BROUÉ'S CONJECTURE FOR THE PRINCIPAL 5-BLOCK OF THE CHEVALLEY GROUP $G_2(4)$

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§1 Preliminaries

1.1. Let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting $p$-modular system for all subgroups of the considered groups, that is, $\mathcal{O}$ is a complete discrete valuation ring with unique maximal ideal $\mathcal{P}$, $\mathcal{K}$ is its quotient field of characteristic 0 and $k$ is its residue field $\mathcal{O}/\mathcal{P}$ of prime characteristic $p$ and we assume that $\mathcal{K}$ and $k$ are both big enough to be splitting fields for all subgroups of the considered groups. The principal $p$-block $B_0(G)$ of a group $G$ is the indecomposable two-sided ideal of the group ring $\mathcal{O}G$ to which the trivial module belongs. In this paper "modules" always mean finitely generated modules. They are left modules, unless stated otherwise. Given a finite-dimensional $k$-algebra $\Lambda$, $\text{mod-}\Lambda$ denotes the category of finitely generated $\Lambda$-modules. All complexes will be cochain complexes. We write $\otimes$ to mean $\otimes_k$. For a subgroup $H$ of a group $G$, let $U$ and $V$ be $\mathcal{O}G$-and $\mathcal{O}H$-modules, respectively. We write $\text{Res}_H^G U$ or $U_{\downarrow H}$ for the restriction of $U$ to $H$, namely

$$\text{Res}_H^G U = U_{\downarrow H} =_{\mathcal{O}H} \mathcal{O}G \otimes_{\mathcal{O}G} U$$

and $V^{\uparrow G}$ for the induction of $V$ to $G$ namely

$$V^{\uparrow G} =_{\mathcal{O}G} \mathcal{O}G \otimes_{\mathcal{O}H} V.$$
We use similar notation for $kG$-modules and $kH$-modules and for ordinary characters. Let $\mathcal{O}_G$ be the trivial $\mathcal{O}G$-module and $k_G$ be the trivial $kG$-module. For $\mathcal{O}G$-module $V$ we write $\overline{V} = k \otimes_{\mathcal{O}} V$. For an $\mathcal{O}$-algebra $B$ we write

$$\overline{B} = k \otimes_{\mathcal{O}} B,$$

1.2. Let $A$ and $B$ be two symmetric $\mathcal{O}$-algebras. According to Rouquier [Ro] we define two types of equivalence. The usual Morita equivalences are a special case of Rickard equivalences. For left $A$-module $U$, we denote by $U^*$ the right $A$-module $\mathrm{Hom}_\mathcal{O}(U, \mathcal{O})$.

**Definition 1.3.** We say that $M$ is an exact $(A, B)$-bimodule if it is projective as an $A$-module and as a right $B$-module.

**Definition 1.4.** Let $C^*$ be a bounded complex of exact $(A, B)$-bimodules. Assume that we have isomorphisms

$$C^* \otimes_B C^* \simeq A \oplus Z_1^*$$

as complexes of $(A, A)$-bimodules

$$C^* \otimes_A C^* \simeq B \oplus Z_2^*$$

as complexes of $(B, B)$-bimodules

where $A$ and $B$ are viewed as complexes concentrated in degree 0 and $Z_1^*$ and $Z_2^*$ are homotopy equivalent to 0. Then we say that $C^*$ induces a Rickard equivalence between $A$ and $B$ or that $C^*$ is a Rickard complex.

**Definition 1.5.** Let $C^*$ be a complex of $(A, B)$-bimodules. Assume that we have isomorphisms

$$C^* \otimes_B C^* \simeq A \oplus Z_1^*$$

as complexes of $(A, A)$-bimodules

$$C^* \otimes_A C^* \simeq B \oplus Z_2^*$$

as complexes of $(B, B)$-bimodules

where $Z_1^*$ and $Z_2^*$ are homotopy equivalent to complexes of projective bimodules. Then we say that $C^*$ induces a stable equivalence between $A$ and $B$.

§ 2 Group ring

2.1. Now we concentrate our attention on group rings. Let $G$ be a finite group with an abelian Sylow $p$-subgroup $P$. We denote by $e$ the block idempotent of the principal block $B_0(G)$ of $\mathcal{O}G$. Let $H$ be a subgroup of $G$ such that $H \supset N_G(P)$. We denote by $f$ the block idempotent of the principal block $B_0(H)$ of $\mathcal{O}H$. 
**Definition 2.2.** A bounded complex $C^\cdot$ of $(OH_f, OGe)$-bimodules is called splendid if its components are $p$-permutation modules whose indecomposable summands have vertices contained in $\Delta P = \{ (x, x) \in H \times G \mid x \in P \}$. Note that any component of a splendid complex is an exact bimodule.

**Definition 2.3.** Let $G$ be a finite group with a Sylow $p$-subgroup $P$, and let $H \leq G$ be a subgroup containing $P$. A splendid Rickard complex for $B_0(G)$ and $B_0(H)$ is a bounded complex $X^\cdot$ of finitely generated $(B_0(H), B_0(G))$-bimodules such that

(i) $X^\cdot \otimes_{B_0(G)} X^\cdot$ is chain homotopy equivalent to $B_0(H)$, considered as a complex of $B_0(H)$-bimodules,

(ii) $X^\cdot \otimes_{B_0(H)} X^\cdot$ is chain homotopy equivalent to $B_0(G)$, considered as a complex of $B_0(G)$-bimodule, and

(iii) $X^\cdot$ is splendid.

In this case we say that $X^\cdot$ induces a splendid Rickard equivalence between $B_0(G)$ and $B_0(H)$. (If $X^\cdot$ is a splendid Rickard complex, then the functor

$$X^\cdot \otimes_{B_0(G)} : D^b(\text{mod-} B_0(G)) \to D^b(\text{mod-} B_0(H))$$

is an equivalence of triangulated categories, and $X^\cdot \otimes_{B_0(G)}$ gives an equivalence between chain homotopy categories, and not just derived categories. $D^b(\text{mod-} B_0(G))$ is a full subcategory of $D(\text{mod-} B_0(G))$ consisting of bounded objects, where $D(\text{mod-} B_0(G))$ is the derived category of the finitely generated module category of $B_0(G)$. We write them $D^b(B_0(G))$ and $D(B_0(G))$ for short.)

**Conjecture 2.4.** Broué's conjecture ([Br2]). Let $G$ be a finite group with an abelian Sylow $p$-subgroup $P$. Then the principal $p$-block $B_0(G)$ of $G$ and the principal $p$-block $B_0(N_G(P))$ of $N_G(P)$ are derived equivalent. (Moreover, they are splendidly Rickard equivalent in the refined version by Rickard.)

### §3 Results

3.1. Broué's conjecture is known to be true for cyclic Sylow $p$-subgroups and for elementary abelian Sylow 3-subgroup of order 9 (see [KK]). Holloway proved that Broué's conjecture is true for some specific groups with elementary abelian Sylow 5-subgroups of order 25 in [H]. In particular, he proved it (actually, the splendid
Rickard equivalence) for the principal 5-blocks of $J_2$ (as well as $2.J_2$). Note that $G_2(4)$ contains a subgroup isomorphic to $J_2$ and these two groups have a common elementary abelian Sylow 5-subgroup $P$ of order 25, and the common normalizer of $P$. We prove the splendid Rickard equivalence of the principal 5-blocks of $G_2(4)$ and $J_2$. See Theorem 3.2. On the other hand, the first author already proved the splendid Morita equivalence between the principal 5-blocks of some family of the Chevalley groups $G_2(2^n)$ including $G_2(4)$. See Theorem 3.3. With Holloway's work we obtain following Corollary 3.4. (In fact the normalizer of $P$ in $G_2(2^n)$ depends on $n$, but the factor group by its maximal normal $p'$-subgroup does not depend on $n$.)

**Theorem 3.2.** (Usami, Yoshida 2003). The principal 5-blocks of $G_2(4)$ and $J_2$ are splendidly Rickard equivalent.

**Theorem 3.3.** (Usami [U] 2001). Assume that

$$5 \text{ divides } 2^n + 1 \text{ but } 5^2 \text{ does not divide it. (3.1)}$$

Then the principal 5-blocks of $G_2(2^n)$ and the principal 5-block of $G_2(4)$ are Morita equivalent. Here a $\Delta(P)$-projective trivial source $G_2(4) \times G_2(2^n)$-module and its $O$-dual induce this Morita equivalence as bimodules, where $P$ is a common abelian Sylow 5-subgroup of $G_2(2^n)$ and $G_2(4)$ and $\Delta(P) = \{(x,x) \in G_2(4) \times G_2(2^n) | x \in P\}$.

**Corollary 3.4.** Broué's conjecture is true for the principal 5-blocks of $G_2(2^n)$ with $n$ satisfying (3.1).

§ 4 General Methods

4.1. With $G$, $P$ and $H$ in 2.1 we proceed according to the following steps:
Step 1. Construct a local splendid Rickard complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for each nontrivial $p$-subgroup $Q$ of $P$.
Step 2. Construct a splendid complex which induces a stable equivalence between $B_0(G)$ and $B_0(H)$.
Step 3. Construct a global splendid Rickard complex between $B_0(G)$ and $B_0(H)$.
Here we introduce a general functor (from global to local) and we also introduce a useful theorem for Step 2.
Definition 4.2. [Br1]. For an $OG$-module $V$ and any $p$-subgroup $P$ of $G$, we set

$$Br_P(V) = V^P / (\sum_Q Tr_Q^P(V^Q) + PV^P)$$

(4.1)

where $V^P$ denotes the set of fixed points of $V$ under $P$ and $Q$ runs over all proper subgroups of $P$ and

$$Tr_Q^P(v) = \sum_{x \in P/Q} x(v)$$

(4.2)

for a $p$-subgroup $Q$ of $P$ and $v \in V^Q$.

Remark 4.3. Let $Q$ be a nontrivial $p$-subgroup of $G$. We can see $Br_Q$ is a functor between the following categories:

$$Br_Q : \{OG\text{-modules}\} \rightarrow \{kN_G(Q)\text{-modules}\}$$

and then

$$Br_Q : \{\text{complexes of } OG\text{-modules}\} \rightarrow \{\text{complexes of } kN_G(Q)\text{-modules}\}.$$

Remark 4.4. With the notation in 2.1 and any nontrivial subgroup $Q$ of $P$ note that

$$Br_{\Delta Q}(OG) = kC_G(Q)$$

and

$$Br_{\Delta Q}(fOGe) = \overline{f}_Q kC_{G}(Q) \overline{e}_Q$$

as bimodules, where $\overline{f}_Q$ and $\overline{e}_Q$ are the principal block idempotents of $kC_H(Q)$ and $kC_G(Q)$ respectively.

Theorem 4.5. (Rouquier and Bou, see Theorem 5.6 in [Ro]). With the notation in 2.1 let $C^*$ be a splendid complex of $(OHf, OGe)$-bimodules. The following assertions are equivalent.

(i) $C^*$ induces a stable equivalence between $OGe$ and $OHf$.

(ii) For every nontrivial subgroup $Q$ of $P$ the complex $Br_{\Delta Q}(C^*)$ induces a Rickard equivalence between $kC_G(Q)\overline{e}_Q$ and $kC_H(Q)\overline{f}_Q$. 
4.6. We can consider a direct summand of a permutation module over $k$ as well as over $O$. Then we can define a splendid complex over $k$ similarly to Definition 2.2, and the definitions of a splendid Rickard complex and splendid Rickard equivalence still make sense if we replace the coefficient ring $O$ by the field $k$. A splendid Rickard equivalence over $O$ induces a splendid Rickard equivalence over $k$ just by applying the functor $k \otimes_O$ to a splendid Rickard complex. Note that any direct summand of a permutation module and any map between such modules can be lifted from $k$ to $O$. Then by Theorem 2.8 in [Ri1] a splendid Rickard complex over $k$ can be lifted to a splendid Rickard complex over $O$ that is unique up to isomorphism. Then it is sufficient to work over $k$ in order to prove the refined version of Broué’s conjecture.

**Theorem 4.7. (Rickard)** (see [Ri2, Theorem 6.1] and [H, Theorem 4.4] ) Suppose that $C^*$ is a complex of $(kH\overline{f}, kG\overline{e})$-bimodules that induces a splendid stable equivalence between $kG\overline{e}$ and $kH\overline{f}$ and let $\{S_1, \ldots, S_r\}$ be a set of representatives for the isomorphism classes of simple $kG\overline{e}$-modules. If there are objects $X_i^*, \ldots, X_r^*$ of $D^b(kH\overline{f})$ such that, for each $1 \leq i \leq r$, $X_i^*$ is stably isomorphic to $C^* \otimes_{kG\overline{e}} S_i$ and such that

(a) $\text{Hom}_{D^b(kH\overline{f})}(X_i^*, X_j^*[m]) = 0$ for $m < 0$,

(b) $\text{Hom}_{D^b(kH\overline{f})}(X_i^*, X_j^*) = \begin{cases} 0 & \text{if } i \neq j, \\ k & \text{if } i = j, \end{cases}$

(c) $X_1^*, \ldots, X_r^*$ generate $D^b(kH\overline{f})$ as a triangulated category,

then there is a splendid Rickard complex $X^*$ that lifts $C^*$ and induces a splendid Rickard equivalence between $kG\overline{e}$ and $kH\overline{f}$ such that, for each $1 \leq i \leq r$, $C^* \otimes_{kG\overline{e}} X_i^* \cong X_i^*$ in $D^b(kH\overline{f})$.

§5 Steps 1 and 2 for Theorem 3.2

5.1. In this section we set

$$G = G_2(4), G \supset J \supset N_G(P) \text{ where } J \cong J_2, \quad (5.1)$$

where $P$ is a common elementary abelian Sylow 5-subgroup of $G$ and $J$ of order 25. We have

$$N_G(P) = N_J(P) \cong P : D_{12}$$
that is, a semi-direct product of $P$ by the dihedral group $D_{12}$ of order 12. Fusion of the subgroups of $P$ is controlled by $N_G(P)$ and

there are, up to conjugacy in $N_G(P)$, two nontrivial cyclic 5-subgroups of $P$, where only one, $Q$ has distinct centralizers in $G$ and $J$.  

(5.2) $Q$ is generated by a 5-element in conjugate class 5C in the character tables of $J_2$ and also of $G_2(4)$ in Atlas [CCNPW]. We fix $Q$ from now on. We set

\[
k \otimes_{O} B_{0}(G) = k \otimes_{O} OGe = kGe
\]

and

\[
k \otimes_{O} B_{0}(J) = k \otimes oOJf = kJf
\]

5.2. Before we go further we review the principal 5-block of $A_5$. $A_5$ contains a subgroup isomorphic to $D_{10}$ which is a normalizer of a fixed cyclic Sylow subgroup of order 5. As $(kD_{10}, kA_5)$-bimodule $B(A_5)$ is indecomposable and its projective cover is

\[
\overline{R}_0 \otimes \overline{P}_0 \oplus \overline{R}_1 \otimes \overline{P}_1 \rightarrow B(A_5) \rightarrow 0
\]

where $\overline{P}_0$ and $\overline{R}_0$ are the projective covers of the trivial $kA_5$-module and the trivial $kD_{10}$-module, respectively, and $\overline{P}_1$ is the projective indecomposable module of the principal block of $kA_5$, that is not isomorphic to $\overline{P}_0$, and $\overline{R}_1$ is the unique projective indecomposable $kD_{10}$-module which is not isomorphic to $\overline{R}_0$. The splendid Rickard equivalence between the principal blocks of $kA_5$ and $kD_{10}$ is induced by the splendid Rickard complex

\[
\cdots 0 \rightarrow 0 \rightarrow \overline{R}_1 \otimes \overline{P}_1 \rightarrow B(A_5) \rightarrow 0 \rightarrow 0 \quad \cdots
\]

which we can obtain by deleting the first term of (5.3). Keeping (5.4) in mind we construct a splendid Rickard complex between $kC_G(Q)\overline{e}_Q$ and $kC_J(Q)\overline{f}_Q$. See (5.6) below. Then we seek a splendid complex $C^*$ which induces a stable equivalence between $B_0(G)$ and $B_0(J)$. (By Theorem 4.5 it is just to find $C^*$ such that $Br_{\Delta(Q)}(C^*)$ is equal to (5.6).)

Lemma 5.3. Let $Q$ be a nontrivial subgroup of $P$ such that $Q$ has distinct centralizers in $G$ and $J$. Then we have the following.

(i) $C_G(Q) = Q \times A_5$ and $C_J(Q) = Q \times D_{10}$.

(ii) Tensoring $(kQ, kQ)$-bimodule $kQ$ to (5.3) we obtain minimal $\Delta(Q)$-projective cover of indecomposable $\overline{f}_QkC_G(Q)\overline{e}_Q \cong kQ \otimes \overline{f}_QkA_5\overline{e}_Q$

\[
kQ \otimes \overline{R}_0 \otimes \overline{P}_0 \oplus kQ \otimes \overline{R}_1 \otimes \overline{P}_1 \rightarrow kQ \otimes \overline{f}_QkA_5\overline{e}_Q \rightarrow 0
\]
(iii) **Deleting the first term of (5.5)** we obtain the following splendid complex which induces the splendid Rickard equivalence between the principal blocks $kC_G(Q)\overline{e}_Q$ and $kC_J(Q)\overline{f}_Q$:

$$
\cdots 0 \to 0 \to kQ \otimes R_1 \otimes \overline{P}_1 \to \overline{f}_Q kC_G(Q)\overline{e}_Q \to 0 \to 0 \cdots
$$

(5.6)

(iv) **The following is the minimal $\Delta Q$-projective cover of $k_{\Delta Q.2}^{\uparrow(Q \times Q).2} \otimes \overline{f}_Q kA_5\overline{e}_Q$**:

$$
k_{\Delta Q.2}^{\uparrow(Q \times Q).2} \otimes \overline{R}_0 \otimes \overline{P}_0 \oplus k_{\Delta Q.2}^{\uparrow(Q \times Q).2} \otimes \overline{R}_1 \otimes \overline{P}_1
\to
k_{\Delta Q.2}^{\uparrow(Q \times Q).2} \otimes \overline{f}_Q kA_5\overline{e}_Q \to 0.
$$

(5.7)

Furthermore we have

$$
k_{\Delta Q.2}^{\uparrow(Q \times Q).2} \downarrow_{Q \times Q} \cong kQ
$$

and then the restriction of (5.7) to $C_J(Q) \times C_G(Q)$ is (5.5).

**Lemma 5.4.**

(i) There exists an exact sequence (with $M^0$ as the Scott module of $J \times G$ with vertex $\Delta P$, Scott$(J \times G, \Delta P)$)

$$
\text{Scott}(J \times G, \Delta Q) \oplus M^{-1} \oplus \text{(some projective bimodule)} \to M^0 \to 0 \quad \text{(exact)}
$$

(5.8)

such that $k \otimes (5.8) :$

$$
\overline{\text{Scott}}(J \times G, \Delta Q) \oplus \overline{M}^{-1} \oplus \text{(some projective bimodule)} \to \overline{M}^0 \to 0 \quad \text{(exact)}
$$

is the minimal $\Delta(Q)$-projective cover of $\overline{M}^0$, where $\overline{M}^{-1}$ is the indecomposable trivial source module with vertex $\Delta(Q)$ which corresponds to the second term with vertex $\Delta Q$ in (5.7).

(ii) **Deleting the Scott module and the projective summand from (5.8) we obtain a splendid complex**

$$
\cdots 0 \to 0 \to M^{-1} \to M^0 \to 0 \to 0 \cdots
$$

which induces a splendid stable equivalence between $B_0(G)$ and $B_0(J)$.  

§ 6 Step 3 for Theorem 3.2

6.1. We obtain a candidate of a splendid Rickard complex between $B_0(G)$ and $B_0(J)$: (We use a perfect isometry between the sets of their ordinary characters to search some candidates.)

\[ X^*: \cdots 0 \rightarrow 0 \rightarrow (\text{a projective bimodule}) \rightarrow (\text{a projective bimodule}) \rightarrow M^{-1} \rightarrow M^0 \rightarrow 0 \rightarrow \cdots \]

Set

\[ X^* \otimes_{\mathcal{O}Ge} S_i = X^* \otimes_{k\overline{G}} S_i = X_i \]

for simple $\mathcal{O}Ge$-modules $\{S_i \mid 1 \leq i \leq 6\}$. We have only to check conditions (a), (b) and (c) in Rickard's Theorem (Theorem 4.7).

参考文献


[U] Y.Usami, Morita equivalent principal 5-blocks of the Chavalley groups $G_2(2^n)$, preprint, Ochanomizu University.

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