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COHOMOLOGY OF GROUPS HAVING 3-SYLOW SUBGROUPS ISOMORPHIC TO THE EXTRASPECIAL 3-GROUP

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1. Introduction

Let us write by E the extraspecial p-group p_+^{1+2} of order p and exponent p for an odd prime p. Let G be a finite group having E as a Sylow p-group, and let BG $(= BP_p^{\wedge})$ the p-completed classifying space of G. In papers [T-Y],[Y1,2], the cohomology and stable splitting for such groups are studied. Results show that there are not so many homotopy types of BG. This fact was first suggested by C.B.Thomas [Th] and D.Green [G].

Indeed, recently, Ruiz and Viruel [R-V] classified all p-local finite groups. Classifying spaces induced from finite groups are homotopic to what are studied in [T-Y]. (While description of $H^*({}^2F_4(2)')_{(3)}$ and $H^*(Fi_{24}')_{(7)}$ contained some errors.)

2. p-local finite groups associated to the extraspecial p group P_+^{1+3}

Recall that the extraspecial p-group p_+^{1+2} has a representation as

$$p_+^{1+2} =< a, b, c | a^p = b^p = c^p = 1, [a, b] = c,\ c \in Center >$$

and denote it simply by E in this paper. We consider p-local finite groups associated to E, which are extensions of groups whose p-Sylow subgroups are isomorphic to E.

The concept of the p-local finite groups arose in the work of Broto-Levi-Oliver [B-L-O 1,2] as a classical concept of finite groups. The p-local finite group is stated as a triple $< S, F, L >$ where S is a p-group, F is a saturated fusion system over a centric linking system L over S (for details see [R-V],[B-L-O 1]). Given a p-local finite group, we can construct its classifying space $B < S, F, L >$ by the realization $|L|_p^{\wedge}$. Of course if $< S, F, L >$ is induced from a finite group G having S as a p-Sylow subgroup, then $B < S, F, L > \cong BG$. However note that in general, there exist p-local finite groups which are not induced from finite groups (exotic cases).

A.Ruiz and A.Viruel recently determined $< p_+^{1+2}, F, L >$ for all odd primes by using the classification of finite simple groups. Indeed, we check the possiblity of existence of finite groups only for simple groups and their extensions. Moreover they find new exotic 7-local finite groups.

The p-local finite groups $< E, F, L >$ are classified by $Out_F(E)$, number of F^{ec}-radical p-subgroup V (for details of definitions, see [R-V]) and $Aut_F(V)$. When a p-local finite group are induced from finite group G, then we see easily that $Out_F(E) \cong W_G(E)(= N_G(E)/E.C_G(E))$ and $Aut_F(V) \cong W_G(V)$. Moreover V is

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F^{ec}-radical if and only if $Aut_F(V) \subset SL_2(\mathbb{F}_p)$ by Lemma 4.1 in [R-V]. When G is a sporadic simple group, V is F^{ec}-radical follows p-pure.

Theorem 2.1. *(Ruiz and Viuel [R-V]) If $p \neq 3,7,5,13$, then p-local finite group $< E, F, L >$ is isomorphic to one of the following (1),(2)*

$$(1) \quad E : W \quad for\ W \subset Out(E) \quad and \quad (|W|, p) = 1$$

$$(2) \quad p^2 : SL_2(\mathbb{F}_p).r \quad for\ r|(p-1), \qquad SL_3(\mathbb{F}_p) : H \quad for\ H \cong \mathbb{Z}/2, \mathbb{Z}/3, S_3.$$

When $p = 3,5,7$ or 13 there are other types

$$(3) \quad {}^2F_4(2)', J_4\ for\ p=3, \quad Th\ for\ p=5, \quad M\ for\ p=13$$

$$(4) \quad He, He:2, Fi'_{24}, Fi_{24}, O'N, O'N:2\ for\ p=7$$

and there exist three exotic 7-local finite groups.

For cases (1), we know that $H^*(E:W) \cong H^*(E)^W$. Except for these extensions and exotic cases, all $H^{even}(G;\mathbb{Z})_{(p)}$ are studied in [T-Y]. It is studied in [Y1] the way to know $H^{odd}(G;\mathbb{Z})_{(p)}$ and $H^*(G;\mathbb{Z}/p)$ from $H^{even}(G;\mathbb{Z})_{(p)}$. The stable splittings for such BG are studied in [Y2]. However there were some error in cohomology of ${}^2F_4(2)', Fi'_{24}$. in this paper, we study cohomology and stable splitting of BG for $p=3$ and 7 mainly .

3. COHOMOLOGY

In this paper we mainly consider the cohomology $H^*(BG;\mathbb{Z})/(p,\sqrt{0})$ where $\sqrt{0}$ is the ideal generated by nilpotent elemet. So we write it simply

$$H^*(BG) = H^*(BG;\mathbb{Z})/(p,\sqrt{0}).$$

Hence it is written by

$$H^*(B\mathbb{Z}/p) \cong \mathbb{Z}/p[y], \quad H^*(B(\mathbb{Z}/p)^2) \cong \mathbb{Z}/p[y_1,y_2] \quad with\ |y| = |y_i| = 2.$$

Let us write $(\mathbb{Z}/p)^2$ by V simply, and a V-subgroup of G mean a subgroup isomorphic to $(\mathbb{Z}/p)^2$.

The cohomology of the extraspecial p group $E = p_+^{1+2}$ is wellknown. In particular recall that

$$(1) \quad H^*(BE) \cong (\mathbb{Z}/p[y_1,y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p[C]) \otimes \mathbb{Z}/p[v]$$

where $|y_i| = 2, |v| = 2p, |C| = 2p-2$ and $Cy_i = y_i^{p-1}$, $C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1}y_2^{p-1}$. The E conjugacy classes of V-subgroups of E are written by

$$A_i = < ab^i, c > \ for\ 0 \le i \le p-1, \quad A_\infty = < b, c > .$$

Letting $H^*(BA_i) \cong \mathbb{Z}/p[y,u]$ and writing $i_{A_i}^*(x) = x|A_i$ for the inclusion $i_{A_i} : A_i \subset E$, the restriction images are given by

$$(2) \quad y_1|A_i = y\ for\ i \in \mathbb{F}_p,\ y_1|A_\infty = 0, \quad y_2|A_i = iy\ for\ i \in \mathbb{F}_p,\ y_2|A_\infty = y,$$

$$C|A_i = y^{p-1}, \quad v|A_i = u^p - y^{p-1}u \quad for\ all\ i.$$

In particular, we can take a $\mathbb{Z}/p$-base b_i of $H^{2p-2}(BE)$ such that

$$b_i|A_j = \delta_{ij}y^{p-1} \quad , e.g.\ b_i = (y_1 - i^{-1}y_2)^{p-1} - C.$$

For an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_p) \cong Out(E)$, its actions are given ([Ly],[T-Y] p491)

$$(3) \quad g^*C = C,\ g^*y_1 = ay_1 + by_2,\ g^*y_2 = cy_1 + dy_2,\ g^*v = (det(g))v,$$

Let us write as $Y_i = y_i^{p-1}$ and $V = v^{p-1}$. Then we get the following additive expression of $H^*(BE)$, which is used in Section .

$$(4) \quad H^*(BE) \cong (\oplus_{i=1}^2 \mathbb{Z}/p[Y_i]\{y_i, ..., y_i^{p-2}, Y_i\}$$

$$\oplus \mathbb{Z}/p[C]\{y_1^i y_2^j | 1 \le i, j \le p-1\} \oplus \mathbb{Z}/p\{1, C\}) \otimes \mathbb{Z}/p[V]\{1, v, ..., v^{p-2}\}.$$

Theorem 3.1. *(Theorem 4.3 in [T-Y], [B-L-O]) Let G have the p-Sylow subgroup E, Then we have the isomorphism*

$$H^*(BG) \cong H^*(BE)^{Out(G)} \cap_{A:F^{ec}-radical} i_A^{*-1} H^*(BA)^{W_G(A)}.$$

4. STABLE SPLITTING

Martino-Priddy prove the following theorem of complete stable splitting

Theorem 4.1. *(Martino-Priddy [M-P]) Let G be a finite group with a p-Sylow subgroup P. The complete splitting of BG is given by*

$$BG \sim \vee rank_p A(Q, M) X_M$$

where indecomposable summands X_M range over isomorphic classes of simple $\mathbb{F}_p[Out(Q]$-modules M and over isomorphism classes of subgroups $Q \subset P$.

For the definition of $A(Q, M)$, see [M-P]. In particular, when Q is not subretract (that is, not a proper retract of a subgroup) of P and when $W_G(Q) \subset Out(Q) \cong GL_n(\mathbb{F}_p)$, the rank of $A(Q, M)$ is compute by

$$rank_p A(Q, M) = \sum dim(\bar{W}_G(Q_i)M)$$

where $\bar{W}_G(Q) = \sum_{x \in W_G(Q)} x$ in $\mathbb{F}_p[GL_n(F_p)]$ and Q_i ranges over representatives of G-conjugacy classes of subgroups isomorphic to Q (see [M-P] Corollary 4.4).

Recall that $Out(E) \cong Out(V = (\mathbb{Z}/p)^2) \cong GL_2(\mathbb{F}_p)$. The simple modules of $G = GL_2(\mathbb{F}_p)$ is wellknown. Let us think V be the natural two-dimensional representation, and det the determinant representation of G. Then there are $p(p-1)$ simple $\mathbb{F}_p[G]$-modules given by $M_{q,k} = S(V)^q \otimes (det)^k$ for $0 \le q \le p-1, 0 \le k \le p-2$. Harris-Kuhn determined the stable splitting of abelian p-groups. In particular, they showed

Theorem 4.2. *(Harris-Kuhn) Let $\tilde{X}_{q,k} = X_{M_{q,k}}$ (resp. $L(1,k)$) identifying $M_{q,k} \in \mathbb{F}_p[V]$ (resp $M_{0,k} \in \mathbb{F}_p[\mathbb{Z}/p]$). There is the complete stable splitting*

$$BV \sim \vee_{q,k} (q+1)\tilde{X}_{q,k} \vee_{q \ne 0} (q+1)L(1,q)$$

where $0 \le q \le p-1$,$0 \le k \le p-2$

The summand $L(1, p-1)$ is usually written by $L(1, 0)$.

It is also known $H^+(L(1,k)) \cong \mathbb{Z}/p[y^{p-1}]\{y^k\}$. Since we have the isomorphism

$$H^{2k}(BV) \cong (\mathbb{Z}/p)^{k+1} \cong H^{2k}((k+1)L(1,k)),$$

we also know if $* \le 2(p-1)$, then $H^*(\tilde{X}_{q,k}) \cong 0$.

Lemma 4.3. *Let $H \subset GL_2(\mathbb{F}_p)$ with $(|H|,p)=1$ and let $G=V:H$. Let us write by $BG \sim \vee_{q,k}\tilde{n}(H)_{q,k}\tilde{X}_{q,k} \vee_{q'} \tilde{m}(H)_{q'}L(1,q')$. Then*

$$\tilde{n}(H)_{q,k} = rank_p H^0(H; M_{q,k}), \quad \tilde{m}(H)_{q'} = rank_p H^{2q'}(BG).$$

In particular $\tilde{n}(H)_{q,0} = rank_p H^{2q}(BG)$.

Proof. Since $H^*(\tilde{X}_{q,k}) \cong 0$ for $* \le 2(p-1)$, it is immediate that $\tilde{m}(H)_{q'} = rank_p H^{2q'}(G)$. To prove the first equation, assume that $H =< x > \cong \mathbb{Z}/s$. Then for $v \in M_{q,k}$,

$$\bar{W}_G(V)v = (1 + (detx)^k x + ... + (detx)^{k(s-1)}x^{s-1})v.$$

Since $(1-(det(x)^k x)^s) = 0$, we get $Ker(1-(detx)^k x) \subset Image(\bar{W}_G(V)$. Since $M_{q,k}$ is a $\mathbb{Z}/p$-module and $(|H|,p)=1$, we know $H^*(H; M_{q,k}) = 0$ for $* > 0$. Hence we get

$$Ker(1-(detx)^k x)/Image(1 + (detx)^k x + ... + (detx)^{k(s-1)}x^{s-1}) \cong H^1(H, M_{q,k}) \cong 0.$$

Thus we have

$$\tilde{m}_{q,k}(H) = rank_p M_{q,k}^H = rank_p H^0(H, M_{q,k}).$$

Next let $1 \to H' \to H \to \mathbb{Z}/s \to 0$. By induction, we assume $Image(\bar{H}') = H^0(H'; M_{q,k})$. Then we can see

$$Image(\bar{H}) = H^0(\mathbb{Z}/s; H^0(H'; M_{q,k})) \cong M_{q,k}^H \cong H^0(H; M_{q,k}).$$

□

Next consider the stable splitting for the extraspecial p-group E.
Diez-Priddy prove the following theorem.

Theorem 4.4. *(Diez-Priddy [D-P]) Let $X_{q,k} = X_{M_{q,k}}$ (resp. $L(2,k)$, $L(1,k)$) identifying $M_{q,k}$ (resp. $M_{p-1,k}$, $\in \mathbb{F}_p[V]$, $\mathbb{F}_p[\mathbb{Z}/p]$). There is the complete stable splitting*

$$BE \sim \vee_{q,k}(q+1)X_{q,k} \vee_k (p+1)L(2,k) \vee_{q\neq 0} (q+1)L(1,q) \vee L(1,p-1)$$

where $0 \le q \le p-1$, $0 \le k \le p-2$.

Remark. Of course $\tilde{X}_{q,k}$ is different from $X_{q,k}$ but $\tilde{X}_{p-1,k} = L(2,k)$.
Recall that

$$H^{2q}(BE) \cong \begin{cases} (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q+1)L(1,q)) & for\ 0 \le k \le p-1 \\ (\mathbb{Z}/p)^{q+2} \cong H^{2p-2}((p+1)L(1,0)) & for\ q = p-1 \end{cases}$$

This showes $H^*(X_{q,k}) \cong 0$ for $* \le 2p-2$ since so is $L(2,k)$. The number $n(G)_{q,k}$ is only depend on $Out(G) = H$. Hence we have the following corollary.

Corollary 4.5. *Let G have the p-Sylow subgroup E and $Out(G) = H$. Let*

$$BG \sim \vee n(G)_{q,k}X_{q,k} \vee m(G,2)_k L(2,k) \vee m(G,1)_k L(1,k).$$

Then $n(G)_{q,k} = \tilde{n}(H)_{q,k}$ and $m(G,1)_k = rank_p H^{2k}(G)$.

Hence next problem is to seek $m(G,2)$. The number $p+1$ for the summand $L(2,k)$ in BE is given as follows. Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $GL_2(\mathbb{F}_p)$ and $U =< u >$: the maximal unipotent subroup. Then for each E-conjugacy class of V-subgroup

$A_i = < ab^i, c >, i \in \mathbb{F}_p \cup \infty$, we see $W_E(A_i) \cong U$. For $y_1^s y_2^l \in M_{q,k}$ (identifying $H^*(BV) \cong S^*(V) = \mathbb{Z}/p[y_1, y_2]$), we can compute

$$\bar{W}_G(A) y_1^s y_2^l = (1 + u + \ldots + u^{p-1}) y_1^s y_2^l = \sum_i (y_1 + i y_2)^s y_2^l$$

$$= \sum_i \sum_s (s,t) i^t y_1^{s-t} y_2^t y_2^l = \sum_s (s,t) \sum_i i^t y_1^{s-t} y_2^{t+l}.$$

Here $\sum_{i=0}^{p-1} i^t = 0$ for $1 \leq t \leq p-2$, and $= -1$ for $t = p-1$. Hence we know

$$dim_p \bar{W}_G(A_i) M_{q,k} = 0 \ for \ 1 \leq q \leq p-2, \quad = 1 \ for \ q = p-1.$$

Thus we know that BE has just one $L(2,k)$ for each E-conjugacy V-subgroup A_i.

Lemma 4.6. *Let A be a F^{ec}-radical subgroup, i.e., $W_G(A) \supset SL_2(\mathbb{F}_p)$. Then $\bar{W}_G(A)(M_{q,k}) = 0$ for all k and $1 \leq q \leq p-1$.*

Proof. The group $SL_2(\mathbb{F}_p)$ is generated by $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We know

$$Ker(1-u) \cong \mathbb{Z}/p[y_1^p - y_2^{p-1} y_1, y_2] \quad and \quad Ker(1-u') \cong \mathbb{Z}/p[y_2^p - y_1^{p-1} y_2, y_1]$$

identifying $S(V)^* \cong \mathbb{Z}/p[y_1, y_2]$. Hence we get $(Ker(1-u) \cap Ker(1-u'))^* \cong 0$ for $0 < * \leq p-1$. □

Proposition 4.7. *Let G has the p-Sylow subgroup E. The number of $L(2,0)$ in BG is given by*

$$m(G,2)_0 = \sharp_G(V) - \sharp_G(F^{ec}V)$$

where $\sharp_G(V)$(resp.$\sharp_G(F^{ec}V)$) is the number of G-conjugacy class of V-subgroups (resp. F^{ec}-radical subroups.)

Proof. We recall

$$H^*(BG) \cong H^*(BE)^{Out(G)} \cap_{A:F^{ec}-radical} i_A^{*-1} H^*(BA)^{W_G(A)}.$$

Let us write $K = E : Out(G)$ and $H^*(BE)^{Out(G)} = H^*(BK)$. From the above lemma, we only need to show

$$m(K,2)_0 = \sharp_K(V) = \sharp_G(V).$$

Let A be a V-subgroup of K and $x \in W_K(A)$. Recall $A = < ab^i, c >$ for some i. Identifying x as an element in $Aut(E)$, we see $x < c > = < c >$ since $< c >$ is the center of E. Hence

$$W_K(A) \subset B = U : (\mathbb{F}_p^*)^2 \ ; \ the \ Borel \ subgroup.$$

So we easily see that $\bar{W}_K(y_1^{p-1}) = \lambda y_2^{p-1}$ for some $\lambda \neq 0$ because $b^* y_2^{p-1} = y_2^{p-1}$ for $b = diagonal \in (\mathbb{F}_p)^{*2}$. That means $m(G,2)_0 = \sharp_K(V) - \sharp_G(F^{ec}V)$.

On the otherhand $m(G,2)_k \leq \sharp_G(V) - \sharp_G(F^{ec}V)$ from the above lemma. Since $\sharp_K(V) \geq \sharp_G(V)$, we see that $\sharp_K(V) = \sharp_G(V)$ and the proposition. □

Lemma 4.8. *Let $\xi \in \mathbb{F}_p^*$ and $(3k, p-1) \neq 1$. If $G \supset E :< diag(\xi, \xi) >$, then BG does not contain the summand $L(2,k)$, i.e., $m(G,2)_k = 0$.*

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Proof. It is sufficient to prove the case $G = E :< diag(\xi,\xi) >$. Let $G = E :< diag(\xi,\xi) >$. Recal $A_i =< ab^i, c >$ and

$$diad(\xi,\xi) : ab^i \mapsto (ab^i)^\xi, \quad c \mapsto c^{\xi^2}.$$

So the Weyle group is $W_G(A_i) = U :< diag(\xi,\xi^2) >$. For $v \in M_{q,k}$, we have

$$\bar{W}_G(A_i)v = \sum_{i=0}^{p-2} (\xi^{3i})^k diag(\xi^i,\xi^{2i})(1+ \ldots + u^{p-1})v.$$

Thus we get the lemma from $\sum_{i=0}^{p-2} \xi^{3ik} = 0$ for $3k \neq 0 \ mod(p-1)$ and $= -1$ otherwise. □

5. COHOMOLOGY AND SPLITTING OF $B(\mathbb{Z}/3)^2)$

In this section, we study the cohomology and stable splitting of BG for G having a 3-Sylow subggroup $(\mathbb{Z}/3)^2 = V$. In this and next sections, p always means 3. Recall $Out(V) \cong GL_2(\mathbb{F}_3)$ and $Out(V)'$ consists the semidihedral group

$$SD_{16} =< x, y | x^8 = y^2 = 1, yxy^{-1} = x^3 > .$$

Each group G having 3-Sylow subgroup V is isomorphic to $E : W$, $W \subset SD_{16}$. There is the SD_{16}-conjugacy calsses of subgroups(here $A \longleftarrow B$ means $A \supset B$)

$$SD_{16} \begin{cases} \longleftarrow Q_8 \longleftarrow \mathbb{Z}/4 \\ \longleftarrow \mathbb{Z}/8 \longleftarrow \mathbb{Z}/4 \longleftarrow bZ/2 \longleftarrow 0 \\ \longleftarrow D_8 \longleftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longleftarrow \mathbb{Z}/2 \end{cases}$$

We can take generators pf subgroups in $GL = 2(\mathbb{F}_3)$ by the matrices

$$\mathbb{Z}/8 =< l >, \ Q_8 =< w, k >, \ D_8 =< w', k >, \ \mathbb{Z}/4 =< w >$$

$$\mathbb{Z}/4 =< k >, \ \mathbb{Z}/2 \oplus \mathbb{Z}/2 =< w', c >, \ \mathbb{Z}/2 =< c >, \ \mathbb{Z}/2 =< w' >$$

$$where \quad l = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$k = l^2 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, w' = wl = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \ c = l^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here we note that k and w is $GL_2(\mathbb{F}_3)$-conjugate, infact $uku^{-1} = w$.

The cohomology of V is given $H^*(BV) \cong \mathbb{Z}/3[y_1, y_2]$, and the following are immediately

$$H^*(BV)^{<c>} \cong \mathbb{Z}/3[y_1^2, y_2^2]\{1, y_1y_2\} \quad H^*(BV)^{<w'>} \cong \mathbb{Z}/3[y_1 + y_2, y_2^2].$$

Let us write $Y_i = y_i^2$ and $t = y_1y_2$. The k-action is given

$$k : Y_1 \mapsto Y_1 + Y_2 + t, \ Y_2 \mapsto Y_1 + Y_2 - t, \ t \mapsto -Y_1 + Y_2.$$

So the following are invariant

$$a = -Y_1 + Y_2 + t, \ a_1 = Y_1(Y_1 + Y_2 + t), \ a_2 = Y_2(Y_1 + Y_2 - t), \ b = t(Y_1 - Y_2).$$

here we note that $a^2 = a_1 + a_2$, $b^2 = a_1a_2$.. We can prove the invariant ring is

$$H^*(BV)^{<k>} \cong \mathbb{Z}/3[a_1, a_2]\{1, a, b, ab\}.$$

Next consider the invariant under $Q_8 =< w, k >$. The action for w is $a \mapsto -a,\ a_1 \leftrightarrow a_2,\ b \mapsto b$. Hence we get

$$H^*(BV)^{Q_8} \cong \mathbb{Z}/3[a_1 + a_2, a_1a_2]\{1, b\}\{1, (a_1 - a_2)a\}.$$

Let us write $S = \mathbb{Z}/3[a_1 + a_2, a_1a_2]$ and $a' = (a_1 - a_2)a$. The action for l is given $l : Y_1 \mapsto Y_2 \mapsto Y_1 + Y_2 + t \mapsto Y_1 + Y_2 - t \mapsto Y_1$. Hence $l : a \mapsto -a,\ a_1 \leftrightarrow a_2,\ b \mapsto -b$. Thefore we get

$$H^*(BV)^{<l>} \cong S\{1, a', ab, (a_1 - a_2)b\}.$$

The action for $w' : Y_1 \mapsto Y_1 + Y_2 + t,\ Y_2 \mapsto Y_2$, implies that $w' : a \mapsto a,\ a_i \mapsto a_i,\ b \mapsto -b$. then we can see

$$H^*(BV)^{D_8} = H^*(BV)^{<k,w'>} \cong \mathbb{Z}/3[a_1, a_2]\{1, a\} \cong S\{1, a, a_1, a'\}$$

Hence we also have

$$H^*(BV)^{SD_{16}} \cong S\{1, a'\}.$$

Recall the Dickson algebra $DA = \mathbb{Z}/3[\tilde{D}_1, \tilde{D}_2] \cong H^*(BV)^{GL_2(\mathbb{F}_3)}$ where

$$\tilde{D}_1 = Y_1^3 + Y_1^2Y_2 + Y_1Y_2^2 + Y_2^3 = (a_2 - a_1)a = a', \quad \tilde{D}_2 = (y_1^3y_2 - y_1y_2^3)^2 = a_1a_2.$$

Using $a^2 = (a_1 + a_2)$ and $\tilde{D}_1^2 = a^6 - a_1a_2a^2$, we can write

$$H^*(BV)^{SD_{16}} \cong \mathbb{Z}/3[a^2, \tilde{D}_2]\{1, \tilde{D}_1\} \cong DA\{1, a^2, a^4\}.$$

Theorem 5.1. *Let $G = (\mathbb{Z}/3)^2 : H$ for $H \subset SD_{16}$. Then BG has the following stable splitting*

$$\overset{\tilde{X}_{0,0}}{\longleftarrow} SD_{16} \left\{ \begin{tikzcd}
\overset{\tilde{X}_{0,1}}{\longleftarrow} Q_8 & & & & & & \\
\overset{\tilde{X}_{2,1}}{\longleftarrow} \mathbb{Z}/8 & & \mathbb{Z}/4 \arrow[ll, "\tilde{X}_{2,0}\vee\tilde{X}_{0,1}\vee L(1,0)"'] \arrow[ull] \arrow[dl] & & \mathbb{Z}/2 \arrow[ll, "2\tilde{X}_{2,0}\vee 2\tilde{X}_{2,1}\vee 2L(1,0)"'] \arrow[dl] & & 0 \arrow[ll, "2\tilde{X}_{1,0}\vee 2\tilde{X}_{1,1}\vee 2L(1,1)"'] \arrow[dl] \\
& \overset{\tilde{X}_{2,0}\vee L(1,0)}{\longleftarrow} D_8 & & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[ll, "\tilde{X}_{2,0}\vee\tilde{X}_{2,1}\vee L(1,0)"'] & & \mathbb{Z}/2 \arrow[ll, "\tilde{X}_{1,0}\vee\tilde{X}_{1,1}\vee L(1,1)"'] &
\end{tikzcd} \right.$$

where $\overset{X_1}{\leftarrow} \ldots. \overset{X_s}{\leftarrow} H$ *means* $B((\mathbb{Z}/3)^2 : H) \sim X_1 \vee \ldots \vee X_s$.

Main parts of the above splittings are given in (6) in [Y2] by direct computations of $\bar{W}_G(V)$ (see p149 in [Y2]). However we get the theorem more easily by using cohomology here. For example, let us consider the case $G = V :< k >$. The cohomology

$$H^0(BG) \cong \mathbb{Z}/3,\ H^2(BG) \cong 0, H^4(BG) \cong \mathbb{Z}/3$$

implies that BG contains just one $\tilde{X}_{0,0}, \tilde{X}_{2,0}, L(1,0)$ but does not $\tilde{X}_{1,0}, L(1,1)$. Since $det(k) = 1$, we also know that $\tilde{X}_{0,1}, \tilde{X}_{2,1}$ are contained. So we can see

$$B(V : \mathbb{Z}/4) \sim \tilde{X}_{0,0} \vee \tilde{X}_{0,1} \vee \tilde{X}_{2,0} \vee \tilde{X}_{2,1} \vee L(1,0).$$

Next consider the case $G' = V :< l >$. The fact $H^4(G) \cong 0$ implies that BG' does not contain $\tilde{X}_{2,0}, L(1,0)$. The determinant $det(l) = -1$, and $l : a \mapsto -a$ shows that BG' contains $X_{2,1}$ but does not $\tilde{X}_{0,1}$. Hence we know $BG' \sim \tilde{X}_{0,0} \vee \tilde{X}_{2,1}$. Moreover we know $BV : SD_{16} \sim \tilde{X}_{0,0}$ since $w : a \to -a$ but $det(w) = 1$. Thus we have the graph

$$\overset{\tilde{X}_{0,0}}{\longleftarrow} SD_{16} \overset{\tilde{X}_{2,1}}{\longleftarrow} \mathbb{Z}/8 \overset{\tilde{X}_{2,0}\vee\tilde{X}_{0,1}\vee L(1,0)}{\longleftarrow} \mathbb{Z}/4$$

Similarly we get the other parts of the above graph.

Corollary 5.2. *Let $S = \mathbb{Z}/3[a_1 + a_2, a_1a_2]$, We have the isomorphisms*

$$H^*(\tilde{X}_{0,0}) \cong S\{1, \tilde{D}_1\}, \quad H^*(\tilde{X}_{0,1}) \cong S\{b, \tilde{D}_1 b\},$$

$$H^*(\tilde{X}_{2,1}) \cong S\{ab, (a_1 - a_2)b\}, \quad H^*(\tilde{X}_{2,0} \vee L(1,0)) \cong S\{a, a_1 - a_2\} \cong DA\{a, a^2, a^3\}.$$

Here we write down the decomposition of cohomology for most typical case

$$H^*(BV)^{<k>} \cong S\{1, a_1 - a_2\}\{1, a\}\{1, b\}$$

$$\cong S\{1, a(a_1 - a_2),\ b, b(a_1 - a_2),\ ab, (a_1 - a_2)b,\ a, (a_1 - a_2)\}$$

$$\cong H^*(\tilde{X}_{0,0}) \oplus H^*(\tilde{X}_{0,1}) \oplus H^*(\tilde{X}_{2,1}) \oplus H^*(\tilde{X}_{2,0} \vee L(1,0)).$$

6. COHOMOLOGY AND SPLITTING OF $B3_+^{1+2}$.

In this section we study the cohomology and stable splitting of BG for G having a 3-Sylow subgroup $E = 3_+^{1+2}$. In the splitting for BE, the summands $X_{q,k}$ are called dominant summands. Moreover the summands $L(2,0) \vee L(1,0)$ is usually written by $M(2)$. The following lemma is almost immediately from Proposition 4.7 and Lemma 4.8.

Lemma 6.1. *If $G \supset E :< diag(-1,-1) >$ idefining $Out(E) \cong GL_2(\mathbb{F}_3)$ and G has E as a 3-Sylow subgroup, then*

$$BG \sim (\textit{dominant summand}) \vee (\sharp_G(V) - \sharp_G(F^{ec}V)(L(2,0) \vee L(1,0)).$$

Theorem 6.2. *If G has a 3-Sylow subgroup E, then BG is homotopic to the classifying space of one of the following groups. Moreover the stable splitting is given by the graph so that $\xleftarrow{X_1} \dots \xleftarrow{X_i} G$ means $BG \sim X_1 \vee \dots \vee X_i$ and $EH = E : H$ for $H \subset SD_{16}$*

$$
\xleftarrow{X_{0,0}} J_4 \left\{
\begin{array}{l}
\xleftarrow{M(2)} ESD_{16} \left\{
\begin{array}{l}
\xleftarrow{X_{0,1}} EQ_8 \\
\xleftarrow{X_{2,1}} E\mathbb{Z}/8 \xleftarrow{X_{2,0}\vee X_{0,1}\vee M(2)} E\mathbb{Z}/4 \xleftarrow{\substack{2X_{2,0}\vee 2X_{2,1}\\ \vee 2M(2)}} E\mathbb{Z}/2 \xleftarrow{\substack{2X_{1,0}\vee 2X_{1,1}\\ \vee 4L(2,1)\vee 2L(1,1)}} E \\
\xleftarrow{X_{2,0}\vee M(2)} ED_8 \xleftarrow{X_{2,0}\vee X_{2,1}\vee M(2)} \qquad E(\mathbb{Z}/2)^2 \xleftarrow{\substack{X_{1,0}\vee X_{1,1}\\ \vee 2L(2,1)\vee L(1,1)}} E\mathbb{Z}/2
\end{array}\right. \\
\xleftarrow{X_{2,0}} {}^2F_4(2)' \xleftarrow{M(2)} M_{24} \xleftarrow{X_{2,0}\vee X_{2,1}} M_{12} \xleftarrow{M(2)} V : GL_2(\mathbb{F}_3)
\end{array}\right.
$$

Proof. All groups except for E and $E :< w' >$ contain $E :< diag(-1,-1) >$. Hence we get the theorem from Corollary 4.5, Theorem 5.1 and Lemma 6.1, except for the place for $H^*(BE :< w' >)$. Let $G = E :< w' >$. Note $w' : y_1 \mapsto y_1 - y_2, y_2 \mapsto -y_2, v \mapsto -v$. Hence $H^2(G) \cong \mathbb{Z}/3[y_1 + y_2\}$. So BG contains one $L(1,1)$. Next consider the number of $L(2,0), L(2,1)$. The G-conjugacy classes of V-subgroups are $A_0 \sim A_2, A_1, A_\infty$. The weyle groups are

$$W_G(A_0) \cong U, \quad W_G(A_1) \cong U :< diag(1,-1) >, \quad W_G(A_\infty) \cong U :< diag(-1,-1) > .$$

By the arguments similar to the proof of Lemma 4.8, we have that

$$\begin{cases} dim(\bar{W}_G(A_i)M_{2,0}) = 1 \quad for\ all i \\ dim(\bar{W}_G(A_i)M_{2,0}) = 1,0,1 \quad for\ i = 0,1,\infty. \end{cases}$$

Thus we know $BG \supset 3L(2,0) \vee 2L(2,1)$.

□

We write down the cohomologies explicitely (see also [T-Y],[Y2]). First we compute $H^*(B(E:H))$. The following cohomologies are easily computed

$$H^*(BE)^{<c>} \cong \mathbb{Z}/3[Y_1] \oplus \mathbb{Z}/3[Y_2]^+ \oplus \mathbb{Z}/3[C]\{1, y_1y_2\}) \otimes \mathbb{Z}/3[v]$$

$$H^*(BE)^{<w>} \cong (\mathbb{Z}/3[Y_1+Y_2]^+ \oplus \mathbb{Z}/3[C]) \otimes \mathbb{Z}/3[v]$$

$$H^*(BE)^{<k>} \cong (\mathbb{Z}/3[C]\{1,a\}) \otimes \mathbb{Z}/3[v] \quad where\ a = -Y_1+Y_2+y_1y_2,\ C^2 = a^2$$

$$H^*(BE)^{<w',c>} \cong (\mathbb{Z}/3[Y_2]^+ \oplus \mathbb{Z}/3[C]\{1,a,a'\}) \otimes \mathbb{Z}/3[D_1] \quad where\ a' = (t+Y_2)v$$

Let $D_1 = C^3 + V,\ D_2 = CV$. Then it is known that

$$D_1|A_i = \tilde{D}_1,\ D_2|A_i = \tilde{D}_2 \quad for\ all\ i \in \mathbb{F}_3 \cup \infty.$$

So we also write $DA \cong \mathbb{Z}/3[D_1, D_2]$. We have

$$H^*(BE)^{Q_8} \cong H^*(BE)^{<k>} \cap H^*(BE)^{<w>} \cong (\mathbb{Z}/3[C]) \otimes \mathbb{Z}/3[v]$$

$$H^*(BE)^{D_8} \cong H^*(BE)^{k} \cap H^*(BE)^{<w'>} \cong \mathbb{Z}/3[C]\{1,a\} \otimes \mathbb{Z}/3[D_1]$$

$$H^*(BE)^{<l>} \cong (H^*(BE)^{<k>})^{<l>} \cong \mathbb{Z}/3[C]\{1,av\} \otimes \mathbb{Z}/3[D_1]$$

Hence we have

$$H^*(BE)^{SD_{16}} \cong \mathbb{Z}/3[C, D_1]$$

Proposition 6.3. *There is isomorphisms for* $|a''| = 4$,

$$H^*({}^2F_4(2)') \cong DA\{1, (D_1 - C^3)a''\}$$

$$H^*(M_{24}) \cong (\mathbb{Z}/3[D_2] \oplus \mathbb{Z}/3[C]\{a''\}) \otimes \mathbb{Z}/3[D_1]$$

Proof. Let $G = M_{24}$. Then G has just two G-conjugacy classes of V-subgroups

$$A_0 \sim A_1 \quad A_2 \sim A_\infty$$

It is known that one is F^{ec}-radical on the other is not. Suppose that A_0 is F^{ec}-radical. Then $W_G(A_0) \cong GL_2(\mathbb{F}_3)$. Let $a'' = a + C$. Then

$$a''|A_0 = (-Y_1 + Y_2 + y_1y_2 + C)|A_0 = 0, \quad a''|A_\infty = -Y.$$

By the Theorem 3.1

$$H^*(BM_{24}) \cong H^*(BE)^{D_8} \cap i_{A_0}^{*-1} H^*(BA_0)^{W_G(A_0)},$$

we get the isomorphism for M_{24}. For $G =^2 F_4()'$, the both conjugacy classes are F^{ec}-subgroups and $W_G(A_2) \cong GL_2(\mathbb{F}_3)$. Hence

$$H^*(B^2F_4(2)') \cong H^*(BM_{24}) \cap i_{A_2}^{*-1} H^*(BA_2)^{GL_2(\mathbb{F}_3)}.$$

We know

$$(D_1 - C^3)a''|A_0 = 0 \quad (D_1 - C^3)a''|A_\infty = -VY = \tilde{D}_2.$$

Thus we get the cohomology of ${}^2F_4(2)'$.

□

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Remark. In [T-Y],[Y2], we take

$$(\mathbb{Z}/2)^2 =< diag(\pm 1, \mp 1) >, \quad D_8 =< diag(\pm -, \mp 1), w > .$$

For this case, the M_{24}-conjugacy classes of V-subgroups are $A_0 \sim A_\infty$, $A_1 \sim A_2$, and we can take $a'' = B = C - Y_1 - Y_2$. This case the expressions of $H^*(M_{12})$,$H^*(V : GL_2(\mathbb{F}_3))$ are more simple (see [T-Y],[Y2]).

Remark. Corollary 6.3 in [T-Y] and Corollary 3.7 in [Y2] are not correct. This came from Theorem 6.1. This theorem is only correct with adding the assumption that there are exactly two G c.c. of $\mathbb{F}_p^2$ subgroups such that one is p-pure and the other in not, which is always satified for sporadic simple groups but not for ${}^2F_4(2)'$.

Corollary 6.4. *We also have the isomorphisms.*

$$H^*(X_{2,0}) \cong DA\{D_2\}, \quad H^*(X_{2,1}) \cong \mathbb{Z}/3[C, D_1]\{c'\} \quad \text{where } (c')^2 = CD_2, c' = (av)$$

$$H^*(X_{0,1}) \cong \mathbb{Z}/3[C, D_1]\{v\}, \quad H^*(M(2)) \cong DA\{C, C^2, C^3\} \quad \text{where } C^4 = CD_1 - D_2.$$

Here we write the typical examples. First note that $C^4 = CD_1 - D_2$ implies

$$\mathbb{Z}/3[C, D_1] \cong DA\{1, C, C^2, C^3\} \cong H^*(X_{0,0}) \oplus H^*(M(2)),$$

$$\mathbb{Z}/3[C, D_1]\{C\} \cong (DA\{C, C^2, C^3,\ D_2\} \cong H^*(M(2)) \oplus H^*(X_{2,0}).$$

The decomposition for $H^*(BE)^{D_8}$ is isomorphic to

$$\mathbb{Z}/3[C, D_1]\{1, a''\} \cong DA\{1,\ C, C^2, C^3,\ (D_1 - C^3)a'',\ a'', a''^2, a''^3\}$$

$$\cong H^*(X_{0,0}) \oplus H^*(M(2)) \oplus H^*(X_{2,0}) \oplus H^*(M(2)).$$

The decomposition for $H^*(BE)^{<k>}$ is

$$(\mathbb{Z}/3[C, D_1]\{1, a,\ v,\ av\} \cong H^*(BE)^{D_8} \oplus H^*(X_{0,1}) \oplus H^*(X_{2,1}).$$

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