(t,x) = \sum_{j,k\in\mathbf{Z}} A_{j,k}(t)\exp(it\Delta)\delta_{ja+kb},$$

where the coefficients $A_{j,k}$ satisfy the following ordinary differential equation:

$$i\frac{dA_{jk}}{dt} = |4\pi t|^{-n(p-1)/2}\tilde{A}_{jk},$$

with

$$\begin{aligned}&\tilde{A}_{jk}\\ &= (2\pi)^{-2}e^{-i(ja+kb)^2/4t}\int_0^{2\pi}\int_0^{2\pi} e^{i(j\theta_1+k\theta_2)}\mathcal{N}\big(\sum_{j'k'} A_{j'k'}e^{-i(j'\theta_1+k'\theta_2)}e^{i(j'a+k'b)^2/4t}\big)\,d\theta_1 d\theta_2.\end{aligned}$$

The above ODE is time-locally solved under some special conditions on μ_{jk} which we want to get rid of.

If $\lambda \in \mathbf{R}$, then we obtain the time global result given below.

Theorem 1.2 *In addition to the assumptions of nonlinearity, we let $\lambda \in \mathbf{R}$. Then, there exists a unique global solution to (1.1) described in the similar way to (1.4) but $\{A_k(t)\}_{k\in\mathbf{Z}} \in C(\mathbf{R};\ell_1^2) \cap C^1(\mathbf{R}\backslash\{0\};\ell_1^2)$.*

2 Proof of Theorem 1.1

In this section, we reduce (1.1) into the system of infinitely many ordinary differential equations of $A_k(t)$. We first prove a simple lemma which gives the useful representation of nonlinearity. This lemma is a by-product of the argument with Takeshi Wada in Osaka University.

Lemma 2.1 *Let $\{A_k\} \in C([-T,T];\ell_1^2)$. Then, we have*

$$\mathcal{N}\big(\sum_{k\in\mathbf{Z}} A_k(t)\exp(it\Delta)\delta_{ka}\big) = |4\pi t|^{-n(p-1)/2}\sum_{k\in\mathbf{Z}}\tilde{A}_k(t)\exp(it\Delta)\delta_{ka}, \tag{2.1}$$

where

$$\tilde{A}_k(t) = (2\pi)^{-1} e^{i(ka)^2/4t} \langle e^{-ik\theta}, \mathcal{N}(\sum_j A_j e^{-ij\theta} e^{-i(ja)^2/4t}) \rangle_\theta,$$

with $\langle f, g\rangle_\theta = \int_0^{2\pi} \bar{f}(\theta) g(\theta) d\theta$.

Proof of Lemma 2.1. Note that

$$\begin{aligned} \exp(it\Delta)f &= (4\pi it)^{-n/2} \int \exp(i|x-y|^2/4t) f(y)dy \\ &= MD\mathcal{F}Mf, \end{aligned}$$

where

$$\begin{aligned} Mg(t,x) &= e^{ix^2/4t} g(x), \\ Dg(t,x) &= (2it)^{-n/2} g(x/2t), \\ \mathcal{F}g(\xi) &= (2\pi)^{-n/2} \int e^{-i\xi x} g(x)dx \quad \text{(Fourier transform of } g). \end{aligned}$$

Then, we see that

$$\begin{aligned} &\mathcal{N}(\sum_k A_j(t) \exp(it\Delta)\delta_{ja}) \\ &= \mathcal{N}((2\pi)^{-n/2} MD \sum_j A_j(t) e^{-ija\cdot x - i(ja)^2/4t}) \\ &= |4\pi t|^{-n(p-1)/2} (2\pi)^{-n/2} MD\mathcal{N}(\sum_j A_j(t) e^{-ija\cdot x - i(ja)^2/4t}). \end{aligned} \tag{2.2}$$

Note that, to show the last equality in (2.2), we make use of the gauge invariance of the nonlinearity. Replacing $a \cdot x$ by θ, we can regard $\mathcal{N}(\sum_j A_j(t) e^{-ij\theta - i(ja)^2/4t})$ as the 2π-periodic function of θ. Therefore, the Fourier series expansion is allowed, i.e.,

$$\begin{aligned} \mathcal{N}(\sum_j A_j(t) e^{-ij\theta - i(ja)^2/4t}) &= \sum_k \tilde{A}_k(t) e^{-i(ka)^2/4t} e^{-ik\theta} \\ &= (2\pi)^{n/2} \sum_k \tilde{A}_k(t) \mathcal{F}M\delta_{ka}. \end{aligned}$$

Plugging this into (2.2), we obtain Lemma 2.1. □

We next consider the reduction of (1.1) into the system of ODE's. By substituting $u = \sum_k A_k(t) \exp(it\Delta)\delta_{ka}$ into (1.1) and noting that $i\partial_t \exp(it\Delta)\delta_{ka} = -\Delta \exp(it\Delta)\delta_{ka}$, Lemma 2.1 yields

$$\sum_k i\frac{dA_k}{dt} \exp(it\Delta)\delta_{ka} = |4\pi t|^{-n(p-1)/2} \sum_k \tilde{A}_k \exp(it\Delta)\delta_{ka}.$$

Recalling that $\exp(it\Delta)\delta_{ka} = (2\pi)^{-n/2}MDe^{-i\theta}M$ and considering the uniqueness of the Fourier series expansion, we arrive at the desired ODE:

$$i\frac{dA_k}{dt} = |4\pi t|^{-n(p-1)/2}\tilde{A}_k \tag{2.3}$$

with the initial condition $A_k(0) = \mu_k$. Now, showing the existence and uniqueness of (1.1) is equivalent to showing those of (2.3). To solve (2.3), let us consider the following integral equation.

$$\begin{aligned} A_k(t) &= \Phi_k((A_k(t))_{k\in\mathbf{Z}}) \\ &\equiv \mu_k - i\int_0^t |4\pi\tau|^{-n(p-1)/2}\tilde{A}_k(\tau)\, d\tau. \end{aligned} \tag{2.4}$$

We here require the contraction property of $(\Phi_k)_{k\in\mathbf{Z}}$.

Lemma 2.2 *Let $I = [-T, T]$ and $(A_k) = (A_k)_{k\in\mathbf{Z}}$. Then, we have*

$$\|\{\tilde{A}_k\}\|_{L^\infty(I;\ell_1^2)} \le C\|\{A_k\}\|^p_{L^\infty(I;\ell_1^2)}, \tag{2.5}$$

$$\begin{aligned} &\|\{\tilde{A}_k^{(1)}\} - \{\tilde{A}_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)} \\ &\le C(\max_{j=1,2}\|\{A_k^{(j)}\}\|_{L^\infty(I;\ell_1^2)})^{p-1}\|\{A_k^{(1)}\} - \{A_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)}. \end{aligned} \tag{2.6}$$

Proof of Lemma 2.2. According to the description of $\tilde{A}_k$ as in Lemma 2.1 and the integration by parts, we see that

$$k\tilde{A}_k = (2\pi)^{-1}ie^{-i(ka)^2/4t}\langle e^{-ik\theta}, \partial_\theta\mathcal{N}(\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t})\rangle_\theta.$$

Then, Parseval's equality yields

$$\begin{aligned} \|\{k\tilde{A}_k\}\|_{\ell_0^2} &= (2\pi)^{-1/2}\|\partial_\theta\mathcal{N}(\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t})\|_{L_\theta^2} \\ &\le C\|\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t}\|^{p-1}_{L_\theta^\infty}\|\sum_j jA_j e^{-ij\theta}e^{i(ja)^2/4t}\|_{L_\theta^2} \\ &\le C\|\{A_j\}\|^p_{\ell_1^2}. \end{aligned}$$

Thus, we obtain (2.5). The proof for (2.6) follows similarly. Since there is a singularity at $u = 0$ of the nonlinearity $\mathcal{N}(u)$, we do not employ ℓ_1^2-norm to measure $\{A_k^{(1)}\} - \{A_k^{(2)}\}$. □

Proof of Theorem 1.1. Let $\|\{\mu_k\}\|_{\ell_1^2} \le \rho_0$. Then, in virture of Lemma 2.2, it is easy to see that, for some $T = T(\rho_0) > 0$, $\{\Phi_k(\{A_j\})\}$ is the contraction map on the closed ball $B_{2\rho_0} = \{\{A_k\}; \|\{A_k\}\|_{L^\infty(I;\ell_1^2)} \le 2\rho_0\}$ with the mertic $\|\{A_k^{(1)}\} - \{A_k^{(2)}\}\|_{L^\infty(I;\ell_0)}$.

Therefore, we first obtain the solution $\{A_k\}$ of (2.4) which belongs to $L^\infty(I;\ell_1^2)$. Since this solution satisfies the integral equation (2.4), we see that it actually belongs to $C(I;\ell_1^2)$ and, moreover, belongs to $C^1(I\backslash\{0\};\ell_1^2)$. □

Remark We can continuate the local solution as long as $\|\{A_k(t)\}\|_{\ell_1^2} < \infty$. This follows by solving

$$A_k(t) \;=\; A_k(t_0) + \int_{t_0}^{t} |4\pi\tau|^{-n(p-1)/2}\tilde{A}_k(\tau)d\tau.$$

The method to construct the solution is similar to the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we derive the a priori estimate of $\{A_k(t)\}$, which yields the global existence of the solution.

Lemma 3.1 *Let $\lambda \in \mathbf{R}$ and let $\{A_k\}$ be the solution to (2.3). Then, we have*

$$\|\{A_k(t)\}\|_{\ell_0^2} = \|\{\mu_k\}\|_{\ell_0^2}, \tag{3.1}$$

$$\|\{kA_k(t)\}\|_{\ell_0^2}^2 + K_{n,p,a}\lambda|t|^{2-n(p-1)/2}\|\sum_{k\in\mathbf{Z}} A_k e^{-ik\theta}e^{i(ka)^2/4t}\|_{L_\theta^{p+1}}^{p+1} \le C_{\{\mu_k\}}\,\langle t\rangle^{2-n(p-1)/2}, \tag{3.2}$$

where $K_{n,p,a} = 8/((4\pi)^{1+n(p-1)/2}a^2(p+1))$ and $\langle t\rangle = (1+t^2)^{1/2}$.

Proof of Lemma 3.1. Then, by multiplying $\bar{A}_k$ on both hand sides of (2.3) and taking summation with respective to $k \in \mathbf{Z}$, (3.1) follows. We next prove (3.2). Let $g(t,\theta) = \sum A_k e^{-ik\theta}e^{i(ka)^2/4t}$ and write

$$\begin{aligned}\frac{d\|\{kA_k(t)\}\|_{\ell_0^2}^2}{dt} &= 2\mathrm{Re}\sum_k \bar{A}_k k^2\frac{dA_k}{dt}\\ &= 2|4\pi t|^{-n(p-1)/2}\mathrm{Im}\sum_k \bar{A}_k k^2\tilde{A}_k.\end{aligned}$$

We here note that

$$\begin{aligned}\sum_k \bar{A}_k k^2\tilde{A}_k &= (2t^2/i\pi a^2)\langle\sum_k A_k e^{-ik\theta}\frac{de^{-i(ka)^2/4t}}{dt},\mathcal{N}(g)\rangle_\theta\\ &= (2t^2/i\pi a^2)\langle\frac{dg}{dt},\mathcal{N}(g)\rangle_\theta\end{aligned}$$

$$
\begin{aligned}
&-(2t^2/i\pi a^2)\sum_k \langle \frac{A_k}{dt} e^{-ik\theta} e^{i(ka)^2/4t}, \mathcal{N}(g)\rangle_\theta \\
= &\ (2t^2/i\pi a^2)\langle \frac{dg}{dt}, \mathcal{N}(g)\rangle_\theta \\
&-(2t^2/\pi a^2)|4\pi t|^{-n(p-1)/2}\sum_k |\langle e^{-ik\theta}, \mathcal{N}(g)\rangle_\theta|^2.
\end{aligned}
$$

Thus,

$$
\frac{d\|\{kA_k\}\|_{\ell_0^2}^2}{dt} = -K_{n,p,a}\lambda|t|^{2-n(p-1)/2}\frac{d\|g\|_{L_\theta^{p+1}}^{p+1}}{dt}
$$

This identity gives, for $t > 0$,

$$
\begin{aligned}
&\frac{d}{dt}(\|\{kA_k\}\|_{\ell_0^2}^2 + K_{n,p,a}\lambda t^{2-n(p-1)/2}\|g\|_{L_\theta^{p+1}}^{p+1}) \\
= &\ (2-n(p-1)/2)K_{n,p,a}\lambda t^{1-n(p-1)/2}\|g\|_{L_\theta^{p+1}}^{p+1} \\
\le &\ (2-n(p-1)/2)t^{-1}(\|\{kA_k\}\|_{\ell_0^2}^2 + K_{n,p,a}\lambda\|g\|_{L_\theta^{p+1}}^{p+1}).
\end{aligned} \tag{3.3}
$$

Let $E(t) = \|\{kA_k\}\|_{\ell_0^2}^2 + K_{n,p,a}\lambda\|g\|_{L_\theta^{p+1}}^{p+1}$. Then, applying Gronwall's inequality to (3.3), we have

$$
E(t) \le E(t_0)(t/t_0)^{2-n(p-1)/2} \quad \text{for } t > t_0 \text{ with } t_0 > 0 \text{ small.} \tag{3.4}
$$

On the other hand, for $t \in (0, t_0)$, the proof for the local existence result as in Theorem 1.1 yields

$$
\begin{aligned}
E(t) &\le \|\{A_k(t)\}\|_{\ell_1^2}^2 + C\|\{A_k(t)\}\|_{\ell_1^2}^{p+1} \\
&\le (2\rho_0)^2 + C(2\rho_0)^{p+1}.
\end{aligned} \tag{3.5}
$$

Combining (3.4) and (3.5), we obtain (3.2). □

Proof of Theorem 1.2. If $\lambda \ge 0$, Lemma 3.1 gives $\|\{A_k\}\|_{\ell_1^2} \le C\langle t\rangle^{1-n(p-1)/4} < \infty$. Thus, the local solution is continuated to the global one. If $\lambda < 0$, Lemma 3.1 (3.2) and Gagliardo-Nirenberg's inequality yield

$$
\|\{kA_k\}\|_{\ell_0^2}^2 \le C\langle t\rangle^{2-n(p-1)/2} + C|t|^{2-n(p-1)/2}(\|g\|_{L_\theta^2}^{\alpha}\|\partial_\theta g\|_{L_\theta^2}^{1-\alpha})^{p+1},
$$

where $1/(p+1) = \alpha/2 + (1-\alpha)(1/2-1)$. We here remark that $(1-\alpha)(p+1) < 2$. Then, by Young's inequality, we have

$$
\|\{kA_k\}\|_{\ell_0^2}^2 \le C_\epsilon\langle t\rangle^N + \epsilon\|\{kA_k\}\|_{\ell_0^2}^2.
$$

This implies that $\|\{A_k(t)\}\|_{\ell_1^2} < \infty$ for any t. Hence, we obtain the global solution. □

References

[1] T. Abe and N. Okazawa, *Global solvability of the complex Ginzburg-Landau equation with distribution-valued initial data, II: 1-dimensional Dirichlet problem*, the 28th conference of evolution euqation (2002, December) at Chuo University.

[2] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial data*, J. Math. Pures Appl. **62**(1983), 73–97.

[3] C. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106**(2001), 627–633.

[4] N. Kita, *Nonlinear Schrödinger equation with δ-functional initial data*, preprint.

[5] Y. Tsutsumi, *The Cauchy problem for the Korteweg-de Vries equation with measure as initial data*, SIAM J. Math. Anal. **20**(1989), 582–588.