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Traveling Waves in a Band Domain with Quasi-Periodically Undulating Boundaries

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1. Introduction

We discuss traveling waves for a curvature-driven motion of plane curves in a band domain Ω_ε, where $\varepsilon > 0$ is a certain small parameter and the boundaries of Ω_ε undulate quasi-periodically as specified below. The law of motion of the curve is given by the equation

$$V = \kappa + A, \tag{1}$$

where V denotes the normal velocity of the curve, κ denotes the curvature and A is a positive constant representing a constant driving force. The domain Ω_ε is defined as follows: Let $g_1(y)$ and $g_2(y)$ be smooth quasi-periodic functions satisfying

$$g_i(y) \geq 0, \quad \inf_y g_i(y) = 0, \quad \sup_y g_i'(y) = \tan \alpha_i, \quad \inf_y g_i'(y) = -\tan \beta_i \quad (i = 1, 2),$$

for some α_i, $\beta_i \in (0, \frac{\pi}{2})$ and $\alpha_i + \beta_i < \frac{\pi}{2}$ $(i = 1, 2)$. Ω_ε is defined by

$$\Omega_\varepsilon := \left\{ (x, y) \in \mathbb{R}^2 \mid -g_{1\varepsilon}(y) < x < g_{2\varepsilon}(y), \quad y \in (-\infty, \infty) \right\}$$

with $g_{i\varepsilon}(y) := 1 + \varepsilon g_i\left(\frac{y}{\varepsilon}\right)$ $(i = 1, 2)$ (see Figure 1).

In this paper, by a solution of (1) we mean a time-dependent simple curve Γ_t in Ω_ε which satisfies (1) and contacts the each boundary of Ω_ε vertically. To avoid sign confusion, the normal to the curve Γ_t will always be chosen toward the direction of the right-hand side region, and the sign of the normal velocity V and the curvature κ will be understood in accordance with this choice of the direction of the normal. Consequently, κ is negative at those points where the curve is concave (see Figure 1).

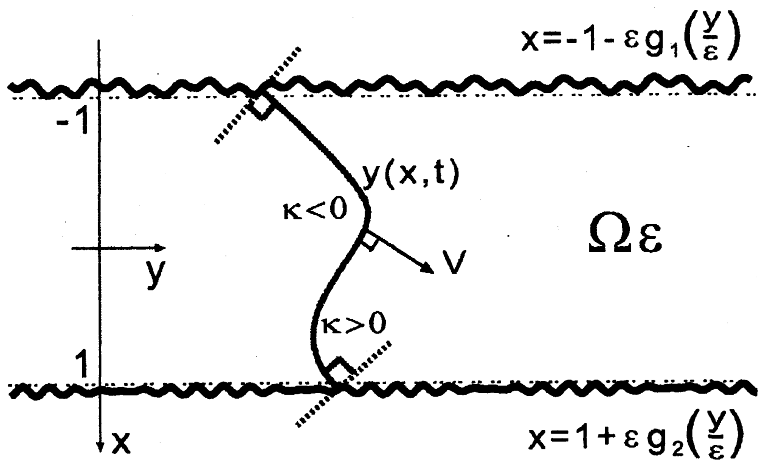

Fig.1 Domain and Curves

We will only consider the case where the curves are expressed as a graph of a certain function $y = y(x,t)$, so (1) is equivalent to

(2) $$y_t = \frac{y_{xx}}{1+y_x^2} + A\sqrt{1+y_x^2}, \qquad t > 0,$$

with boundary conditions

(3) $$y_x(x,t)\big|_{(-g_{1\varepsilon}(y),\ y)} = g_1'(y/\varepsilon), \qquad y_x(x,t)\big|_{(g_{2\varepsilon}(y),\ y)} = -g_2'(y/\varepsilon),$$

and restrictions

(4) $$-g_{1\varepsilon}(y) < x < g_{2\varepsilon}(y).$$

Let $\Omega_0 = \{(x,y) \in \mathbb{R}^2 \mid -1 < x < 1\}$ be a straight band domain which is formally a limit of Ω_ε as $\varepsilon \to 0$. For Ω_0 one can easily see that equation (1) has a traveling wave solution $\Gamma_t = \{(x, y_0 + At) \mid -1 < x < 1\}$ which moves at a constant speed A remaining its shape (a line segment).

On the other hand, for Ω_ε, traveling wave solutions of (1) in the usual sense do not exist in general. For such undulating band domains, the notion of traveling waves has to be extended to the more general one in the same way as in [1].

Case 1. Periodic traveling waves. In the case where g_1 and g_2 are 1-periodic functions, a solution $\mathcal{Y}^\varepsilon(x,t)$ of (2)-(4) is called a periodic traveling wave if

$$\mathcal{Y}^\varepsilon(x, t+T_\varepsilon) = \mathcal{Y}^\varepsilon(x,t) + \varepsilon$$

for some $T_\varepsilon > 0$. Such a periodic traveling wave propagates in y-direction with average speed $c_\varepsilon = \varepsilon/T_\varepsilon$, changing its profile periodically in time.

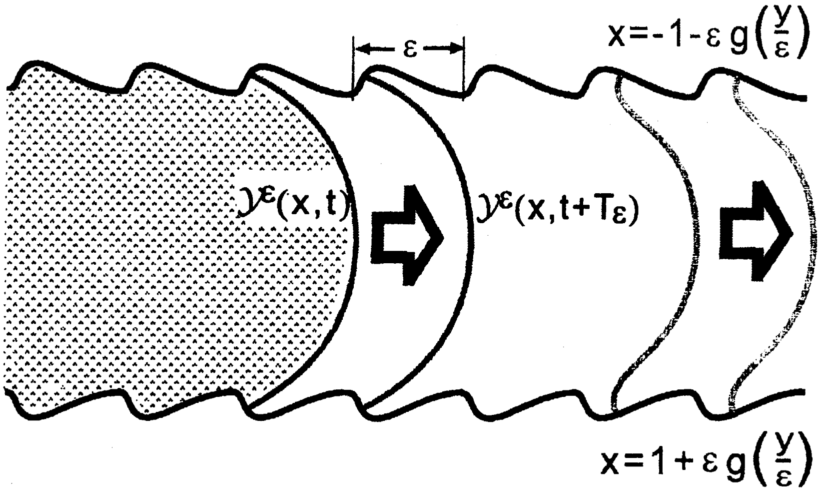

Fig.2 Periodic travling wave

Case 2. Quasi-periodic traveling waves. Roughly speaking, a quasi-periodic traveling wave for (1) is a curve which moves rightward changing its profile and speed quasi-periodically in time. To give a precise definition of quasi-periodic traveling waves, we introduce some notation and terminology. For any solution $y(x,t)$ of (2)-(4), we call

$\xi(t) := y(0,t)$ the *current position*;

$\sigma_{\xi(t)}b = b(y+\xi(t)) \in \mathcal{H}_b$ the *current landscape*, where

$$b(y) = (\ g_{1\varepsilon}(y),\ g_{2\varepsilon}(y)\ ) \quad \text{and} \quad \mathcal{H}_b := \overline{\{\sigma_r b \,|\, r \in \mathbb{R}\}}^{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})} \ \simeq\ \mathbb{T}^m \text{ for some } m \in \mathbb{N};$$

$y(x,t) - \xi(t)$ the *current profile.*

Definition. A solution $\mathcal{Y}^\varepsilon(x,t)$ of (2)-(4) is called a *quasi-periodic traveling wave* if there exists $v(z,s) \in C(\mathcal{H}_b \times \mathbb{R},\ \mathbb{R})$ such that

$$\mathcal{Y}^\varepsilon(x,t)\ -\ \xi^\varepsilon(t)\ =\ v(\sigma_{\xi^\varepsilon(t)}b,\ x),$$

where $\xi^\varepsilon(t) = \mathcal{Y}^\varepsilon(0,t)$ is the current position of $\mathcal{Y}^\varepsilon(x,t)$. This means that the current profile depends continuously on the current landscape. A quasi-periodic traveling wave is called *regular* if $\inf_t\ \xi^\varepsilon_t(t) > 0$.

Note that this definition agrees with that of traveling waves for the homogeneous and the periodic cases. Moreover, we say that a quasi-periodic traveling wave has the *average speed* c_ε

if

$$\frac{\xi(t+T)-\xi(t)}{T} \to c_\varepsilon \quad \text{as } T \to \infty \quad \text{uniformly in } t.$$

Recently, Matano has proved the existence of a regular quasi-periodic traveling wave having average speed for (2)-(4) on the assumption that $A > (\sin\alpha_1 + \sin\alpha_2)/2$. Moreover, one can discuss the uniqueness and stability of the traveling wave by using an argument similar to that in [3, 4].

The goal of this paper is to determine the homogenization limit of quasi-periodic traveling waves for (2)-(4). Our result is the following:

Main Theorem. *Assume that* $A > (\sin\alpha_1 + \sin\alpha_2)/2$ *and let* $\mathcal{Y}^\varepsilon(x,t)$ *be the quasi-periodic traveling wave of (2)-(4), then*

(i) The average speed c_ε *satisfies*

$$c_0 < c_\varepsilon < c_0 + \mathcal{M}(\alpha_1,\alpha_2,A)\,\varepsilon^{1/2} \tag{5}$$

for small ε*, where* $c_0 = c_0(\alpha_1,\alpha_2,A) < A$ *is independent of* ε*, and is given by*

$$2 + F(\alpha_1,c_0) + F(\alpha_2,c_0) = 0,$$

with

$$F(\alpha,c) := \frac{\alpha}{c} - \frac{2A}{c\sqrt{A^2-c^2}}\arctan\left(\sqrt{\frac{A+c}{A-c}}\cdot\tan\frac{\alpha}{2}\right).$$

(ii) $\mathcal{Y}^\varepsilon(x,t)$ *converges (locally in* $C^{2,1}$*) to a homogenization limit* $\varphi(x;c_0)+c_0 t$*, where* $\varphi(x;c_0)$ *is defined by*

$$\varphi(v;c_0) = -\frac{1}{c_0}\log\left|\frac{A - c_0\cos(\arctan v)}{A-c_0}\right|,$$
$$x(v;c_0) = F(\arctan v,\ c_0) - 1 - F(\alpha_1,c_0),$$

by a parameter $v \in (-\tan\alpha_2,\ \tan\alpha_1)$.

Remark 1. (i) The above theorem implies that the effect of spatial inhomogeneity appears in the homogeneization limit, although Ω_ε tends to Ω_0 as $\varepsilon \to 0$. Indeed, the homogenized traveling wave has non-planar profile φ and its propagation speed c_0 is less than A.

(ii) The function $\varphi(x;c_0)$ satisfies

$$c_0 = \frac{\varphi_{xx}}{1+\varphi_x^2} + A\sqrt{1+\varphi_x^2}, \qquad x \in (-\chi_1,\chi_2)$$

for some $\chi_1,\ \chi_2 > 1$, and $\varphi_x(-1) = \tan\alpha_1,\ \varphi_x(1) = -\tan\alpha_2$ (cf. [2]).

2. Proof of Main Theorem

In this section, by constructing a lower solution and an upper solution we prove Main Theorem in the symmetric case: $g_1 = g_2$. The proof for the general case is similar and we omit it. In what follows, we write $g = g_1 (= g_2)$, $\alpha = \alpha_1 (= \alpha_2)$ and $\varphi = \varphi(\cdot; c_0)$.

By Remark 1 (ii), we obtain

Lemma 2.1 $\underline{y}(x,t) := \varphi(x; c_0) + c_0\, t$ is a lower solution of (2)-(4), and $c_0 < c_\varepsilon$.

Let $\mathcal{Y}^\varepsilon(x,t)$ be a periodic traveling wave of (2)-(4). We note that $\mathcal{Y}^\varepsilon(x,t)\big|_{[-1,1]}$ is nothing but the solution of

$$\begin{cases} \tilde{y}_t = \dfrac{\tilde{y}_{xx}}{1+\tilde{y}_x^2} + A\sqrt{1+\tilde{y}_x^2}, & -1 < x < 1,\ t > 0, \\ \tilde{y}(\pm 1, t) = \mathcal{Y}^\varepsilon(\pm 1, t), & t > 0, \\ \tilde{y}(x, 0) = \mathcal{Y}^\varepsilon(x, 0), & -1 < x < 1. \end{cases} \tag{6}$$

Without loss of generality, we may assume $\varphi(\pm 1) = 0$, $\mathcal{Y}^\varepsilon(x,0) \le \varphi(x)$ for $x \in [-1,1]$ and $\mathcal{Y}^\varepsilon(x_0, 0) = \varphi(x_0)$ for some $x_0 \in [-1,1]$. Take $L > \frac{4\sqrt{c_0}\, e^2}{\cos\alpha}$ and define

$$v(x,t) = L\varepsilon^{\frac{1}{2}} \left(1 - e^{-\frac{\pi^2}{4}t} \sin\frac{\pi(1+x)}{2}\right), \qquad x \in [-1,1],\ t \ge 0.$$

Lemma 2.2. $\bar{y}(x,t) := v(x,t) + \varphi(x) + c_0 t$ is an upper solution of (6) on $t \in [0,1]$, and hence

$$\bar{y}(x,t) \ge \mathcal{Y}^\varepsilon(x,t), \qquad x \in [-1,1],\ t \in [0,1]. \tag{7}$$

Sketch of the Proof. To prove the Lemma, it suffices to show that

$$\bar{y}_t \ge \frac{\bar{y}_{xx}}{1+\bar{y}_x^2} + A\sqrt{1+\bar{y}_x^2}, \qquad x \in [-1,1],\ t \ge 0, \tag{8}$$

and

$$\mathcal{Y}^\varepsilon(\pm 1, t) < \bar{y}(\pm 1, t), \qquad t \in [0,1]. \tag{9}$$

The inequality (8) can be easily verified by our construction. Now we show that

$$\mathcal{Y}^\varepsilon(-1, t) < \bar{y}(-1, t), \qquad t \in [0,1]. \tag{10}$$

The other inequality in (9) can be treated similarly.

Suppose that $\tilde{t} < 1$ and

$$\mathcal{Y}^\varepsilon(-1,t) < \bar{y}(-1,t), \qquad t \in [0, \tilde{t}].$$

Let $y_0 \in (0,1)$ be such that $g'(y_0) = \tan\alpha$ and $g(y_0) = \frac{\vartheta}{\varepsilon} = O(1)$. Let $\zeta(x)$ be an arc with curvature $-A$ and satisfying $\zeta(-1-\vartheta) = 0$, $\zeta'(-1-\vartheta) = \tan\alpha$. Then we have

$$\zeta(x) = -\frac{1}{A}\cos\alpha + \frac{1}{A}\sqrt{\cos^2\alpha + 2A\sin\alpha\cdot(1+\vartheta+x) - A^2(1+\vartheta+x)^2}.$$

Since

$$\varphi(-1+l\sqrt{\varepsilon}\,) = \tan\alpha\cdot l\sqrt{\varepsilon} + \left(\frac{c_0}{2\cos^2\alpha} - \frac{A}{2\cos^3\alpha}\right) l^2\varepsilon + \cdots,$$

for $l = \frac{l\pi\cos^2\alpha}{6c_0} e^{-\frac{\pi^2}{4}}$, we have

$$\zeta(-1+l\sqrt{\varepsilon}\,) = \tan\alpha\cdot(l\sqrt{\varepsilon}+\vartheta) - \frac{A}{2\cos^3\alpha}\,(l\sqrt{\varepsilon}+\vartheta)^2 + \cdots \geq \tan\alpha\cdot\vartheta + \varphi(-1+l\sqrt{\varepsilon}) - M\varepsilon$$

for small ε, where $M = \frac{l^2 c_0}{\cos^2\alpha}$.

Suppose that $\zeta(x) + H(\tilde{t})$ intersects $\bar{y}(x,\tilde{t})$ at $x = -1 + l\sqrt{\varepsilon}$ for some $H(\tilde{t})$, that is,

$$\zeta(-1+l\sqrt{\varepsilon}\,) + H(\tilde{t}) = \bar{y}(-1+l\sqrt{\varepsilon},\tilde{t}\,).$$

Then we obtain

$$\begin{aligned} H(\tilde{t}) &= v(-1+l\sqrt{\varepsilon},\tilde{t}\,) + \varphi(-1+l\sqrt{\varepsilon}\,) + c_0\tilde{t} - \zeta(-1+l\sqrt{\varepsilon}) \\ &\leq v(-1+l\sqrt{\varepsilon},\tilde{t}) - \tan\alpha\cdot\vartheta + M\varepsilon + c_0\tilde{t} \\ &\leq L\sqrt{\varepsilon} - L\frac{\pi l}{3}\, e^{-\frac{\pi^2}{4}}\varepsilon - \tan\alpha\cdot\vartheta + M\varepsilon + c_0\tilde{t} \\ &= \bar{y}(-1,\tilde{t}) - L\frac{\pi l}{3}\, e^{-\frac{\pi^2}{4}}\varepsilon - \tan\alpha\cdot\vartheta + M\varepsilon. \end{aligned}$$

On the other hand, there exists a $\delta \in [0,\varepsilon)$ such that the arc $\zeta(x) + H(\tilde{t}) + \delta$ intersects $\partial\Omega_\varepsilon$ at some point (x^*, y^*), where

$$x^* = -1 - \vartheta \qquad \text{and} \qquad g'_\varepsilon(y^*) = g'(y^*/\varepsilon) = \tan\alpha.$$

This implies that the arc $\zeta(x) + H(\tilde{t}) + \delta$ is a stationary curve of (2)-(4) on $[-1-\vartheta, -1+l\sqrt{\varepsilon}\,]$.

Since

$$\mathcal{Y}^\varepsilon(-1+l\sqrt{\varepsilon},\tilde{t}) \leq \bar{y}(-1+l\sqrt{\varepsilon},\tilde{t}) \leq \zeta(-1+l\sqrt{\varepsilon}) + H(\tilde{t}) + \delta,$$

we have $\mathcal{Y}^\varepsilon(x,\tilde{t}) \leq \zeta(x) + H(\tilde{t}) + \delta$ for $x \in [-1-\vartheta,\ -1+l\sqrt{\varepsilon}\,]$. Especially,

$$\begin{aligned} \mathcal{Y}^\varepsilon(-1,\tilde{t}) &\leq \zeta(-1) + H(\tilde{t}) + \delta \leq \tan\alpha\cdot\vartheta + H(\tilde{t}) + \varepsilon \\ &\leq \bar{y}(-1,\tilde{t}) + \left[M + 1 - L\frac{\pi l}{3}e^{-\frac{\pi^2}{4}}\right]\cdot\varepsilon \leq \bar{y}(-1,\tilde{t}) - 2\varepsilon \end{aligned}$$

by the choice of l and L. Therefore we have

$$\bar{y}(-1,\tilde{t}+t) \geq \bar{y}(-1,\tilde{t}) > \mathcal{Y}^\varepsilon(-1,\tilde{t}) + \varepsilon \geq \mathcal{Y}^\varepsilon(-1,\tilde{t}+t), \qquad t \in [0,T_\varepsilon].$$

This means that $\mathcal{Y}^\varepsilon(-1,t) < \bar{y}(-1,t)$ on $t \in [0,\tilde{t}+T_\varepsilon]$.

Repeating the above discussion finite times, we get (10). This proves the Lemma.

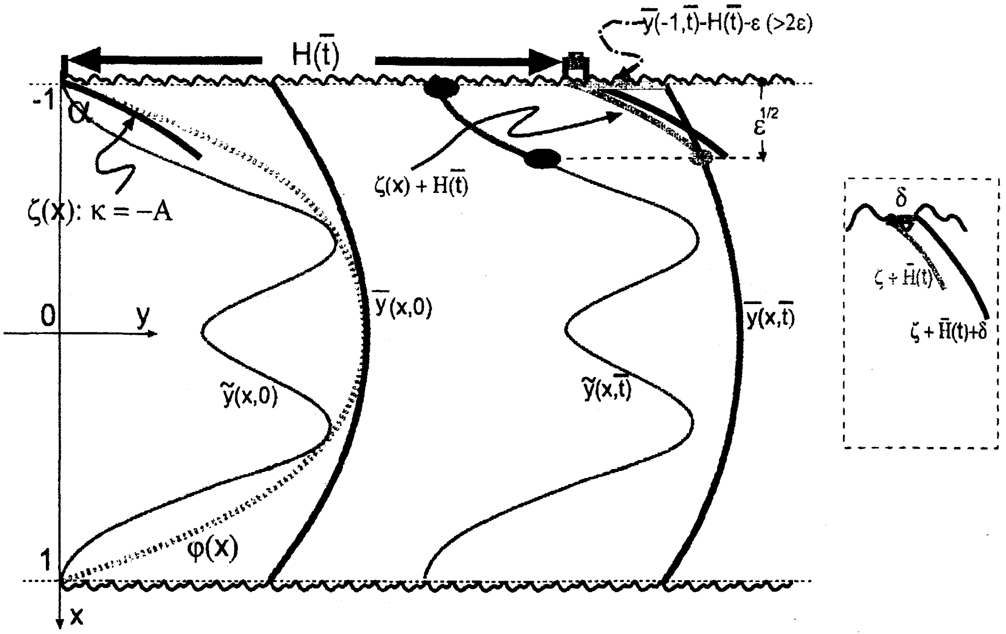

Fig. 3 Upper Solution

Proof of Main Theorem. By Lemma 2.1 we only need the upper bound of c_ε. Denote by $[\chi]$ the integer part of $\chi > 0$. By Lemma 2.2 we have

$$\mathcal{Y}^\varepsilon(x,1) - \varphi(x) \le \bar{y}(x,1) - \varphi(x) = v(x,1) + c_0 \le \left[\frac{L\varepsilon^{\frac{1}{2}} + c_0}{\varepsilon} + 1 \right] \cdot \varepsilon.$$

On the other hand,

$$\mathcal{Y}^\varepsilon \left(x, \left[\frac{L\varepsilon^{\frac{1}{2}} + c_0}{\varepsilon} + 1 \right] \cdot T_\varepsilon \right) \le \varphi(x) + \left[\frac{L\varepsilon^{\frac{1}{2}} + c_0}{\varepsilon} + 1 \right] \cdot \varepsilon,$$

and "equality" holds at some $x_0 \in [-1,1]$. Therefore we obtain

$$1 \le \left[\frac{L\varepsilon^{\frac{1}{2}} + c_0}{\varepsilon} + 1 \right] \cdot T_\varepsilon \le \left(\frac{L\varepsilon^{\frac{1}{2}} + c_0}{\varepsilon} + 1 \right) \cdot T_\varepsilon,$$

and hence

$$c_\varepsilon = \frac{\varepsilon}{T_\varepsilon} \le c_0 + L\varepsilon^{\frac{1}{2}} + \varepsilon$$

This proves (5).

Statement (ii) follows from the comparison theorem, standard parabolic estimates and (5).

Remark 2. To give a scent to the readers for the relation between c_0 and A, as well as that between c_0 and α, we consider the problem in a band domain with ratchet boundaries (see Figure 4).

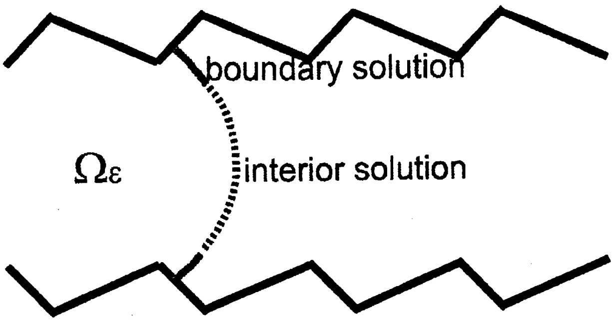

Fig.4 Band domain with ratchet boundaries

We divide the traveling wave into two parts: the part near the boundaries (we call it *boundary solution*), and the part away from the boundaries (we call it *interior solution*). Via a rather intriguing asymptotic expansion approach, we find that the interior solution is approximately a traveling wave with constant speed and profile, while the behavior of the boundary solution is complex. In one period the motion of the boundary solution consists of *three* stages (see Figure 5).

Stage 1 – Contact points (where the solution curves contact with the boundary) are on PQ. In this stage the profile of the solution is like φ and the propagation speed is of order $O(1)$.

Stage 2 – Contact points are on QR. In this stage, the contact point $\gamma(t)$ moves rapidly from Q to R in a short time $O(\varepsilon^2)$, while the interior solution almost remains stationary.

Stage 3 – Contact points stay at R. In this stage the propagation speed of the boundary solution varies from of order $O(\frac{1}{\varepsilon})$ to of order $O(1)$.

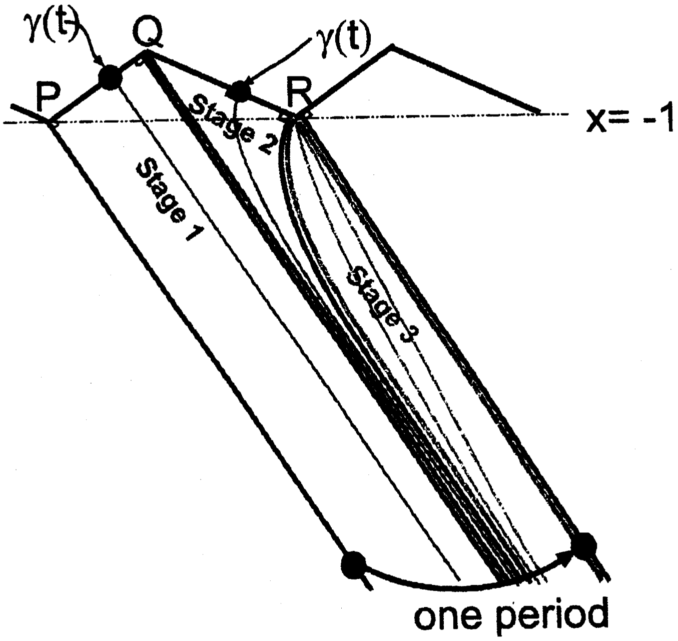

Fig.5 Three Stages of Boundary Behavior

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