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BROUÉ'S CONJECTURE FOR THE PRINCIPAL 5-BLOCK OF THE CHEVALLEY GROUP $G_2(4)$

YOKO USAMI (宇佐美　陽子, お茶の水女子大, 理)

Department of Mathematics, Ochanomizu University

NORIHIDE YOSHIDA (吉田　憲秀, 千葉大学, 理)

Department of Mathematics, Faculty of Science,

Chiba University

§1 Preliminaries

1.1. Let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting p-modular system for all subgroups of the considered groups, that is, $\mathcal{O}$ is a complete discrete valuation ring with unique maximal ideal $\mathcal{P}, \mathcal{K}$ is its quotient field of characteristic 0 and k is its residue field $\mathcal{O}/\mathcal{P}$ of prime characteristic p and we assume that $\mathcal{K}$ and k are both big enough to be splitting fields for all subgroups of the considered groups. The principal p-block $B_0(G)$ of a group G is the indecomposable two-sided ideal of the group ring $\mathcal{O}G$ to which the trivial module belongs. In this paper " modules" always mean finitely generated modules. They are left modules, unless stated otherwise. Given a finite-dimensional k-algebra Λ, mod-Λ denotes the category of finitely generated Λ-modules. All complexes will be cochain complexes. We write $\otimes$ to mean $\otimes_k$. For a subgroup H of a group G, let U and V be $\mathcal{O}G$-and $\mathcal{O}H$-modules, respectively. We write $\mathrm{Res}_H^G U$ or $U_{\downarrow H}$ for the restriction of U to H, namely

$$\mathrm{Res}_H^G U = U_{\downarrow H} = {}_{\mathcal{O}H}\, \mathcal{O}G \otimes_{\mathcal{O}G} U$$

and $V^{\uparrow G}$ for the induction of V to G namely

$$V^{\uparrow G} = {}_{\mathcal{O}G}\, \mathcal{O}G \otimes_{\mathcal{O}H} V.$$

We use similar notation for kG-modules and kH-modules and for ordinary characters. Let $\mathcal{O}_G$ be the trivial $\mathcal{O}G$-module and k_G be the trivial kG-module. For $\mathcal{O}G$-module V we write $\overline{V} = k \otimes_{\mathcal{O}} V$. For an $\mathcal{O}$-algebra B we write

$$\overline{B} = k \otimes_{\mathcal{O}} B,$$

1.2. Let A and B be two symmetric $\mathcal{O}$-algebras. According to Rouquier [Ro] we define two types of equivalence. The usual Morita equivalences are a special case of Rickard equivalences. For left A-module U, we denote by U^* the right A-module $\mathrm{Hom}_{\mathcal{O}}(U, \mathcal{O})$.

Definition 1.3. We say that M is an *exact*(A, B)-bimodule if it is projective as an A-module and as a right B-module.

Definition 1.4. Let $C^\bullet$ be a bounded complex of exact (A, B)-bimodules. Assume that we have isomorphisms

$$C^\bullet \otimes_B C^{\bullet *} \simeq A \oplus Z_1^\bullet \text{ as complexes of } (A, A)\text{-bimodules}$$

$$C^{\bullet *} \otimes_A C^\bullet \simeq B \oplus Z_2^\bullet \text{ as complexes of } (B, B)\text{-bimodules}$$

where A and B are viewed as complexes concentrated in degree 0 and $Z_1^\bullet$ and $Z_2^\bullet$ are homotopy equivalent to 0. Then we say that $C^\bullet$ induces a *Rickard equivalence* between A and B or that $C^\bullet$ is a *Rickard complex.*

Definition 1.5. Let $C^\bullet$ be a complex of (A, B)-bimodules. Assume that we have isomorphisms

$$C^\bullet \otimes_B C^{\bullet *} \simeq A \oplus Z_1'^\bullet \text{ as complexes of } (A, A)\text{-bimodules}$$

$$C^{\bullet *} \otimes_A C^\bullet \simeq B \oplus Z_2'^\bullet \text{ as complexes of } (B, B)\text{-bimodules}$$

where $Z_1'^\bullet$ and $Z_2'^\bullet$ are homotopy equivalent to complexes of projective bimodules. Then we say that $C^\bullet$ induces a *stable equivalence* between A and B.

§ 2 Group ring

2.1. Now we concentrate our attention on group rings. Let G be a finite group with an abelian Sylow p-subgroup P. We denote by e the block idempotent of the principal block $B_0(G)$ of $\mathcal{O}G$. Let H be a subgroup of G such that $H \supset N_G(P)$. We denote by f the block idempotent of the principal block $B_0(H)$ of $\mathcal{O}H$.

Definition 2.2. A bounded complex $C^\bullet$ of $(\mathcal{O}Hf, \mathcal{O}Ge)$-bimodules is called splendid if its components are p-permutation modules whose indecomposable summands have vertices contained in $\Delta P = \{\ (x,x) \in\ H \times G \mid x \in P\ \}$. Note that any component of a splendid complex is an exact bimodule.

Definition 2.3. Let G be a finite group with a Sylow p-subgroup P, and let $H \leq G$ be a subgroup containg P. A splendid Rickard complex for $B_0(G)$ and $B_0(H)$ is a bounded complex $X^\bullet$ of finitely generated $(B_0(H), B_0(G))$-bimodules such that

(i) $X^\bullet \otimes_{B_0(G)} X^{\bullet *}$ is chain homotopy equivalent to $B_0(H)$, considered as a complex of $B_0(H)$-bimodules,
(ii) $X^{\bullet *} \otimes_{B_0(H)} X^\bullet$ is chain homotopy equivalent to $B_0(G)$, considered as a complex of $B_0(G)$-bimodule, and
(iii) $X^\bullet$ is splendid.

In this case we say that $X^\bullet$ induces a splendid Rickard equivalence between $B_0(G)$ and $B_0(H)$. (If $X^\bullet$ is a splendid Rickard complex, then the functor

$$X^\bullet \otimes_{B(G)} ? : D^b(\text{ mod-}B_0(G)) \to D^b(\text{ mod-}B_0(H))$$

is an equivalence of triangulated categories, and $X^\bullet \otimes_{B_0(G)}$? gives an equivalence between chain homotopy categories, and not just derived categories. $D^b(\text{ mod-}B_0(G))$ is a full subcategory of $D(\text{ mod-}B_0(G))$ consisting of bounded objects, where $D(\text{ mod-}B_0(G))$ is the dirived category of the finitely generated module category of $B_0(G)$. We write them $D^b(B_0(G))$ and $D(B_0(G))$ for short.)

Conjecture 2.4. Broué's conjecture ([Br2]). Let G be a finite group with an abelian Sylow p-subgroup P. Then the principal p-block $B_0(G)$ of G and the principal p-block $B_0(N_G(P))$ of $N_G(P)$ are derived equivalent. (Moreover, they are splendidly Rickard equivalent in the refined version by Rickard.)

§ 3 Results

3.1. Broué's conjecture is known to be true for cyclic Sylow p-subgroups and for elementary abelian Sylow 3-subgroup of order 9 (see [KK]). Holloway proved that Broué's conjecture is true for some specific groups with elementary abelian Sylow 5-subgroups of order 25 in [H]. In particular, he proved it (actually, the splendid

Rickard equivalence) for the principal 5-blocks of J_2 (as well as $2.J_2$). Note that $G_2(4)$ contains a subgroup isomorphic to J_2 and these two groups have a common elementary abelian Sylow 5-subgroup P of order 25, and the common normalizer of P. We prove the splendid Rickard equivalence of the principal 5-blocks of $G_2(4)$ and J_2. See Theorem 3.2. On the other hand, the first author already proved the splendid Morita equivalence between the principal 5-blocks of some family of the Chevalley groups $G_2(2^n)$ including $G_2(4)$. See Theorem 3.3. With Holloway's work we obtain following Corollary 3.4. (In fact the normalizer of P in $G_2(2^n)$ depends on n, but the factor group by its maximal normal p'-subgroup does not depend on n.)

Theorem 3.2. (Usami, Yoshida 2003). *The principal 5-blocks of $G_2(4)$ and J_2 are splendidly Rickard equivalent.*

Theorem 3.3. (Usami [U] 2001). *Assume that*

$$5 \text{ divides } \; 2^n + 1 \text{ but } 5^2 \text{ does not divide it.} \tag{3.1}$$

Then the principal 5-blocks of $G_2(2^n)$ and the principal 5-block of $G_2(4)$ are Morita equivalent. Here a $\Delta(P)$-projective trivial source $G_2(4) \times G_2(2^n)$-module and its $\mathcal{O}$-dual induce this Morita equivalence as bimodules, where P is a common abelian Sylow 5-subgroup of $G_2(2^n)$ and $G_2(4)$ and $\Delta(P) = \{(x,x) \in G_2(4) \times G_2(2^n) | x \in P\}$.

Corollary 3.4. *Broué's conjecture is true for the principal 5-blocks of $G_2(2^n)$ with n satisfying* (3.1).

§ 4 General Methods

4.1. With G , P and H in 2.1 we proceed according to the following steps :
Step 1. Construct a local splendid Rickard complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for each nontrivial p-subgroup Q of P.
Step 2. Construct a splendid complex which induces a stable equivalence between $B_0(G)$ and $B_0(H)$.
Step 3. Construct a global splendid Rickard complex between $B_0(G)$ and $B_0(H)$.
Here we introduce a general functor (from global to local) and we also introduce a useful theorem for Step 2.

Definition 4.2. [Br1]. For an $\mathcal{O}G$-module V and any p-subgroup P of G, we set

$$\mathrm{Br}_P(V) = V^P / (\sum_Q Tr_Q^P(V^Q) + \mathcal{P}V^P) \tag{4.1}$$

where V^P denotes the set of fixed points of V under P and Q runs over all proper subgroups of P and

$$Tr_Q^P(v) = \sum_{x \in P/Q} x(v) \tag{4.2}$$

for a p-subgroup Q of P and $v \in V^Q$.

Remark 4.3. Let Q be a nontrivial p-subgroup of G. We can see Br_Q is a functor between the following categories :

$$\mathrm{Br}_Q : \{\mathcal{O}G\text{-modules}\} \rightarrow \{kN_G(Q)\text{-modules}\}$$

and then

$$\mathrm{Br}_Q : \{\text{ complexes of } \mathcal{O}G\text{-modules}\} \rightarrow \{\text{ complexes of } kN_G(Q)\text{-modules}\}.$$

Remark 4.4. With the notation in 2.1 and any nontrivial subgroup Q of P note that

$$\mathrm{Br}_{\Delta Q}(\mathcal{O}G) = kC_G(Q)$$

and

$$\mathrm{Br}_{\Delta Q}(f\mathcal{O}Ge) = \overline{f}_Q kC_G(Q)\overline{e}_Q$$

as bimodules, where $\overline{f}_Q$ and $\overline{e}_Q$ are the principal block idempotents of $kC_H(Q)$ and $kC_G(Q)$ respectively.

Theorem 4.5. (Rouquier and Bou, see Theorem 5.6 in [Ro]). *With the notation in 2.1 let $C^\bullet$ be a splendid complex of $(\mathcal{O}Hf, \mathcal{O}Ge)$-bimodules.The following assertions are equivalent.*

(i) *$C^\bullet$ induces a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.*
(ii) *For every nontrivial subgroup Q of P the complex $Br_{\Delta Q}(C^\bullet)$ induces a Rickard equivalence between $kC_G(Q)\overline{e}_Q$ and $kC_H(Q)\overline{f}_Q$.*

4.6. We can consider a direct summand of a permutation module over k as well as over $\mathcal{O}$. Then we can define a splendid complex over k similarly to Definition 2.2, and the definitions of a splendid Rickard complex and splendid Rickard equivalence still make sense if we replace the coefficient ring $\mathcal{O}$ by the field k. A splendid Rickard equivalence over $\mathcal{O}$ induces a splendid Rickard equivalence over k just by applying the functor $k\otimes_{\mathcal{O}}?$ to a splendid Rickard complex. Note that any direct summand of a permutation module and any map between such modules can be lifted from k to $\mathcal{O}$. Then by Theorem 2.8 in [Ri1] a splendid Rickard complex over k can be lifted to a splendid Rickard complex over $\mathcal{O}$ that is unique up to isomorphism. Then it is sufficient to work over k in order to prove the refined version of Broué's conjecture.

Theorem 4.7. (Rickard)(see [Ri2, Theorem 6.1] and [H, Theorem 4.4]) *Suppose that $C^\bullet$ is a complex of $(kH\overline{f}, kG\overline{e})$-bimodules that induces a splendid stable equivalence between $kG\overline{e}$ and $kH\overline{f}$ and let $\{S_1, \ldots, S_r\}$ be a set of representatives for the isomorphism classes of simple $kG\overline{e}$-modules. If there are objects $X_1^\bullet, \ldots, X_r^\bullet$ of $D^b(kH\overline{f})$ such that, for each $1 \leq i \leq r$, $X_i^\bullet$ is stably isomorphic to $C^\bullet \underset{kG\overline{e}}{\otimes} S_i$ and such that*

(a) $\mathrm{Hom}_{D^b(kH\overline{f})}(X_i^\bullet, X_j^\bullet[m]) = 0$ *for* $m < 0$,

(b) $\mathrm{Hom}_{D^b(kH\overline{f})}(X_i^\bullet, X_j^\bullet) = \begin{cases} 0 & \text{if } i \neq j, \\ k & \text{if } i = j, \end{cases}$ *and*

(c) $X_1^\bullet, \ldots, X_r^\bullet$ *generate* $D^b(kH\overline{f})$ *as a triangulated category,*

then there is a splendid Rickard complex $X^\bullet$ that lifts $C^\bullet$ and induces a splendid Rickard equivalence between $kG\overline{e}$ and $kH\overline{f}$ such that, for each $1 \leq i \leq r$, $C^\bullet \underset{kG\overline{e}}{\otimes} S_i \cong X_i^\bullet$ in $D^b(kH\overline{f})$.

§5 Steps 1 and 2 for Theorem 3.2

5.1. In this section we set

$$G = G_2(4), G \supset J \supset N_G(P) \text{ where } J \cong J_2, \tag{5.1}$$

where P is a common elementary abelian Sylow 5-subgroup of G and J of order 25. We have

$$N_G(P) = N_J(P) \cong P : D_{12}$$

, that is, a semi-direct product of P by the dihedral group D_{12} of order 12. Fusion of the subgroups of P is controlled by $N_G(P)$ and

$$\text{there are, up to conjugacy in } N_G(P)\text{, two nontrivial cyclic 5-subgroups of } P\text{, where only one, } Q \text{ has distinct centralizers in } G \text{ and } J. \tag{5.2}$$

Q is generated by a 5-element in conjugate class $5C$ in the character tables of J_2 and also of $G_2(4)$ in Atlas [CCNPW]. We fix Q from now on. We set

$$k \otimes_{\mathcal{O}} B_0(G) = k \otimes_{\mathcal{O}} \mathcal{O}Ge = kG\overline{e} \text{ and } k \otimes_{\mathcal{O}} B_0(J) = k \otimes_{\mathcal{O}} \mathcal{O}Jf = kJ\overline{f}.$$

5.2. Before we go further we review the principal 5-block of A_5. A_5 contains a subgroup isomorphic to D_{10} which is a normalizer of a fixed cyclic Sylow subgroup of order 5. As (kD_{10}, kA_5)-bimodule $\overline{B}(A_5)$ is indecomposable and its projective cover is

$$\overline{R}_0 \otimes \overline{P}_0 \;\oplus\; \overline{R}_1 \otimes \overline{P}_1 \;\rightarrow\; \overline{B}(A_5) \;\rightarrow\; 0 \tag{5.3}$$

where $\overline{P}_0$ and $\overline{R}_0$ are the projective covers of the trivial kA_5-module and the trivial kD_{10}-module, respectively, and $\overline{P}_1$ is the projective indecomposable module of the principal block of kA_5, that is not isomorphic to $\overline{P}_0$, and $\overline{R}_1$ is the unique projective indecomposable kD_{10}-module which is not isomorphic to $\overline{R}_0$. The splendid Rickard equivalence between the principal blocks of kA_5 and kD_{10} is induced by the splendid Rickard complex

$$\cdots\; 0 \rightarrow 0 \rightarrow \overline{R}_1 \otimes \overline{P}_1 \;\rightarrow \overline{B}(A_5) \rightarrow 0 \rightarrow 0 \;\cdots \tag{5.4}$$

which we can obtain by deleting the first term of (5.3). Keeping (5.4) in mind we construct a splendid Rickard complex between $kC_G(Q)\overline{e}_Q$ and $kC_J(Q)\overline{f}_Q$. See (5.6) below. Then we seek a splendid complex $C^\bullet$ which induces a stable equivalence between $B_0(G)$ and $B_0(J)$. (By Theorem 4.5 it is just to find $C^\bullet$ such that $\mathrm{Br}_{\Delta(Q)}(C^\bullet)$ is equal to (5.6).)

Lemma 5.3. *Let Q be a nontrivial subgroup of P such that Q has distinct centralizers in G and J. Then we have the following.*

(i) $C_G(Q) = Q \times A_5$ and $C_J(Q) = Q \times D_{10}$.

(ii) *Tensoring (kQ, kQ)-bimodule kQ to* (5.3) *we obtain minimal $\Delta(Q)$-projective cover of indecomposable $\overline{f}_Q kC_G(Q)\overline{e}_Q \cong kQ \otimes \overline{f}_Q kA_5\overline{e}_Q$:*

$$kQ \otimes \overline{R}_0 \otimes \overline{P}_0 \oplus kQ \otimes \overline{R}_1 \otimes \overline{P}_1 \longrightarrow kQ \otimes \overline{f}_Q kA_5\overline{e}_Q \longrightarrow 0 \tag{5.5}$$

(iii) *Deleting the first term of* (5.5) *we obtain the following splendid complex which induces the splendid Rickard equivalence between the principal blocks* $kC_G(Q)\overline{e}_Q$ *and* $kC_J(Q)\overline{f}_Q$.

$$\cdots 0 \to 0 \to kQ \otimes \overline{R}_1 \otimes \overline{P}_1 \to \overline{f}_Q kC_G(Q)\overline{e}_Q \to 0 \to 0 \cdots \tag{5.6}$$

(iv) *The following is the minimal* ΔQ*-projective cover of* $k_{\Delta Q.2}{\uparrow^{(Q\times Q).2}} \otimes \overline{f}_Q kA_5\overline{e}_Q$:

$$\begin{aligned} k_{\Delta Q.2}{\uparrow^{(Q\times Q).2}} \otimes \overline{R}_0 \otimes \overline{P}_0 \oplus k_{\Delta Q.2}{\uparrow^{(Q\times Q).2}} \otimes \overline{R}_1 \otimes \overline{P}_1 \\ \to k_{\Delta Q.2}{\uparrow^{(Q\times Q).2}} \otimes \overline{f}_Q kA_5\overline{e}_Q \to 0. \end{aligned} \tag{5.7}$$

Furthermore we have

$$k_{\Delta Q.2}{\uparrow^{(Q\times Q).2}}{\downarrow_{Q\times Q}} \cong kQ$$

and then the restriction of (5.7) *to* $C_J(Q) \times C_G(Q)$ *is* (5.5).

Lemma 5.4. (i) *There exists an exact sequence (with* M^0 *as the Scott module of* $J \times G$ *with vertex* ΔP, Scott$(J \times G, \Delta P)$)

$$\text{Scott}(J\times G\,,\,\Delta Q) \oplus M^{-1} \oplus (\text{some projective bimodule }) \longrightarrow M^0 \longrightarrow 0 \text{ (exact)} \tag{5.8}$$

such that $k \underset{\mathcal{O}}{\otimes} (5.8)$:

$$\overline{\text{Scott}}(J\times G\,,\,\Delta Q) \oplus \overline{M}^{-1} \oplus (\text{some projective bimodule }) \longrightarrow \overline{M}^0 \longrightarrow 0 \text{ (exact)}$$

is the minimal $\Delta(Q)$*-projective cover of* $\overline{M}^0$, *where* $\overline{M}^{-1}$ *is the indecomposable trivial source module with vertex* $\Delta(Q)$ *which corresponds to the second term with vertex* ΔQ *in* (5.7).

(ii) *Deleting the Scott module and the projective summand from* (5.8) *we obtain a splendid complex*

$$\cdots 0 \to 0 \to M^{-1} \longrightarrow M^0 \to 0 \to 0 \cdots$$

which induces a splendid stable equivalence between $B_0(G)$ *and* $B_0(J)$.

§6 Step 3 for Theorem 3.2

6.1. We obtain a candidate of a splendid Rickard complex between $B_0(G)$ and $B_0(J)$: (We use a perfect isometry between the sets of their ordinary characters to search some candidates.)

$$X^\bullet : \cdots 0 \longrightarrow 0 \longrightarrow \text{(a projective bimodule)} \longrightarrow \text{(a projective bimodule)} \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow 0 \longrightarrow \cdots$$

Set

$$X^\bullet \underset{\mathcal{O}Ge}{\otimes} S_i = \overline{X}^\bullet \underset{kG\overline{e}}{\otimes} S_i = X_i$$

for simple $\mathcal{O}Ge$-modules $\{S_i \mid 1 \leq i \leq 6\}$. We have only to check condidions (a), (b) and (c) in Rickard's Theorem (Theorem 4.7).

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