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Eigenvalues and elementary divisors of Cartan matrices of finite groups

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1 Introduction

Let G be a finite group and let F be an algebraically closed field of characteristic $p > 0$. Let B be a block of FG with defect group D of order p^d. Let $l(B)$ be the number of irreducible Brauer characters in B and $k(B)$ be the number of ordinary irreducible characters in B. Let $C_B = (c_{ij})$ be the Cartan matrix of B and $D_B = (d_{ij})$ be the decomposition matrix of B. Since C_B is an indecomposable nonnegative matrix, it has the Frobenius-Perron eigenvalue (i.e. unique largest eigenvalue) $\rho(B)$. We denote by $R = R_B$ the set of eigenvalues of C_B and by $E = E_B$ the set of $(\mathbf{Z})$-elementary divisors of C_B. We are concerned with behavior of eigenvalues of C_B, in particular with when it is an integer. In [KW] and [KMW] we found that there are some relations between eigenvalues and elementary divisors of C_B in some cases. Furthermore we had also some questions there. In this article, first we show some results in [KW] and [KMW] and next we mention a new conjecture which includes a part of the questions and show that the conjecture is true in cyclic blocks with $l(B) \leq 5$ and in tame blocks and furthermore we show some examples in cases that G is a symmetric group, a simple group or a near simple group.

(1) Properties of C_B
  (a) $C_B = {}^tD_B \cdot D_B$.
  (b) C_B is a nonnegative, indecomposable matrix over $\mathbf{Z}$.
  (c) C_B is a symmetric matrix.
  (d) C_B is positive definite.
  (e) $\det C_B = p^r \geq p^d = |D|$.

(2) Properties of elementary divisors of C_B

Let us set $E = E_B = \{e_1, \cdots, e_{l(B)}\}$.

(a) There exists unique largest elementary divisor $e_1 = |D|$ and others $e_i < |D|$ which are a power of p.

(b) $e_i = |C_G(x_i)|_p$ for some p-regular element in G.

(c) $\prod_{i=1}^{l(B)} e_i = \det C_B$.

(d) If $B \sim B'$ (Rickard equivalent i.e. derived equivalent, see [B, 4.B]), then $E_B = E_{B'}$. (This comes from that if B and B' are Rickad equivalent there exists a perfect isometry between B and B' and further there exists a matrix $V \in GL(l(B), \mathbf{Z})$ such that $C_{B'} = {}^tVC_BV$ ([B, 4.11 Theorem])).

(3) Properties of eigenvalues of C_B

Let us set $R = R_B = \{\rho_1, \cdots, \rho_{l(B)}\}$. An eigenvalue $\rho \in R_B$ need not to be an integer, but they are positive. $\rho(B)$ need not only to be larger but also smaller than $|D|$.

(a1) There exsts unique largest eigenvalue $\rho_1 = \rho(B)$ and others $\rho_i < \rho_1$. There exists a positive vector $x \in \mathbf{R}^{l(B)}$ such that $C_Bx = \rho(B)x$ which we call a Frobenius eigenvector.

(a2) If $\rho \in R$, then there exists an algebraic integer λ such that $|D| = \rho \cdot \lambda$ (i.e. $\rho \mid |D|$ as algebraic integer). This comes from $|D|C_B^{-1} \in \mathrm{Mat}(l(B), \mathbf{Z})$.

(b) What group structural property like (2b) does ρ have? What happens if $\rho \in \mathbf{Z}$?

(c) $\prod_{i=1}^{l(B)} \rho_i = \det C_B$.

(d) If $B \sim B'$ (Rickard equiv.), then R_B and $R_{B'}$ need not to be equal. But of course if $B \sim B'$ (Morita equiv.), then $R_B = R_{B'}$.

Example 1. Let $G = S_4$ be the symmetric group of degree 4, $p = 2$, and $B = B_1$ be the principal block of G.

Then $C_B = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, $\rho(B) = \dfrac{7+\sqrt{17}}{2} < |D| = 8.$

Let $G = S_5$ be the the symmetric group of degree 5, $p = 2$, and $B = B_1$ be the principal block of G.

Then $C_B = \begin{pmatrix} 8 & 4 \\ 4 & 3 \end{pmatrix}$, $\rho(B) = \dfrac{11+\sqrt{89}}{2} > |D| = 8.$

In each case $\rho(B)$ is not an integer. In former case $\rho(B) < |D|$, but in latter case $\rho(B) > |D|$. It is known that $B_1(S_4)$ and $B_1(S_5)$ are Rickard equivalent. So this is also an example that $R_B \neq R_{B'}$ even if B and B' are Rickard equivalent.

2 Qestions and facts

We had the following two questions and we proved it is actually true in some cases in [KW] and [KMW].

Q 1. If $\rho(B) \in \mathbf{Z}$, then does $\rho(B) = |D|$ hold?

Q 2. If $\rho(B) = |D|$, then does $R_B = E_B$ hold?

These questions are answered affirmatively in the following cases.

Fact 1. If $D \lhd G$, then $\rho(B) = |D|$ and $R_B = E_B = \{|C_D(x_1)|, \cdots, |C_D(x_{l(B)})|\}$, where $\{x_1, \cdots, x_{l(B)}\}$ is a representative of p-regular classes of G associated with B. In this case, $f = {}^t(f_1, \cdots, f_{l(B)})$ is a Frobenius eigenvactor of C_B, where $f_i = \varphi_i(1)$ for $\varphi_i \in \mathrm{IBr}(B)$.

Fact 2. If D is cyclic, then **Q1** and **Q2** are true. In this case, if $\rho(B) \in \mathbf{Z}$, then B and its Brauer correspondent b are Morita equivalent. Then $C_B = C_b$, so $\rho(B) = |D|$ and $R_B = E_B = \{|D|, 1, \cdots, 1\}$ by Fact 1. Furthermore, $\tilde{f} = {}^t(\tilde{f}_1, \cdots, \tilde{f}_{l(B)})$ is a Frobenius eigenvector of C_B, where $\tilde{f}_i = \tilde{\varphi}_i(1)$ for $\tilde{\varphi}_i \in \mathrm{IBr}(b)$. Here b is the Brauer correspondent of B.

Fact 3. If B is tame (i.e. $p = 2$ and $D \simeq$ dihedral, generalized quaternion or semidihedral), then **Q1** and **Q2** are true. In this case, if $\rho(B) \in \mathbf{Z}$, then B and b are Morita equivalent, and further $\rho(B) = |D|$ and B is one of the following three cases.

(i) $l(B) = 1$,

(ii) $D \simeq E_4$ (i.e. Klein's four group) and $C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$,

(iii) $D \simeq Q_8$ (i.e. the quaternion group of order 8) and $C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

In cases (ii),(iii) $R_B = E_B = \{|D|, 1, 1\}$ and $\tilde{f} = {}^t(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ is a Frobenius eigenvector, where $\tilde{f}_i = \tilde{\varphi}_i(1)$, for $\tilde{\varphi}_i \in \mathrm{IBr}(b)$. Here b is the Brauer correspondent of B.

Remark 1. If $\rho(B) \in \mathbf{Z}$, B and its Brauer correspondent b are not Morita equivalent in general. Let $G = SL(2,3) \cdot E_{27}$ (semidirect product, the center of $SL(2,3)$ acts trivially), $p = 3$, and B be a non-principal block. Then $D \simeq \mathbf{Z}_3 \wr \mathbf{Z}_3, l(B) = 1$ and $k(B) = 12$. But $k(b) = 17$. So B and b are not Morita equivalent.

Fact 4. If G be a p-solvable group, then **Q2** is true, but **Q1** is not yet proved to be true. However, if $l(B) = 2$, then Q1 is true. We tried to compute many cases of finite simple groups with small $l(B)$. Then **Q1** and **Q2** seem to be true.

3 Conjectures

Kiyota has conjectured the following on **Q1** just after [KW] was published.

Conjecture(K) (Kiyota). Let $N(\rho)$ be the norm of an algebraic integer ρ. Then $|D| \mid N(\rho(B))$.

If Conjecture (K) is true, then **Q1** is true. Because, if $\rho(B) \in \mathbf{Z}$, then since $N(\rho(B)) = \rho(B)$ we have $|D| \mid \rho(B)$ by Conjecture. On the other hand, by the property (a2) of eigenvalues of C_B in §1 $\rho(B) \mid |D|$ as integer. This means $\rho(B) = |D|$.

Verifying Conjecture (K) for symmetric groups and some simple groups, the following more explicit relation between eigenvalues and elementary divisors of C_B seems to exist.

Let $f_B(x)$ be the characteristic polynomial of C_B. Let $f_B = f_1 \cdot f_2 \cdots f_r$ be a $\mathbf{Z}$-irreducible decomposition of $f_B(x)$. Let $R_i := \{\rho_{i1}, \cdots, \rho_{in_i}\}$, $1 \leq i \leq r$ be the set of all roots of $f_i(x)$. So we denote and write as $R = \{\rho_{11}, \cdots, \rho_{1n_1}\ ;\ \rho_{21}, \cdots, \rho_{2n_2}\ ;\ \cdots\ ;\ \rho_{r1}, \cdots, \rho_{rn_r}\}$. Then for each i, $N(\rho_{ij}) = \prod_{k=1}^{n_i} \rho_{ik} = |f_i(0)|$ for any $j = 1, 2, \cdots n_i$.

Conjecture. There is a direct decomposition $E = E_1 \sqcup \cdots \sqcup E_r$ as set such that the following three conditions are satisfied.

(i) $|R_i| = |E_i|$ for $1 \leq i \leq r$.

(ii) Let $E_i = \{e_{i1}, \cdots, e_{in_i}\}$, then $\prod_{k=1}^{n_i} e_{ik} = N(\rho_{ij})$.

(iii) Let $\rho(B) \in R_1$. Then $|D| \in E_1$. In particular, $|D| \mid N(\rho(B))$.

Remark 2. Assume Conjecture is true. Then the following (1), (2) hold.

(1) If eigenvalue $\rho \in \mathbf{Z}$, then $\rho \in E$ by (i), (ii).

(2) If $\rho(B) \in \mathbf{Z}$, then $\rho(B) = |D|$ by (iii).

These do not hold for the Cartan matrix of a general algebra. For example, there exists an indecomposable cellular algebra A with the Cartan matrix $C_A = \begin{pmatrix} m & 1 \\ 1 & m \end{pmatrix}$ for $m > 2$ by [X]. In this case $R = \{m+1, m-1\}, E = \{m^2-1, 1\}$. This algebra A comes from the Brauer tree algebra with two exceptional vertices. So this cannot be a cyclic block of a finite group algebra.

Remark 3. First we conjectured that if $\rho(B) \in R_1$, then $\deg f_1 \geq \deg f_i$ for all $1 \leq i \leq r$. But this does not hold in general. For example, let $G = SL(2,32)$ or $Sz(32)$, and $p = 2$, B be the principal block. Then $\deg f_1 = 7$, but $\deg f_2 = 12$ as is mentioned below.

4 Cyclic blocks with $l(B) \leq 5$ and tame blocks

It is difficult to verify Conjecture in cyclic blocks in general. We have the following.

Theorem 1. *Suppose B is a cyclic block with $l(B) \leq 5$. Then Conjecture is true. Furthermore, if $\rho(B) \in R_1$, then $\deg f_1 \geq \deg f_i$ for $1 \leq i \leq r$.*

Remark 4. It is clear if $l(B) = 1$. Fact 2 implies that it is also clear in the case $l(B) = 2$. We may consider the cases $l(B) = 3, 4$ or 5. There are 32 cases of Brauer trees considering a position of an exceptional vertex. In each case the Cartan matrix contains one parameter (i.e. the multiplicity m). We can determine the characteristic polynomial $f_B(x)$ and decompose into $\mathbf{Z}$-irreducible polynomials by the $\mathbf{Z}$-elementary transformation. Furthermore, we can prove each $f_i(x)$ is actually irreducible. See [W].

Theorem 2. *Suppose B is a tame block. Then Conjecture is true. Furthermore, if $\rho(B) \in R_1$, then $\deg f_1 \geq \deg f_i$ for $1 \leq i \leq r$.*

Remark 5. Since B is tame, $l(B) = 1, 2$ or 3. By Fact 3 we may consider the case $l(B) = 3$. There are 12 cases and in each case the Cartan matrix contains one parameter. It is easier to calculate eigenvalues and elementary divisors and furthermore characteristic polynomials than cyclic blocks. So we can also easily prove each component of $f_B(x)$ is actually $\mathbf{Z}$-irreducible similarly to cyclic blocks. See [W].

5 Examples : Symmetric groups and some simple groups

We calculate these by using MAPLE. In order to see our conjecture we especially pick up some examples in which $f_B(x)$ decomposes into various $\mathbf{Z}$-irreducible components. We denote by $d(B)$ the defect of B i.e. the order of a defect group D of B is $p^{d(B)}$. In this section we always denote by B_1 the principal block of FG. Let S_n be the symmetric group of degree n. Suppose $p = 2$. Then $f_{B_1}(x)$ is $\mathbf{Z}$-irreducible for $7 \leq n \leq 14$ and other non-principal blocks are as well. Our conjecture is trivially true if $f_B(x)$ is $\mathbf{Z}$-irreducible. So we start with S_n for $p = 3$.

1 Symmetric Group

[1] $p = 3$

$$(1)\ S_9,\ C_{B_1} = \begin{pmatrix} 8 & 5 & 2 & 2 & 2 & 4 & 2 & 4 & 1 & 4 \\ 5 & 8 & 4 & 2 & 4 & 4 & 2 & 4 & 2 & 2 \\ 2 & 4 & 6 & 4 & 2 & 4 & 2 & 2 & 1 & 0 \\ 2 & 2 & 4 & 8 & 1 & 4 & 5 & 4 & 2 & 2 \\ 2 & 4 & 2 & 1 & 6 & 4 & 2 & 2 & 3 & 1 \\ 4 & 4 & 4 & 4 & 4 & 8 & 4 & 5 & 2 & 2 \\ 2 & 2 & 2 & 5 & 2 & 4 & 8 & 4 & 4 & 4 \\ 4 & 4 & 2 & 4 & 2 & 5 & 4 & 8 & 4 & 4 \\ 1 & 2 & 1 & 2 & 3 & 2 & 4 & 4 & 6 & 2 \\ 4 & 2 & 0 & 2 & 1 & 2 & 4 & 4 & 2 & 6 \end{pmatrix}, \quad l(B_1) = 10,\ d(B_1) = 4,$$

$f_{B_1}(x) = (x^5 - 48x^4 + 53x^3 - 2232x^2 + 3780x - 3^7)(x^5 - 24x^4 + 194x^3 - 600x^2 + 612x - 3^4)$, $\quad N(\rho(B_1)) = 3^7$,

$R_{B_1} = \{\rho_{11} = \rho_{B_1}, \ldots, \rho_{15}\ ;\ \rho_{21}, \ldots, \rho_{25}\}$,
$E_{B_1} = \{3^4, 3^2, 3, 1, 1; 3^2, 3, 3, 1, 1\}$,

$$(2)\ S_{11},\ C_{B_2} = \begin{pmatrix} 3 & 1 & 2 & 0 & 1 \\ 1 & 3 & 2 & 1 & 0 \\ 2 & 2 & 5 & 2 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 1 & 0 & 2 & 1 & 3 \end{pmatrix}, \qquad l(B_2) = 5,\quad d(B_2) = 2,$$

$f_{B_2}(x) = (x - 9)(x - 3)^2(x - 1)^2$,
$R_{B_2} = E_{B_2} = \{9; 3; 3; 1; 1\}$

In this case $D \simeq E_9$ is elementary abelian and $\rho(B_2) \in \mathbf{Z}$. We also had a question in [KMW] that when D is abelian, if $\rho(B) \in \mathbf{Z}$, then are B and its Brauer correspondent b Morita equivalent? [CK] implies that B_2 above and its Brauer correspondent b_2 are indeed Morita equivalent.

[2] $p = 5$

(1) S_{10}, B_1

$$C_{B_1} = \begin{pmatrix} 4 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 4 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 0 & 2 & 4 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 2 & 4 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 4 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 4 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 4 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 2 & 4 \end{pmatrix}, \quad l(B_1) = 14, \quad d(B_1) = 2,$$

$f_{B_1}(x) = (x^8 - 34x^7 + 427x^6 - 2557x^5 + 7867x^4 - 12347x^3 + 9077x^2 - 2490x + 5^3)$
$(x^6 - 21x^5 + 155x^4 - 511x^3 + 775x^2 - 525x + 5^3), \quad N(\rho(B_1)) = 5^3,$

$R_{B_1} = \{\rho_{11}, \ldots, \rho_{18} \; ; \; \rho_{21}, \ldots, \rho_{26}\}$
$E_{B_1} = \{5^2, 5, 1, 1, 1, 1, 1, 1 \; ; \; 5, 5, 5, 1, 1, 1\}$

2 Simple Groups

[1] $SL(2, 32), p = 2, B_1$

C_{B_1} is a 31×31 matrix

$f_{B_1}(x) = (x^7 - 122x^6 + 1761x^5 - 6388x^4 + 8243x^3 - 4242x^2 + 811x - 32)$
$(1 - 60x + 1262x^2 - 11852x^3 + 56383x^4 - 142712x^5 + 194980x^6 - 142712x^7 + 56383x^8 - 11852x^9 + 1262x^{10} - 60x^{11} + x^{12})^2,$

$f_1(x) = x^7 - 122x^6 + \cdots + 811x - 32, \quad \rho(B_1)$ is a root of $f_1(x)$, and $\deg f_1 = 7 < \deg f_2 = 12, \quad N(\rho(B_1)) = 2^5 = |D|$

$R_{B_1} = \{\rho_{11} = \rho(B_1), \ldots, \rho_{17} \; ; \; \rho_{21}, \ldots, \rho_{2,12} \; ; \; \rho_{31}, \ldots, \rho_{3,12}\},$
$E_{B_1} = \{2^5, 1, \ldots, 1 \; ; \; 1, \ldots, 1 \; ; \; 1, \ldots, 1\}$

[2] $Sz(32), p = 2, B_1$

C_{B_1} is a 31×31 matrix

$f_{B_1}(x) = (-1024 + 96143x - 2369654x^2 + 7551363x^3 - 6304380x^4 + 969293x^5 - 25582x^6 + x^7)(1 - 226x + 17582x^2 - 562646x^3 + 7240879x^4 - 27930100x^5 + 42692404x^6 - 27108652x^7 + 7239375x^8 - 712458x^9 + 25246x^{10} - 286x^{11} + x^{12})^2$,

$f_1(x) = -1024 + 96143x \cdots - 25582x^6 + x^7$, $\rho(B_1)$ is a root of $f_1(x)$, and $\deg f_1 = 7 < \deg f_2 = 12$, $\quad N(\rho_(B_1)) = 2^{10} = |D|$

$R_{B_1} = \{\rho_{11} = \rho(B_1), \ldots, \rho_{17} \; ; \; \rho_{21}, \ldots, \rho_{2,12} \; ; \; \rho_{31}, \ldots, \rho_{3,12}\}$,
$E_{B_1} = \{2^{10}, 1, \ldots, 1 \; ; \; 1, \ldots, 1 \; ; \; 1, \ldots, 1\}$

These are the examples that $\deg f_1$ is not larger than or equal to the degrees of others. But in cases $SL(2, 2^n)$ for $2 \leq n \leq 4$ and $Sz(8)$, we have $\deg f_1 \geq \deg f_i$ for $1 \leq i \leq r$.

[3] $U_3(4), p = 2, B_1$

$$C_{B_1} = \begin{pmatrix}
16 & 12 & 12 & 12 & 12 & 4 & 4 & 8 & 8 & 8 & 8 & 2 & 2 & 2 & 2 \\
 & 26 & 14 & 18 & 18 & 6 & 8 & 8 & 14 & 10 & 8 & 1 & 4 & 6 & 2 \\
 & & 26 & 18 & 18 & 6 & 8 & 14 & 8 & 8 & 10 & 4 & 1 & 2 & 6 \\
 & & & 26 & 14 & 8 & 6 & 8 & 10 & 8 & 14 & 2 & 6 & 1 & 4 \\
 & & & & 26 & 8 & 6 & 10 & 8 & 14 & 8 & 6 & 2 & 4 & 1 \\
 & & & & & 6 & 1 & 2 & 2 & 6 & 6 & 2 & 2 & 0 & 0 \\
 & & & & & & 6 & 6 & 6 & 2 & 2 & 0 & 0 & 2 & 2 \\
 & & & & & & & 10 & 6 & 4 & 4 & 2 & 0 & 2 & 3 \\
 & & & & & & & & 10 & 4 & 4 & 0 & 2 & 3 & 2 \\
 & & & & & & & & & 10 & 6 & 3 & 2 & 2 & 0 \\
 & & & & & & & & & & 10 & 2 & 3 & 0 & 2 \\
 & & & & & & & & & & & 3 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 3 & 0 & 0 \\
 & & & & & & & & & & & & & 3 & 0 \\
 & & & & & & & & & & & & & & 3
\end{pmatrix}$$

$l(B_1) = 15$, $d(B_1) = 6$,

$f_{B_1}(x) = (x^5 - 126x^4 + 1379x^3 - 3682x^2 + 2716x - 2^8)$
$(x^3 - 19x^2 + 42x - 4)(x^3 - 19x^2 + 26x - 4)^2(x - 1), \qquad N(\rho(B_1)) = 2^8$

$R_{B_1} = \{\rho_{11} = \rho(B_1), \ldots, \rho_{15} \ ; \ \rho_{21}, \rho_{22}, \rho_{23} \ ; \ \rho_{31}, \rho_{32}, \rho_{33} \ ; \rho_{41}, \rho_{42}, \rho_{43} \ ; \ 1\}$,

$E_{B_1} = \{2^6, 2^2, 1, 1, 1 \ ; \ 2^2, 1, 1 \ ; \ 2^2, 1, 1 \ ; \ 2^2, 1, 1 \ ; \ 1\}$

[4] $U_3(16), p = 2, B_1$

$C_{B_1} =$

$$\begin{pmatrix}
440 & 160 & 160 & 160 & 238 & 238 & 238 & 238 & 238 & 238 & 126 & 126 & 28 & 28 & 28 & 40 & 40 & 40 & 40 & 40 & 40 \\
160 & 80 & 60 & 60 & 90 & 90 & 86 & 86 & 106 & 106 & 50 & 50 & 6 & 17 & 8 & 22 & 22 & 12 & 12 & 14 & 14 \\
160 & 60 & 80 & 60 & 106 & 106 & 90 & 90 & 86 & 86 & 50 & 50 & 8 & 6 & 17 & 14 & 14 & 22 & 22 & 12 & 12 \\
160 & 60 & 60 & 80 & 86 & 86 & 106 & 106 & 90 & 90 & 50 & 50 & 17 & 8 & 6 & 12 & 12 & 14 & 14 & 22 & 22 \\
238 & 90 & 106 & 86 & 152 & 144 & 132 & 128 & 132 & 128 & 74 & 68 & 12 & 12 & 22 & 22 & 22 & 28 & 30 & 17 & 20 \\
238 & 90 & 106 & 86 & 144 & 152 & 128 & 132 & 128 & 132 & 68 & 74 & 12 & 12 & 22 & 22 & 22 & 30 & 28 & 20 & 17 \\
238 & 86 & 90 & 106 & 132 & 128 & 152 & 144 & 132 & 128 & 74 & 68 & 22 & 12 & 12 & 17 & 20 & 22 & 22 & 28 & 30 \\
238 & 86 & 90 & 106 & 128 & 132 & 144 & 152 & 128 & 132 & 68 & 74 & 22 & 12 & 12 & 20 & 17 & 22 & 22 & 30 & 28 \\
238 & 106 & 86 & 90 & 132 & 128 & 132 & 128 & 152 & 144 & 74 & 68 & 12 & 22 & 12 & 28 & 30 & 17 & 20 & 22 & 22 \\
238 & 106 & 86 & 90 & 128 & 132 & 128 & 132 & 144 & 152 & 68 & 74 & 12 & 22 & 12 & 30 & 28 & 20 & 17 & 22 & 22 \\
126 & 50 & 50 & 50 & 74 & 68 & 74 & 68 & 74 & 68 & 44 & 36 & 8 & 8 & 8 & 10 & 13 & 10 & 13 & 10 & 13 \\
126 & 50 & 50 & 50 & 68 & 74 & 68 & 74 & 68 & 74 & 36 & 44 & 8 & 8 & 8 & 13 & 10 & 13 & 10 & 13 & 10 \\
28 & 6 & 8 & 17 & 12 & 12 & 22 & 22 & 12 & 12 & 8 & 8 & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 6 & 6 \\
28 & 17 & 6 & 8 & 12 & 12 & 12 & 12 & 22 & 22 & 8 & 8 & 0 & 6 & 0 & 6 & 6 & 0 & 0 & 2 & 2 \\
28 & 8 & 17 & 6 & 22 & 22 & 12 & 12 & 12 & 12 & 8 & 8 & 0 & 0 & 6 & 2 & 2 & 6 & 6 & 0 & 0 \\
40 & 22 & 14 & 12 & 22 & 22 & 17 & 20 & 28 & 30 & 10 & 13 & 0 & 6 & 2 & 9 & 6 & 3 & 2 & 3 & 2 \\
40 & 22 & 14 & 12 & 22 & 22 & 20 & 17 & 30 & 28 & 13 & 10 & 0 & 6 & 2 & 6 & 9 & 2 & 3 & 2 & 3 \\
40 & 12 & 22 & 14 & 28 & 30 & 22 & 22 & 17 & 20 & 10 & 13 & 2 & 0 & 6 & 3 & 2 & 9 & 6 & 3 & 2 \\
40 & 12 & 22 & 14 & 30 & 28 & 22 & 22 & 20 & 17 & 13 & 10 & 2 & 0 & 6 & 2 & 3 & 6 & 9 & 2 & 3 \\
40 & 14 & 12 & 22 & 17 & 20 & 28 & 30 & 22 & 22 & 10 & 13 & 6 & 2 & 0 & 3 & 2 & 3 & 2 & 9 & 6 \\
40 & 14 & 12 & 22 & 20 & 17 & 30 & 28 & 22 & 22 & 13 & 10 & 6 & 2 & 0 & 2 & 3 & 2 & 3 & 6 & 9
\end{pmatrix}$$

$l(B_1) = 21, \quad d(B_1) = 9,$

$f_{B_1}(x) = (x^6 - 1567x^5 + 48357x^4 - 312687x^3 + 397528x^2 - 139688x + 2^{12})(x^4 - 72x^3 +$
$134x^2 - 40x + 1)^2(x^3 - 29x^2 + 88x - 8)(x^2 - 6x + 1)^2, \quad N(\rho(B)) = 2^{12}$

$R_{B_1} = \{\rho_{11} = \rho(B_1), \ldots, \rho_{16} \ ; \ \rho_{21}, \ldots, \rho_{24} \ ; \ \rho_{31}, \ldots, \rho_{34} \ ; \ \rho_{41}, \rho_{42}, \rho_{43} \ ; \ \rho_{51}, \rho_{52} \ ; \ \rho_{61}, \rho_{62}\}$,

$E_{B_1} = \{2^9, 2^3, 1, 1, 1, 1 \ ; \ 1, 1, 1, 1 \ ; \ 1, 1, 1, 1 \ ; \ 2^3, 1, 1 \ ; \ 1, 1 \ ; \ 1, 1\}$

[5] *Held*

(1) $p=2, B_1$

$$C_{B_1}=\begin{pmatrix} 54 & 87 & 87 & 23 & 23 & 106 & 106 & 96 & 20 & 33 & 33 \\ 87 & 183 & 174 & 40 & 39 & 225 & 225 & 200 & 48 & 66 & 65 \\ 87 & 174 & 183 & 39 & 40 & 225 & 225 & 200 & 48 & 65 & 66 \\ 23 & 40 & 39 & 13 & 10 & 50 & 50 & 44 & 10 & 17 & 16 \\ 23 & 39 & 40 & 10 & 13 & 50 & 50 & 44 & 10 & 16 & 17 \\ 106 & 225 & 225 & 50 & 50 & 291 & 290 & 258 & 62 & 87 & 87 \\ 106 & 225 & 225 & 50 & 50 & 290 & 291 & 258 & 62 & 87 & 87 \\ 96 & 200 & 200 & 44 & 44 & 258 & 258 & 236 & 52 & 80 & 80 \\ 20 & 48 & 48 & 10 & 10 & 62 & 62 & 52 & 16 & 16 & 16 \\ 33 & 66 & 65 & 17 & 16 & 87 & 87 & 80 & 16 & 31 & 30 \\ 33 & 65 & 66 & 16 & 17 & 87 & 87 & 80 & 16 & 30 & 31 \end{pmatrix}$$

$l(B_1)=11,\ \ d(B_1)=10,$

$f_{B_1}(x)=(x^7-1328x^6+54487x^5-740336x^4+3658208x^3-6014592x^2+3499520x-2^{19})(x^3-13x^2+36x-16)(x-1),\quad N(\rho(B_1))=2^{19},$

$R_{B_1}=\{\rho_{11}=\rho(B_1),\ldots,\rho_{17}\ ;\ \rho_{21},\rho_{22},\rho_{23}\ ;\ 1\},$
$E_{B_1}=\{2^{10},2^3,2^3,2^3,1,1,1\ ;\ 2^2,2,2\ ;\ 1\}$

(2) $p=3, B_2$

$$C_{B_2}=\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 3 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 & 3 \end{pmatrix}, \qquad l(B_2)=7,\ \ d(B_2)=2,$$

$f_{B_2}(x)=(x-9)(x-3)(x-1)^5,\quad N(\rho(B_1))=9,$
$R_{B_2}=E_{B_2}=\{9;3;1;1;1;1;1\}$

In this case [KKW] has also proved that B_2 and its Brauer correspondent b_2 are Morita equivalent.

3 Central Extensions and Automorphism Groups of Simple Groups

(i) Suppose $\tilde{G}$ is a central extension of G by a p-subgroup $\tilde{Q} \subseteq Z(\tilde{G})$ i.e. $\tilde{G}/\tilde{Q} \simeq G$. Let $\tilde{B}$ be a p-block of $\tilde{G}$ corresponding to a p-block B of G. Then $C_{\tilde{B}} = |\tilde{Q}|C_B$. Therefore, $R_{\tilde{B}} = \{|\tilde{Q}|\rho \mid \rho \in R_B\}$ and also $E_{\tilde{B}} = \{|\tilde{Q}|e \mid e \in E_B\}$. Thus $f_B(x) = f_1(x) \cdots f_r(x)$ is the $\mathbf{Z}$-irreducble decomposition if and only if $f_{\tilde{B}}(x) = \tilde{f}_1(x) \cdots \tilde{f}_r(x)$ is the $\mathbf{Z}$-irreducible decomposition, where $\tilde{R}_i = \{|\tilde{Q}|\rho \mid \rho \in R_i\}$ is the set of roots of $\tilde{f}_i(x)$. So we can reduce our conjecture to B in this case.

(ii) Suppose also $\tilde{G}$ is a central extension of G by a p'-subgroup $\tilde{Z} \subseteq Z(\tilde{G})$ i.e. $\tilde{G}/\tilde{Z} \simeq G$. Let $\tilde{B}$ be a p-block of $\tilde{G}$ with $\tilde{Z} \subseteq \mathrm{Ker}\tilde{B}$. Then $\tilde{B}$ is 1-1 corresponding to B. But if Ker $\tilde{B}$ does not contain $\tilde{Z}$, then there is no p-block B of G such that $\pi(\tilde{B}) = B$ for the canonical epimorphism $\pi : F\tilde{G} \longrightarrow FG$.

The following is an example of this case. Here J_3 is the Janko's third simple group and $\tilde{G} = 3.J_3$ is the triple cover of J_3. We consider $\tilde{G}$ and $p = 2$. So for the principal block $\tilde{B}_1$, Ker $\tilde{B}_1$ contains $\tilde{Z}$, but Ker $\tilde{B}_6$ does not contain $\tilde{Z}$. We simply write B_1 and B_6 here, instead of $\tilde{B}_1$ and $\tilde{B}_6$. Then the decomposition of each $f_B(x)$ is similar but different.

[1] $3.J_3, p = 2$

$$C_{B_1} = \begin{pmatrix} 84 & 16 & 16 & 24 & 32 & 32 & 26 & 20 & 20 & 14 \\ & 7 & 6 & 8 & 9 & 9 & 8 & 3 & 4 & 6 \\ & & 7 & 8 & 9 & 9 & 8 & 4 & 3 & 6 \\ & & & 19 & 18 & 18 & 12 & 4 & 4 & 12 \\ & & & & 21 & 20 & 12 & 6 & 6 & 12 \\ & & & & & 21 & 12 & 6 & 6 & 12 \\ & & & & & & 13 & 6 & 6 & 7 \\ & & & & & & & 7 & 4 & 2 \\ & & & & & & & & 7 & 2 \\ & & & & & & & & & 9 \end{pmatrix}$$

$l(B_1) = 10, \quad d(B_1) = 7,$
$f_{B_1}(x) = (x^7 - 190x^6 + 5905x^5 - 48250x^4 + 133354x^3 - 129660x^2 + 4340x - 2^{11})$
$(x^2 - 4x + 2)(x - 1)$
$R_{B_1} = \{\rho_1 = \rho(B_1), \ldots, \rho_7 \ ; \ \rho_8, \rho_9 \ ; \ 1\},$
$E_{B_1} = \{2^7, 2^3, 2, 1, 1, 1, 1 \ ; \ 2, 1 \ ; \ 1\}$

$$C_{B_6} = \begin{pmatrix} 150 & 83 & 83 & 62 & 37 & 37 & 14 & 24 & 6 & 28 \\ & 51 & 48 & 34 & 22 & 21 & 8 & 12 & 5 & 16 \\ & & 51 & 34 & 21 & 22 & 8 & 12 & 5 & 16 \\ & & & 29 & 14 & 14 & 5 & 10 & 2 & 10 \\ & & & & 12 & 11 & 3 & 6 & 2 & 7 \\ & & & & & 12 & 3 & 6 & 2 & 7 \\ & & & & & & 4 & 2 & 1 & 3 \\ & & & & & & & 5 & 0 & 4 \\ & & & & & & & & 2 & 1 \\ & & & & & & & & & 7 \end{pmatrix} \quad l(B_1) = 10, \quad d(B_1) = 7,$$

$f_{B_6}(x) = (x^8 - 319x^7 + 6059x^6 - 43392x^5 + 144539x^4 - 229082x^3 + 161462x^2 - 42736x + 2^{11})(x^2 - 4x + 2)$
$R_{B_6} = \{\rho_1 = \rho(B_1), \ldots, \rho_8 \ ; \ \rho_9, \rho_{10}\}$,
$E_{B_6} = \{2^7, 2^3, 2, 1, 1, 1, 1, 1 \ ; \ 2, 1\}$

We also consider the automorphism group of a simple group. Suppose $G \triangleright H$ and $|G : H|$ is prime to p. Furthermore, suppose that B is the unique p-block of G covering a given block b of H. In general, $l(B)$ and $l(b)$ are different. But if $|G : H| = q$ (a prime number different from p), then $\rho(B) = \rho(b)$ by [KW]. So in particular, $(f_B)_1(x) = (f_b)_1(x)$. The following is an example of this case. Here $H = J_2$ is the Janko's second simple group and $G = J_2.2$ is an automorphism group of J_2 with $|G : H| = 2$ and we consider $p = 3$. If $|G : H| = p$, then $\rho(b) < \rho(B) < p\rho(b)$ by [KW], but the pattern of the roots of $(f_B)_1(x)$ and $(f_b)_1(x)$ seems to be same.

[2] $J_2.2$, $p = 3$

(1) J_2, $p = 3$,

$$C_{B_1} = \begin{pmatrix} 9 & 6 & 6 & 2 & 2 & 1 & 1 & 6 \\ 6 & 9 & 6 & 1 & 2 & 4 & 2 & 6 \\ 6 & 6 & 9 & 2 & 1 & 2 & 4 & 6 \\ 2 & 1 & 2 & 4 & 2 & 0 & 1 & 4 \\ 2 & 2 & 1 & 2 & 4 & 1 & 0 & 4 \\ 1 & 4 & 2 & 0 & 1 & 3 & 1 & 2 \\ 1 & 2 & 4 & 1 & 0 & 1 & 3 & 2 \\ 6 & 6 & 6 & 4 & 4 & 2 & 2 & 9 \end{pmatrix} \quad l(B_1) = 8, \quad d(B_1) = 3,$$

$f_{B_1}(x) = (\mathbf{x^5 - 43x^4 + 429x^3 - 1410x^2 + 1206x - 3^5})(x^3 - 7x^2 + 10x - 3)$,
$R_{B_1} = \{\rho_{11} = \rho(B_1), \ldots, \rho_{15} \ ; \ \rho_{21}, \rho_{22}, \rho_{23}\}$, $N(\rho(B_1)) = 3^5$,
$E_{B_1} = \{3^3, 3, 3, 1, 1 \ ; \ 3, 1, 1\}$

(2) $J_2.2$, $p=3$,

$$C_{B_1} = \begin{pmatrix} 6 & 6 & 6 & 2 & 1 & 3 & 3 \\ 3 & 6 & 6 & 2 & 1 & 3 & 3 \\ 6 & 6 & 15 & 3 & 6 & 6 & 6 \\ 2 & 2 & 3 & 6 & 1 & 4 & 4 \\ 1 & 1 & 6 & 1 & 4 & 2 & 2 \\ 3 & 3 & 6 & 4 & 2 & 6 & 3 \\ 3 & 3 & 6 & 4 & 2 & 3 & 6 \end{pmatrix} \qquad l(B_1)=7, \quad d(B_1)=3,$$

$f_{B_1}(x) = (\mathbf{x^5 - 43x^4 + 429x^3 - 1410x^2 + 1206x - 3^5})(x-3)^2$, $N(\rho(B_1)) = 3^5$,
$R_{B_1} = \{\rho_{11} = \rho(B_1), \ldots, \rho_{15} \ ; \ 3 \ ; \ 3\}$,
$E_{B_1} = \{3^3, 3, 3, 1, 1 \ ; \ 3 \ ; \ 3\}$

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