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HALL SUBGROUPS OF M-GROUPS NEED NOT BE M-GROUPS

群馬大学・教育学部　福島　博 (HIROSHI FUKUSHIMA)
(Gunma University)

ABSTRACT. In this paper, we shall give examples of M-groups that have a Hall subgroup that is not an M-group.

1. INTRODUCTION

A character of a finite group G is monomial if it is induced from a linear character of a subgroup of G. A group G is an M-group if all its complex irreducible characters (the set $\mathrm{Irr}(G)$) are monomial.

In the 1960's, Dornhoff [3] proved that a normal Hall subgroup of an M-group must be an M-group. Based on this work, it was conjectured that normal subgroups of M-groups are M-groups and Hall subgroups of M-groups are M-groups. In the early 1970's, Dade [1] and van der Waall [5] independently showed that normal subgroups of M-groups need not be M-groups. In this paper, we shall give examples of M-groups that have a Hall subgroup that is not an M-group.

Let $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$. We write $C_G(\theta)$ to denote the stabilizer of θ in G. We also write $\mathrm{Irr}(G|\theta) = \{\chi \in \mathrm{Irr}(G) | [\chi_N, \theta] \neq 0\}$.

2. PRELIMINARY LEMMAS

We begin with some preliminary lemmas.

Lemma 2.1 *Let E be an extra-special p-group, let σ be an automorphism of E of order q where $q \neq p$ is a prime, and let $G = E\langle\sigma\rangle$. If σ acts irreducibly on $E/Z(E)$, then G is not an M-group.*

Proof. We know that $|E| = p^{2n+1}$ for some positive integer n. This implies that the nontrivial irreducible p-modules for σ must have dimention $2n$, and thus, σ must centralize $Z(E)$. For any nonlinear character $\psi \in \mathrm{Irr}(E)$, we see that $\psi(1) = p^n$ and ψ is invariant under the action of σ. By Corollary 6.28 of [4], ψ has an extention $\chi \in \mathrm{Irr}(G)$.

We claim that χ is not monomial. Suppose χ were monomial. Then there would be a subgroup $H \subseteq G$ and a linear character $\lambda \in \mathrm{Irr}(H)$ so that $\lambda^H = \chi$. We see that $|G : H| = \chi(1) = p^n$. It follows that $|H| = p^{n+1}q$, and so by conjugating, we may assume that $\sigma \in H$. This implies that $H \cap E/Z(E)$ is a σ-submodule of $E/Z(E)$, but this is contradiction, since $Z(E) \subset H \cap E \subset E$ and $E/Z(E)$ is irreducible under the action of σ. Therefore, we conclude that χ is not monomial, and hence, G is not an M-group. □

Lemma 2.2 *Let N be a normal subgroup of G, and let φ be a linear character of N. Write T for the stabilizer of φ in G, and assume that φ extends to $\hat{\varphi} \in Irr(T)$. Given a character $\eta \in Irr(T/N)$, the character $(\hat{\varphi}\eta)^G$ is monomial if and only if η is monomial. Furthermore, every character in $Irr(G|\varphi)$ is monomial if and only if T/N is an M-group.*

Proof. By Gallagher's theorem, we know that $\hat{\varphi}\eta \in \mathrm{Irr}(T|\varphi)$, and by Clifford's theorem, $\chi = (\hat{\varphi}\eta)^G$ is irreducible. Using Lemma 4.1 of [2] (or Problem 6.11 of [4]), we see that χ is monomial if and if $\hat{\varphi}\eta$ is monomial. Suppose S is a subgroup of T and $\sigma \in \mathrm{Irr}(S)$ so that $\sigma^T = \eta$. It is known that $(\hat{\varphi}_S\sigma)^T = \hat{\varphi}\eta$ (see Problem 5.3 of [4]). It follows that $\hat{\varphi}\eta$ is monomial if and only if η is monomial. The last conclusion is an immediate consequence of the previous one, so this proves the lemma. □

Lemma 2.3 *Let p be an odd prime so that 3 divides $p+1$. Let E be an extra-special p-group of order p^5 and exponent p. Then E has an automorphism σ of order 3 with centralizer $Z(E)$, and E has a maximal abelian subgroup A that is normal in E and σ- invariant. If G is the semi-direct product $E\langle\sigma\rangle$, then G is an M-group.*

Proof. We know that 3 divides p^2-1. We can view E as the central product of E_1 and E_2 where E_1 and E_2 are both extra-special groups of order p^3 and exponent p. Since 3 divides p^2-1, it follows that E_1 and E_2 each have an automorphism of order 3 with each having centralizer $Z = Z(E)$. Applying these to the central product, we get the automorphism σ of E with order 3 and centralizer Z. We let $T = \langle\sigma\rangle$, and we note that $C_E(T) = Z$. It suffices to find abelian T-invariant subgroup $A \subseteq E$ of index p^2. (Note that this will prove that G is an M-group.)

Let U/Z be an irreducible T-submodule of E/Z. Then U/Z has order p^2 since 3 does not divide $p-1$. If U is abelian, then we are done, so we assume U is nonabelian. Let $V = C_E(U)$. Then $U \cap V = Z$, and E/Z is the direct sum of the irreducible modules U/Z and V/Z. Note that V must be nonabelian.

Take x to be an element of U that does not lie in Z, and write $y = x^\sigma$. Then x and y generate U, so they do not commute. Let $z = [x, y]$, and observe that $Z = \langle z\rangle$. Then $y^\sigma \in x^{-1}y^{-1}Z$.

Let r be an element of V that does not lie in Z, and write $s = r^\sigma$. We see that r and s generate V, so they do not commute, and hence, $[r, s]$ is a nonidentity element of Z. We see that $s^\sigma \in r^{-1}s^{-1}Z$. Suppose that $[r, s] = z^{-1}$. We observe that $(xr)^\sigma = ys$, and we compute $[xr, ys] = [x, y][r, s] = zz^{-1} = 1$. Also, $(ys)^\sigma \in (x^{-1}y^{-1})(r^{-1}s^{-1})Z = (xr)^{-1}(ys)^{-1}Z \subseteq \langle xr, ys\rangle$. We conclude that $\langle xr, ys\rangle$ is a σ-invariant abelian subgroup of E of index p^2, and we will be done.

The idea is to choose r properly. We pick any element $v \in V - Z$, and let $w = v^\sigma$. Note that $w^\sigma \in v^{-1}w^{-1}Z$. We know that $[v, w] = z^a$ for some integer a with $1 \leq a \leq p-1$. We consider elements of the form v^iw^j, and we see that $(v^iw^j)^\sigma Z = w^iv^{-j}w^{-j}Z = v^{-j}w^{i-j}Z$. It follows that

$$[v^iw^j, (v^iw^j)^\sigma] = [v^iw^j, v^{-j}w^{i-j}] = (z^a)^{i(i-j)-j(-j)} = z^{a(i^2-ij+j^2)}.$$

We need to show that as i and j vary over Z/pZ, the quantity $i^2 - ij + j^2$ takes on all values in Z/pZ.

For any value $b \in Z/pZ$, we consider the equation $i^2 - ij + j^2 = b$. We take the equation $i^2 - ij + j^2 - b = 0$, and we solve for i. By the quadratic formula, we can do this if the discriminant is a square. The discriminant is $j^2 - 4(j^2 - b) = 4b - 3j^2$. We want to find k so that $4b - j^2 = k^2$. As j and k vary through the p possible values in Z/pZ, we see that $4b - 3j^2$ and k^2 each take on $(p+1)/2$ different values. Since there are only p possible values in Z/pZ, there must be an overlap between these two sets. We now fix b so that $ab = -1$ modulo p. The work we have just done shows that we can find i and j so that $i^2 - ij + j^2 = b$ modulo p. We take $r = v^i w^j$, and we see that $s = r^\sigma$. The work we did earlier shows that $[r, s] = z^{a(i^2-ij+j^2)} = z^{ab} = z^{-1}$. We now conclude that E has a normal σ-invariant subgroup of index p^2, so the lemma is proved. □

3. The Construction

We suppose that p and q are distinct odd primes so that p divides $q-1$ and 3 divides $p+1$. Then $q = 1 + pk$ for some integer k. Hence $(q^p - 1)/(q-1) = 1 + q + \cdots + q^{p-1} \equiv 1 + (1+pk) + \cdots + (1 + (p-1)pk) = p + ((p-1)p^2k/2) \pmod{p^2}$. Thus $(q^p-1)/(q-1) = pr$, where $r \equiv 1 \pmod{p}$. In particular, $(p, r) = 1$. Next we claim that 3 does not divide r. It is known that the gcd of $q-1$ and $(q^p-1)/(q-1)$ must divide p, so if 3 divides $q-1$, then 3 will not divide $(q^p-1)/(q-1)$. On the other hand, we know that the order of q modulo 3 must divide 2, so if 3 does not divide $q-1$, then the order of q modulo 3 is 2. Since 2 does not divide p, it cannot be the q^p is congruent to 1 modulo 3, so 3 does not divide $(q^p-1)/(q-1)$.

We mention that there exist pairs of primes with the properties mentioned in the previous paragraph. One such pair of primes is $p = 5$ and $q = 11$. Observe that $(11^5 - 1)/(11-1) = 5 \cdot 3221$.

Let F be the finite field of order q^p. Take V to be the additive group of F, so V is an elementary abelian q-group. Let N be the subgroup of of order $(q^p-1)/(q-1) = pr$ in the multiplicative group of F. Multiplication in F provides a natural action of N on V via automorphisms. The orbits in these action correspond to the cosets of the subgroup of order $q-1$ in the multiplicative group of F. Fix $s, t \in N$ so that $o(s) = p$ and $o(t) = r$, and note that $N = \langle st \rangle$. Let a be a generator for the Galois group of F over the field of order q so that a has order p. The Galois action provides a natural action for a on V and N. Note that the fixed field for a is the field of order q, so each orbit of N on V is stabilized by a. Since p divides $q-1$, it follows when s is viewed as an element of F that s lies in the fixed field, so a will centralize s.

Let Q be an extra-special p-group of order p^5 and exponent p. Let Z be the center of Q, and suppose that Z is generated by z. We can fix the generators of Q to be x_0, y_0, x_1, y_1 so that $x_0^p = y_0^p = x_1^p = y_1^p = z^p = 1$ and $[x_0, y_0] = [x_1, y_1] = z$. We define $Q_0 = \langle x_0, y_0 \rangle$ and $Q_1 = \langle x_1, y_1 \rangle$. Let K be an elementary abelian group of order p^2 that is generated by x_2 and y_2. It is not difficult to see that $E = Q \times K$ has an automorphism σ of oredr 3 that centralizes Z and is defined by

$$x_0^\sigma = y_0,\ y_0^\sigma = x_0^{-1}y_0^{-1},\ x_1^\sigma = y_1,\ y_1^\sigma = x_1^{-1}y_1^{-1},\ x_2^\sigma = y_2,\ y_2^\sigma = x_2^{-1}y_2^{-1}.$$

We define M to be the semi-direct product arising from σ acting on E.

We set $U = V\langle t \rangle$. We define an action of Q on U whose kernel is Q_0 by $u^{x_1} = u^a$ and $u^{y_1} = u^s$ for all $u \in U$. We define an action of K on U by $u^{x_2} = u^{a^{-1}}$ and $u^{y_2} = u^s$ for all $u \in U$. We set $U_0 = U \times U^\sigma \times U^{\sigma^2}$. We define an action of M on U_0 by $(u^{\sigma^i})^x = u^{(\sigma^i x \sigma^{-i})\sigma^i}$ for all $u \in U, x \in M$, and $i = 1, 2$. Our group G is the resulting semi-direct product of M acting on U_0. Observe that $|U| = q^p r$ and $|M| = p^7 3$, so $|G| = q^{3p} r^3 p^7 3$. Take $V_0 = V \times V^\sigma \times V^{\sigma^2}$, and let H be the semi-direct product of M acting on V_0. We see that $|H| = q^{3p} p^7 3$ and $|G : H| = r^3$, so H is a Hall subgroup of G. (Obviously, q does not divide r, the choice of p precludes p from dividing r, and we showed that 3 does not divide r; so $(r^3, q^{3p} p^7 3) = 1$.) We will show that G is an M-group and H is not an M-group. Also, we take L to be the semi- direct product of M acting on $\langle t \rangle \times \langle t^\sigma \rangle \times \langle t^{\sigma^2} \rangle$. Observe that L acts coprimely on V_0.

Lemma 3.1 *H is not an M-group*

Proof. Let $A = Q_0 \langle x_1 x_2, y_1 y_2^{-1} \rangle$. It is not difficult to see that A is the kernel of the action of E on $\mathrm{Irr}(V)$. Furthermore, since E/A is abelian, it must have a regular orbit in $\mathrm{Irr}(V)$, so we can find a character $\lambda \in \mathrm{Irr}(V)$ with $C_E(\lambda) = A$. Let $\varphi = \lambda \times \lambda^\sigma \times \lambda^{\sigma^2}$. We see that $C_E(\varphi) = A \cap A^\sigma \cap A^{\sigma^2}$. It follows that $Q_0 \subseteq C_E(\varphi)$. Also, A is not σ-invariant, so $|E : C_E(\varphi)| > |E : A| = p^2$. We obtain $|C_E(\varphi) : Q_0| < p^2$, and we conclude that $C_E(\varphi) = Q_0$. Since σ will stabilize φ, we have $C_M(\varphi) = Q_0 \langle \sigma \rangle$, and $C_M(\varphi)$ is not an M-group by Lemma 2.1.

Let T be the stabilizer of φ in H. It follows that $T = V_0C_M(\varphi)$. Since $(|V_0|, |T : V_0|) = 1$, we know φ extends to $\hat{\varphi} \in \mathrm{Irr}(T)$. Since $T/V_0 \cong C_M(\varphi)$, we can find a character $\eta \in \mathrm{Irr}(T/V_0)$ which is not monomial. We know that $(\hat{\varphi}\eta)^H$ is irreducible and it is not monomial by Lemma 2.2. Therefore, H is not an M-group. □

Lemma 3.2 *G is an M-group*

Proof. Using Lemma 2.3, it is not difficult to show that $M \cong G/U_0$ is an M-group. To show G is an M-group, it suffices to show that every character in $\mathrm{Irr}(G)$ whose kernel does not contain U_0 is monomial.

Suppose $\chi \in \mathrm{Irr}(G)$ and U_0 is not contained in $\mathrm{Ker}(\chi)$. For now, we will assume that V_0 is contained in $\mathrm{Ker}(\chi)$. Let φ be an irreducible constituent of χ_{U_0}, and notice that $\varphi \in \mathrm{Irr}(U_0/V_0)$. Let T be the stabilizer of φ in G, and observe that $(|U_0 : V_0|, |T : U_0|) = 1$, so φ extends to $\hat{\varphi} \in \mathrm{Irr}(T)$. By Lemma 2.2, it suffices to prove that T/U_0 is an M-group. If 3 does not divide $|T : U_0|$, then T/U_0 is a p-group, and we are done. Thus, we may assume that 3 divides $|T : U_0|$, and by conjugating, we may assume that $\sigma \in T$. This implies that $\varphi = \nu \times \nu^{\sigma} \times \nu^{\sigma^2}$ for some character $\nu \in \mathrm{Irr}(U/V)$. Observe that $T = U_0C_M(\varphi)$ and $C_M(\varphi) = C_E(\varphi)\langle\sigma\rangle$. Furthermore, we have $C_E(\varphi) = C_E(\nu) \cap C_E(\nu)^{\sigma} \cap C_E(\nu)^{\sigma^2} = C_E(\nu) \cap C_E(\nu)^{\sigma}$. It is not difficult to see that $|E : C_E(\nu)| = p$ and $C_E(\nu) \cap C_E(\nu)^{\sigma}$ is an extra-special p-group of order p^5. Thus, $C_M(\varphi)$ is an M- group by Lemma 2.3.

Finally, we assume that V_0 is not contained in the kernel of χ. Let δ be an irreducible constituent of χ_{V_0}. Let S be the stabilizer of δ in G. Again, $(|V_0|, |S : V_0|) = 1$, so δ extends to $\hat{\delta} \in \mathrm{Irr}(T)$, and we see using Lemma 2.2 that it suffices to show that S/V_0 is an M-group. If 3 does not divide $|S : V_0|$, then $S/(S \cap U_0)$ is a p-group and $(S \cap U_0)/V_0$ is abelian. By Theorem 6.23 of [4], this will force S/V_0 to be an M-group. We suppose that 3 does not divide $|S : V_0|$, and by conjugating, we may assume that $\sigma \in S$. This implies that $\delta = \lambda \times \lambda^{\sigma} \times \lambda^{\sigma^2}$ for some nonprincipal character $\lambda \in \mathrm{Irr}(V)$.

We know that $\langle t \rangle$ acts Frobeniusly on V, so no nonidentity element in $\langle t \rangle$ will stabilize λ. It follows that $C_{\langle t \rangle E}(\lambda)$ is a p-subgroup, so we can find an element

$h \in \langle t \rangle$ so that $A = C_{\langle t \rangle E}(\lambda^h) \subseteq E$. Recall that the orbits in V under the action of $\langle t \rangle E$ have size pr. It is not difficult to see that the orbits in V have the same size, so $|\langle t \rangle E : A| = pr$, and hence, A has index p in E.

Let $\delta' = \lambda^h \times (\lambda^\sigma)^{h^\sigma} \times (\lambda^{\sigma^2})^{h^{\sigma^2}}$, and we observe that $\delta' = \delta^{(h, h^\sigma, h^{\sigma^2})}$. Obseve that $C_E(\delta') = A \cap A^\sigma \cap A^{\sigma^2} = A \cap A^\sigma$. We note that A is not σ-invariant so $A \cap A^\sigma \subset A$. On the other hand, A is normal in E of index p, so $A \cap A^\sigma$ will be normal in E of index p^2, and $|A \cap A^\sigma| = p^5$. Since K is not contained in A, we have $K \times Q_0 \neq A \cap A^\sigma$, and we conclude that $C_E(\delta') = A \cap A^\sigma$ is an extra-special group of order p^5. Now, $C_M(\delta') = C_L(\delta')$ is a conjugate of $C_L(\delta)$ that lies in M, so $C_M(\delta') = C_E(\delta')\langle \sigma \rangle$, and $C_M(\delta')$ is an M-group by Lemma 2.3. It follows that $S/V_0 \cong C_M(\delta)$ is an M- group. This proves the theorem. □

4. Another Construction

We observe that Lemma 2.3 is still true if E is replaced by a central product of two quaternion groups of order 8. We can change the construction in Section 3 by taking $p = 2$ and q to be an odd prime so that $q + 1 = 2r$ where r is relatively prime to 6. (The first such prime $q = 13$.) We take Q to be the central product of two quaternion groups of order 8, and we make the appropriate changes in the generators of Q. The rest of the argument will go through for this construction.

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DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, GUNMA UNIVERSITY MAEBASHI, GUNMA 371-8510, JAPAN