Blocks of Central $p$-Group Extensions of Finite Groups

This is a joint work with Naoko Kunugi. A result stated here will be published with a complete proof, see [1]. The result in [1] is, actually, inspired by a result stated in their paper [3] of Usami and Nakabayashi, where they prove our theorem for principal block algebras.

Here, we consider the following setting-up.

First of all, let $G$ and $G'$ be finite groups which have a common central $p$-subgroup $Z$ for a prime number $p$, and let $\overline{A}$ and $\overline{A}'$ respectively be $p$-blocks of $G/Z$ and $G'/Z$ induced by $p$-blocks $A$ and $A'$ respectively of $G$ and $G'$, both of which have the same defect group. Let $(\mathcal{O}, K, k)$ be a splitting $p$-modular system for all subgroups of $G$ and $G'$, that is, $\mathcal{O}$ is a complete discrete valuation ring of rank one with its quotient field $K$ of characteristic zero and with its residue field $k$ of characteristic $p$, and both $K$ and $k$ are splitting fields for all subgroups of $G$ and $G'$.

Then, we may have the following natural question. Namely,

**Question.** If $\overline{A}$ and $\overline{A}'$ are of a certain equivalence, then so are $A$ and $A'$?

Our main result is in fact the following.

**Theorem** (Koshitani-Kunugi). Keep the notation above. Assume that $G$ and $G'$ have a common subgroup $H$ satisfying $H \supseteq P \supseteq Z$ for a $p$-subgroup $P$ of $H$ and a central $p$-subgroup $Z$ of $G$ and $G'$. Let $A$ and $A'$, respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that $P$ is a defect group of $A$ and $A'$. Set $\overline{G} = G/Z$, $\overline{G}' = G'/Z$, $\overline{P} = P/Z$ and $\overline{H} = H/Z$, and let $\pi : \mathcal{O}G \rightarrow \mathcal{O}\overline{G}$ and $\pi' : \mathcal{O}G' \rightarrow \mathcal{O}\overline{G}'$ be the canonical $O$-algebra-epimorphisms induced by the canonical group-epimorphisms $G \rightarrow \overline{G}$ and
$G' \to \overline{G'}$, respectively. Write $\overline{A} = \pi(A)$ and $\overline{A'} = \pi'(A')$. Then, it is well-known that $\overline{A}$ and $\overline{A'}$, respectively, are again block algebras of $\mathcal{O}\overline{G}$ and $\mathcal{O}\overline{G'}$ such that $\overline{P}$ is a defect group of $\overline{A}$ and $\overline{A'}$.

If there is an $(\overline{A}, \overline{A'})$-bimodule $\overline{M}$ such that $\overline{A} \otimes_{\mathcal{O}\overline{H}} \overline{A'} = \overline{M} \oplus \text{(projective)}$ and $\overline{M}$ realizes a Morita equivalence between $\overline{A}$ and $\overline{A'}$, then $A$ and $A'$ are also Morita equivalent via an $(A, A')$-bimodule $M$ such that $M | A \otimes_{\mathcal{O}H} A'$.

**Remark.** Theorem above is, actually, pretty much usable to prove Broué's abelian defect group conjecture for non-principal block algebras. For instance, see [2].

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**References**

