数理解析研究所講究録 1360 巻 2004 年 88-104

An application of proper forcings with models as side conditions

Teruyuki Yorioka *
Graduate School of Science and Technology
Kobe University
Rokkodai, Nada-ku, Kobe 657-8501, Japan
yorioka@kurt.scitec.kobe-u.ac.jp

October 29, 2003

Abstract

The method in the title has been introduced by Stevo Todorčević. In this note, we give one application of this method, i.e. we show that it is consistent that Martin's Axiom holds and there exist $(\mathfrak{c},\mathfrak{c})$-gaps but no $(\omega_1,\mathfrak{c})$-gaps.

1 Introduction

Proper forcing notions have been introduced by Saharon Shelah. These are very useful forcing notions to lead consistency results. Forcing notions in the title is one of types of proper forcing notions which has been introduced by Stevo Todorčvić ([19]). A condition of a forcing notion of this type consists of two parts: a working part D and a side part $\mathcal{N}$ which is a finite $\in$-chain of countable elementary submodels of some large enough structure $H(\theta)$. To define such a forcing notion, we always require that $\mathcal{N}$ *separates* D, i.e.

$$\forall x \neq y \in D \exists N \in \mathcal{N}(\{x,y\} \cap N \text{ has exactly one element}).$$

(See also [10].) Todorčević used the method to show that the conjecture (S) is true under the Proper Forcing Axiom. Zapletal also applied it to study a strongly almost disjoint family ([24]).

The topic of this note is gaps in $\mathcal{P}(\omega)/\mathrm{fin}$, in particular specific types of gaps in $\mathcal{P}(\omega)/\mathrm{fin}$, namely $(\omega_1,\mathfrak{c})$-gaps and $(\mathfrak{c},\mathfrak{c})$-gaps (where $\mathfrak{c}$ is the size of the continuum). The subject of gaps in $\mathcal{P}(\omega)/\mathrm{fin}$ has been investigated by many

*The author acknowledges support received from the Fields Institute and from the Kobe Technical Club to participate in the Thematic Program on Set Theory and Analysis, from Sep.1 to Dec.31, 2002.

mathematicians for a long time. We already know many ZFC results and many consistency results about gaps. It is one of the classical results that there exist some types of gaps: (ω_1, ω_1)-gaps and $(\omega, \mathfrak{b})$-gaps (where $\mathfrak{b}$ is the (un)bounding number). (These are due to Hausdorff [7] and Rothberger [14], see also [15].) Concerning the existence of (κ, λ)-gaps, we know that it is consistent with ZFC that if there exists a (κ, λ)-gap where κ and λ are regular cardinals with $\kappa \leq \lambda$, then either ($\kappa = \omega$ and $\lambda = \mathfrak{c}$) or $\kappa = \lambda = \omega_1$. In this paper, we give one result of the existence of (κ, λ)-gaps under *Martin's Axiom* (MA) for regular cardinals κ and λ. This subject has also been studied in the past.

It is one of the classical results that any (ω, ω)-pregap is separated. And if κ and λ are regular cardinals so that κ or λ is not ω_1, then for any (κ, λ)-gap $(\mathcal{A}, \mathcal{B})$ there is a ccc forcing notion which forces that $(\mathcal{A}, \mathcal{B})$ is separated (see [13], [15]). (So any such gap is not indestructible.) Therefore under MA if $(\mathcal{A}, \mathcal{B})$ is an (κ, λ)-gap, then $\kappa = \lambda = \omega_1$ or $(\kappa = \mathfrak{c} \vee \lambda = \mathfrak{c})$ holds. In fact, we know that there is a $(\omega, \mathfrak{b})$-gap (see also [15], [20]), so under MA there always exists an $(\omega, \mathfrak{c})$-gap. In [7], Hausdorff has proved that there always exists an (ω_1, ω_1)-gap. In particular, his proof gives that there exists an indestructible (ω_1, ω_1)-gap, hence under MA, there exists an (ω_1, ω_1)-gap. (In fact under MA, every (ω_1, ω_1)-gap is indestructible.) So the remaining problems are about the existence of $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps under MA.

In [11], Kunen has proved that the following statements are consistent with ZFC:

1. MA $+\exists(\mathfrak{c}, \mathfrak{c})$-gaps $+\ \exists(\omega_1, \mathfrak{c})$-gaps, and
2. MA $+\neg\exists(\mathfrak{c}, \mathfrak{c})$-gaps $+\ \neg\exists(\omega_1, \mathfrak{c})$-gaps.

In this note, we see the outline of a proof of the following theorem, which answers a problem addressed in [15], using forcing notions with models as side conditions. For the detail proof, see [23].

Theorem 1.1 ([23], Theorem 1.2). *If* PFA *is consistent, then it is also consistent that* MA *holds and there exist* $(\mathfrak{c}, \mathfrak{c})$*-gaps but no* $(\omega_1, \mathfrak{c})$*-gaps.*

Following definitions are needed to explain the above theorem and its proof.

Definition and Notation 1.2. *Let a and b be elements of $\mathcal{P}(\omega)$, and $\mathcal{A}$ and $\mathcal{B}$ subsets of $\mathcal{P}(\omega)$.*

1. *$a \perp b$ denotes that $a \cap b$ is finite.*
2. *$a \subseteq^* b$ denotes that $a \setminus b$ is finite.*
3. *$\mathcal{A}^{\perp} := \{c \subseteq \omega; \forall a \in \mathcal{A}(a \perp c)\}$.*
4. *$(\mathcal{A}, \mathcal{B})$ is called a pregap if for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $a \perp b$, i.e. $\mathcal{B} \subseteq \mathcal{A}^{\perp}$.*
5. *A pregap $(\mathcal{A}, \mathcal{B})$ is separated if there is $c \in \mathcal{A}^{\perp}$ such that $b \subseteq^* c$ for all $b \in \mathcal{B}$.*

6. *A pregap $(\mathcal{A}, \mathcal{B})$ is countably separated if there is a sequence $\langle c_n; n \in \omega \rangle$ of elements of $\mathcal{P}(\omega)$ such that for all $\langle a, b \rangle \in \mathcal{A} \times \mathcal{B}$ there is an $n \in \omega$ with $a \perp c_n$ and $b \subseteq^* c_n$.*

7. *$(\mathcal{A}, \mathcal{B})$ is called a gap if it is a pregap and not separated.*

8. *If $\mathrm{ot}(\mathcal{A}, \subseteq^*) = \kappa$ and $\mathrm{ot}(\mathcal{B}, \subseteq^*) = \lambda$, then $(\mathcal{A}, \mathcal{B})$ is called a (κ, λ)-pregap (or a (κ, λ)-gap) if it is a pregap (or gap).*

9. *An (ω_1, ω_1)-gap $(\mathcal{A}, \mathcal{B})$ is called indestructible if for every forcing extension in which cardinalities are preserved, $(\mathcal{A}, \mathcal{B})$ is still a gap.*

10. *For a collection $\mathcal{P}$ of forcing notions and a cardinal $\kappa \leq \mathfrak{c}$, $\mathrm{MA}_\kappa(\mathcal{P})$ means that for every $\mathbb{P} \in \mathcal{P}$ and κ many dense sets $\{\mathcal{D}_\alpha; \alpha < \kappa\}$, there exists a filter G in $\mathbb{P}$ which meets all $\mathcal{D}_\alpha$.*

11. *Martin's Axiom (MA) is the statement $\mathrm{MA}_{<\mathfrak{c}}(\mathrm{ccc})$.*

    The Proper Forcing Axiom (PFA) is the statement $\mathrm{MA}_{\aleph_1}(\mathrm{proper})$.

12. *For a collection $\mathcal{P}$ of forcing notions, we denote by $\mathfrak{m}(\mathcal{P})$ the least cardinal κ so that $\mathrm{MA}_\kappa(\mathcal{P})$ fails. If $\mathcal{P}$ is a singleton $\{\mathbb{P}\}$, then $\mathfrak{m}(\mathcal{P})$ is denoted by $\mathfrak{m}(\mathbb{P})$.*

1.1 Preparation

Definition 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{P}(\omega)$.*

1. $\mathcal{B}^+ := \{c \subseteq \omega; \exists b \in \mathcal{B}(c \subseteq^* b)\}$.

2. $\mathcal{A} \otimes \mathcal{B} := \{\langle a, b \rangle \in \mathcal{A} \times \mathcal{B}; a \cap b = \emptyset\}$.

3. (Todorčević [19]) *Coloring:* $[\mathcal{A} \otimes \mathcal{B}]^2 = K_0 \,\dot\cup\, K_1$, *where*

$$\{\langle a, b \rangle, \langle a', b' \rangle\} \in K_0 :\iff (a \cap b') \cup (a' \cap b) \neq \emptyset.$$

For $X, Y \subseteq \mathcal{A} \otimes \mathcal{B}$, we write $X \star Y := \{\{x, y\} \in [\mathcal{A} \otimes \mathcal{B}]^2; x \in X \;\&\; y \in Y \;\&\; x \neq y\}$.

For $X \subseteq \mathcal{A} \otimes \mathcal{B}$ and $i = 0$ or 1, X is called K_i-homogeneous if $X \star X \subseteq K_i$.

$\mathcal{P}(\omega)$ is identified with the Cantor space. Now we fix a linear order $<_{\mathcal{P}(\omega)}$ in $\mathcal{P}(\omega)$ and then we identify $[\mathcal{A} \otimes \mathcal{B}]^2$ with the topological space $\{\langle a, b \rangle \in \mathcal{P}(\omega) \times \mathcal{P}(\omega); a <_{\mathcal{P}(\omega)} b\}$. Then we notice that $[\mathcal{A} \otimes \mathcal{B}]^2 = (\mathcal{A} \times \mathcal{B} \cup \mathcal{B} \times \mathcal{A}) \cap \{\langle a, b \rangle \in \mathcal{P}(\omega) \times \mathcal{P}(\omega); a <_{\mathcal{P}(\omega)} b\}$ and K_0 is open in this topology. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$, $\mathcal{A}$ is called σ-directed if for every countable subset X of $\mathcal{A}$, there is $a \in \mathcal{A}$ so that for all $x \in X$, $x \subseteq^* a$. The following propositions are well-known.

Proposition 1.4. (Folklore, [5]) *If both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed, $(\mathcal{A}, \mathcal{B})$ is separated iff $(\mathcal{A}, \mathcal{B})$ is countably separated.*

Proposition 1.5. (Folklore, [5]) *Let $(\mathcal{A}, \mathcal{B})$ is a pregap. Then $\mathcal{A}\otimes\mathcal{B}^+$ is a union of countably many K_1-homogeneous subsets iff $(\mathcal{A}, \mathcal{B})$ is countably separated.*

Proposition 1.6. (Kunen [11], see also [15], [19] or [20]) *Let $(\mathcal{A}, \mathcal{B})$ is an (ω_1, ω_1)-pregap. Suppose that $\{a_\alpha; \alpha < \omega_1\} \subseteq \mathcal{A}$, $\{b_\alpha; \alpha < \omega_1\} \subseteq \mathcal{B}$, $a_\alpha \cap b_\alpha = \emptyset$ for all $\alpha < \omega_1$ and $\{\langle a_\alpha, b_\alpha\rangle; \alpha < \omega_1\}$ is K_0-homogeneous. Then $(\{a_\alpha; \alpha < \omega_1\}, \{b_\alpha; \alpha < \omega_1\})$ forms a gap and is indestructible, i.e. still forms a gap in any extension with a forcing doesn't collapse $\aleph_1$.*

2 A proof of the theorem

Suppose that PFA holds in the ground model $\mathbf{V}$. Then $\mathfrak{c} = \aleph_2$ and there is a decreasing sequence $\langle X_\alpha; \alpha < \omega_2\rangle$ of elements of $\mathcal{P}(\omega)$ which is a generator of an ultrafilter, i.e.

1. $\forall \alpha < \beta < \omega_2 (X_\beta \subseteq^* X_\alpha)$
2. $\forall Y \subseteq \omega \exists \alpha < \omega_1 (X_\alpha \subseteq^* Y \vee X_\alpha \subseteq^* \omega \setminus Y)$

Let $\mathcal{U}$ be the ultrafilter generated by $\langle X_\alpha; \alpha < \omega_2\rangle$, and $\mathcal{U}^*$ the dual ideal of $\mathcal{U}$. We define a forcing notion $\mathbb{P}(\mathcal{U})(= \mathbb{P}) := \bigcup_{X\in\mathcal{U}^*} 2^X$, for conditions f, g in $\mathbb{P}$ $f \leq_{\mathbb{P}} g$ iff $g \subseteq^* f$. And we let $\mathbb{P}'(\mathcal{U})(= \mathbb{P}') := \bigcup_{X\in\mathcal{U}^*} 2^X$ be Grigorieff forcing ([6]), i.e. for conditions f, g in $\mathbb{P}'$, $f \leq_{\mathbb{P}'} g$ iff $g \subseteq f$. ($\mathbb{P}$ and $\mathbb{P}'$ have the same underlying set. The only difference is the ordering, but $\mathbf{1}(= \emptyset)$ is the strongest condition in both $\mathbb{P}$ and $\mathbb{P}'$.) We must note that $\mathbb{P}'(\mathcal{U})$ is proper if $\mathcal{U}$ is a fat p-filter (by Shelah, see [16]). Now, since $\mathcal{U}$ satisfies the properties of fat-ness and p-filter, $\mathbb{P}'(\mathcal{U})$ is a proper forcing notion.

The following proposition is very similar to [21] and [22].

Proposition 2.1. *$\mathbb{P}$ is σ-closed, ω_2-Baire and adds an (ω_2, ω_2)-gap (under* PFA*).*

Proof. For the first two statements, see [21]. (Since the length of the generating sequence of $\mathcal{U}$ is ω_2, if $\mathfrak{m}(\mathbb{P}') = \aleph_2$, then it follows that $\mathbb{P}$ is ω_2-Baire.)

For the last statement, let G be a $\mathbb{P}$-generic filter over $\mathbf{V}$. Then we may take a condition $f_\alpha \in G \cap 2^{X_\alpha}$ for every $\alpha < \omega_2$, and let $a_\alpha := \{n \in X_\alpha; f_\alpha(n) = 0\}$ and $b_\alpha := \{n \in X_\alpha; f_\alpha(n) = 1\}$. Then it is trivial that $(\{a_\alpha; \alpha < \omega_2\}, \{b_\alpha; \alpha < \omega_2\})$ is an (ω_2, ω_2)-gap by the genericity. $\square$

Therefore in the extension with $\mathbb{P}$ over $\mathbf{V}$, there are no new reals and MA holds. So to finish the proof, we have only to show that $\mathbb{P}$ adds no (ω_1, ω_2)-gaps (under PFA). To prove this, we will use the method in the title.

2.1 A proof of non-existence of (ω_1, ω_2)-gaps in the extension with $\mathbb{P}$ over $\mathbf{V}$

Assume that in the extension with $\mathbb{P}$, there exists an (ω_1, ω_2)-gap, whose $\mathbb{P}$-name is $(\dot{\mathcal{A}}, \dot{\mathcal{B}})$, i.e.

$$\Vdash_{\mathbb{P}} \text{“ } (\dot{\mathcal{A}}, \dot{\mathcal{B}}) \text{ is an } (\omega_1, \omega_2)\text{-gap”}.$$

Since $\mathbb{P}$ is ω_2-Baire in $\mathbf{V}$, there are $\mathcal{A} \in \mathbf{V}$ and $f \in \mathbb{P}$ such that $f \Vdash_{\mathbb{P}}$" $\dot{\mathcal{A}} = \check{\mathcal{A}}$ ". So by the homogeneity of $\mathbb{P}$, without loss of generality, we may assume that $\Vdash_{\mathbb{P}}$" $\dot{\mathcal{A}} = \check{\mathcal{A}}$ ".

We recall that

$$\Vdash_{\mathbb{P}} \text{“ } \dot{\mathcal{B}}^{+} = \{c \subseteq \omega; \exists b \in \dot{\mathcal{B}}(c \subseteq^{*} b)\} \text{ ”.}$$

Then

$$\Vdash_{\mathbb{P}} \text{“ } (\check{\mathcal{A}}, \dot{\mathcal{B}}^{+}) \text{ also forms a gap ”.}$$

In $\mathbf{V}$, for all $f \in \mathbb{P}$, let $\mathcal{B}^{+}(f) := \{b \subseteq \omega; \exists g \leq_{\mathbb{P}} f(g \Vdash_{\mathbb{P}} \text{“ } \check{b} \in \dot{\mathcal{B}}^{+} \text{ ”})\}$. It is trivial that for conditions f, g in $\mathbb{P}$, $f \Vdash_{\mathbb{P}}$" $\check{\mathcal{B}}^{+}(f) \supseteq \dot{\mathcal{B}}^{+}$ " and if $f \leq_{\mathbb{P}} g$, then $\mathcal{B}^{+}(f) \subseteq \mathcal{B}^{+}(g)$. The next proposition is used to show Lemma 2.6.

Proposition 2.2 ([23], Proposition 2.7). *For every $f \in \mathbb{P}$, $\mathcal{A} \otimes \mathcal{B}^{+}(f)$ is not a union of countably many K_1-homogeneous subsets.*

We will find $\tilde{f} \in \mathbb{P}$ and $\mathcal{X} \subseteq \mathcal{A} \otimes \mathcal{B}^{+}(\mathbf{1})$ such that

1. $\mathcal{X}$ is uncountable and K_0-homogeneous, and
2. for all $\langle a, b\rangle \in \mathcal{X}$, $\tilde{f} \Vdash_{\mathbb{P}}$" $\check{b} \in \dot{\mathcal{B}}^{+}$ ",

which completes the proof, because then

$$\tilde{f} \Vdash_{\mathbb{P}} \text{“ } \check{\mathcal{X}} \text{ forms an } (\omega_1, \omega_1)\text{-indestructible gap in } (\check{\mathcal{A}}, \dot{\mathcal{B}}^{+}) \text{ ”,}$$

which is a contradiction.

In fact we can get an uncountable K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^{+}(\mathbf{1})$ applying OCA. But now we need the condition $\tilde{f}$ as above to get a contradiction. To get the desired objects, we consider the extension by the following forcing notion $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$. This is an example of a forcing notion with models as side conditions.

2.2 An example of proper forcings with models as side conditions

Definition 2.3. *A condition of $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ is a triple $p = \langle f_p, X_p, \mathfrak{N}_p\rangle$ satisfying the following statements:*

(a) *f_p is a member of $\bigcup_{X \in \mathcal{U}^*} 2^X$,*

(b) *X_p is a finite K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^{+}(\mathbf{1})$,*

(c) *$\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\mathfrak{c}^{+})(= H(\aleph_3))$ containing everything we need for our discussion, e.g. $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, etc $\cdots$ (i.e. $\mathfrak{N}_p$ can be enumerated by $\{N_i; i < n\}$ such that for all $i < n-1$, $N_i \in N_{i+1}$ and N_i is an elementary submodel of N_{i+1} (say $N_i \prec N_{i+1}$)),*

(d) *for any* $x\ (= \langle a_x, b_x\rangle) \in X_p$, $f_p \Vdash_{\mathbb{P}}$ "$\check{b}_x \in \dot{\mathcal{B}}^+$ ",

(e) *for any* $x, y \in X_p$ *with* $x \neq y$ *there exists* $N \in \mathfrak{N}_p$ *so that* $|N \cap \{x, y\}| = 1$, *(define* $x \lhd y :\Longleftrightarrow \exists N \in \mathfrak{N}_p(x \in N \&\ y \notin N)$*),*

(f) *for all* $N \in \mathfrak{N}_p$, f_p *is* $(N, \mathbb{P}')$*-generic, and*

(g) *for every* $x \in X_p$ *and* $N \in \mathfrak{N}_p$ *with* $x \notin N$,

$$f_p \Vdash_{\mathbb{P}'} \text{“}\forall Y \in \check{N}[\dot{G}](Y \subseteq \check{\mathcal{A}} \otimes \mathcal{B}^+\check{(}\mathbf{1}) \ \&\ Y \star Y \subseteq K_1 \Rightarrow \check{x} \notin Y)\text{”}.$$

For conditions $p, q \in \mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$,

$$p \leq_{\mathbb{Q}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})} q :\Longleftrightarrow f_p \supseteq f_q \ (i.e.\ f_p \leq_{\mathbb{P}'} f_q) \ \&\ X_p \supseteq X_q \ \&\ \mathfrak{N}_p \supseteq \mathfrak{N}_q.$$

We note that $\mathfrak{N}_p$ is an element of $H(\aleph_3)$ because every element of $\mathfrak{N}_p$ is a countable subset of $H(\aleph_3)$ and $\mathfrak{N}_p$ is finite. We must show that $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ is proper and adds desired objects. To apply the PFA, we need to show the following lemma.

Lemma 2.4. *For* $\mathcal{A}$, $\dot{\mathcal{B}}$ *and* $\mathcal{U}$, $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ *is proper.*

Proof. Let θ be a large enough regular cardinal, $M \prec H(\theta)$ a countable elementary submodel containing everything needed for our discussion, e.g. $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, $H(\aleph_3)$ etc, and $p = \langle f_p, X_p, \mathfrak{N}_p\rangle \in M$ a condition of $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})(= \mathbb{Q})$.

Since $\mathbb{P}'$ is proper, we can choose an extension $f_q \leq_{\mathbb{P}'} f_p$ such that f_q is $(M \cap H(\aleph_3), \mathbb{P}')$-generic. (We note that $M \cap H(\aleph_3)$ is an elementary submodel of $H(\aleph_3)$.) Then let $q := \langle f_q, X_p, \mathfrak{N}_p \cup \{M \cap H(\aleph_3)\}\rangle$, which is a condition of $\mathbb{Q}$. (We note that $\mathfrak{N}_p \subseteq M \cap H(\aleph_3)$, in fact $\mathfrak{N}_p \in M \cap H(\aleph_3)$ holds since $p \in M$ and $\mathfrak{N}_p \in H(\aleph_3)$.) Show that q is $(M, \mathbb{Q})$-generic, i.e. for every dense open subset $\mathcal{D} \in M$ in $\mathbb{Q}$ and an extension $r \in \mathbb{Q}$ of q there exists a condition $s \in \mathcal{D} \cap M$ such that r and s are compatible in $\mathbb{Q}$ (i.e. $\mathcal{D} \cap M$ is predense in $\mathbb{Q}$ below q).

Taking such $\mathcal{D} \in M$ and $r \leq_{\mathbb{Q}} q$, without loss of generality, we may assume that r is in $\mathcal{D}$. Let $X_r \setminus M = \{x_i; i \leq n\}$ where $x_i \lhd x_{i+1}$ for $i < n$ and $N_0 := M \cap H(\aleph_3)$, and pick $N_i \in \mathfrak{N}_r$ such that $x_{i-1} \in N_i$ but $x_i \notin N_i$ for $1 \leq i \leq n$. We choose rational open intervals $U_i \subseteq \mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$ such that

- $x_i \in U_i$ for $i \leq n$,
- $U_i \cap U_j = \emptyset$ and $U_i \star U_j \subseteq K_0$ for every $i, j \leq n$ with $i \neq j$.

(We recall that K_0 is open, so this can be done.) We note that all rational open intervals are in any model of ZFC because those codes consists of finite elements. Let G be $\mathbb{P}'$-generic over $H(\theta)$ with $f_r \in G$.

Claim 2.5. *In* $H(\theta)[G]$, *there are rational open intervals* $V_i^0, V_i^1 \subseteq U_i$ *and* $y_i \in V_i^1 \cap \mathbf{V}$ *for* $i \leq n$ *and* $s \in \mathcal{D} \cap \mathbf{V}$ *such that*

1. $x_i \in V_i^0$ *for all* $i \leq n$,

2. $V_i^0 \cap V_i^1 = \emptyset$ *and* $V_i^0 \star V_i^1 \subseteq K_0$ *for all* $i \leq n$,

3. $f_s \in G$,

4. $X_s = (X_r \cap M) \cup \{y_i; i \leq n\}$ *and for any* $x \in X_r \cap M$ *and* $i < j \leq n$, *and* $x \lhd y_i \lhd y_j$,

5. $\mathfrak{N}_s$ *is an end extension of* $\mathfrak{N}_r \cap M$.

Proof. By induction on $i \leq n$, we construct rational open intervals $V_{n-i}^0, V_{n-i}^1 \subseteq U_{n-i}$, $y_{n-i}^{n-j} \in V_{n-j}^1 \cap \mathbf{V}$ for $j \leq i$ and $s_{n-j} \in \mathcal{D} \cap \mathbf{V}$ such that

1'. $x_{n-i} \in V_{n-i}^0$,

2'. $V_{n-i}^0 \cap V_{n-i}^1 = \emptyset$ and $V_{n-i}^0 \star V_{n-i}^1 \subseteq K_0$,

3'. $f_{s_{n-i}} \in G$,

4'. $X_{s_{n-i}} = (X_{s_{n-i}} \cap N_{n-i}) \cup \{y_{n-i}^{n-j}; j \leq i\}$ and for any $x \in X_{s_{n-i}} \cap N_{n-i}$ and $j < k \leq n$, $x \lhd y_{n-i}^{n-k} \lhd y_{n-i}^{n-j}$, and

5'. $\mathfrak{N}_{s_{n-i}}$ is an end extension of $\mathfrak{N}_{s_{n-i+1}} \cap N_{n-i}$.

Construction Assume that we have already constructed $V_{n-i}^0, V_{n-i}^1, y_{n-i}^{n-j}, s_{n-j}$ for all $j < i$.

Let

$$Y_{n-i} := \{x \in U_{n-i} \cap \mathbf{V};\ \exists z_n \in V_n^1 \cap \mathbf{V} \cdots \exists z_{n-i+1}^1 \in V_{n-i+1}^1 \cap \mathbf{V} \exists s \in \mathcal{D} \cap \mathbf{V} \text{ s.t.}$$

- $f_s \in G$
- $X_s = (X_{s_{n-i+1}} \cap N_{n-i}) \cup \{x\} \cup \{z_{n-j}; j < i\}$
- $\forall z \in X_{s_{n-i+1}} \cap N_{n-i} \forall j < k \leq i (z \lhd x \lhd z_{n-k} \lhd z_{n-j})$
- $\mathfrak{N}_s$ is an end extension of $\mathfrak{N}_{s_{n-i+1}} \cap N_{n-i}$ $\}$.

Then $Y_{n-i} \in N_{n-i}[G]$ and $x_{n-i} \in Y_{n-i}$ by *3'*, *4'* and *5'*. Since $x_{n-i} \in Y_{n-i}$, by (**g**), Y_{n-i} is not K_1-homogeneous. Let $\overline{Y_{n-i}} := \{x \in Y_{n-i}; \exists y \in Y_{n-i} \setminus \{x\}(\{x,y\} \in K_0)\}$. Then $Y_{n-i} \setminus \overline{Y_{n-i}}$ is in $N_{n-i}[G]$ and K_1-homogeneous, hence x_{n-i} belongs to $\overline{Y_{n-i}}$ by (**g**) again. Therefore there exists $y_{n-i}^{n-i} \in Y_{n-i} \setminus \{x_{n-i}\}$ such that $\{x_{n-i}, y_{n-i}^{n-i}\}$ is in K_0. Then we take rational open intervals $V_{n-i}^0, V_{n-i}^1 \subseteq U_{n-i}$ such that $x_{n-i} \in V_{n-i}^0$, $V_{n-i}^0 \cap V_{n-i}^1 = \emptyset$ and $V_{n-i}^0 \star V_{n-i}^1 \subseteq K_0$. By $y_{n-i}^{n-i} \in Y_{n-i}$, there are $y_{n-i}^n \in V_n^1 \cap \mathbf{V}, \cdots, y_{n-i}^{n-i+1} \in V_{n-i+1}^1 \cap \mathbf{V}$ and $s_{n-i} \in \mathcal{D} \cap \mathbf{V}$ satisfying *3'*, *4'* and *5'*, which completes a construction. Put $y_i := y_0^i$ for $i \leq n$ and $s := s_0$, then these are as desired. $\dashv$

Since M is an elementary submodel of $H(\theta)$, $M[G]$ is an elementary submodel of $H(\theta)[G]$. So by the previous claim, there are $y_i \in V_i^1 \cap M[G] \cap \mathbf{V}$ for $i \leq n$ and $s \in \mathcal{D} \cap M[G] \cap \mathbf{V}$ satisfying *3*, *4* and *5* of the claim. Then we take a condition $g \in G$ which decides all values of V_i^0, V_i^1, y_i for all $i \leq n$ and s. By the separability of $\mathbb{P}'$, g is an extension of f_s in $\mathbb{P}'$. We may assume that $g \leq_{\mathbb{P}'} f_r$ because both g and f_r are in a filter G. Then we note that g is also a common extension of f_r and f_s in $\mathbb{P}$. By the construction, $\langle g, X_s \cup X_r, \mathfrak{N}_r \cup \mathfrak{N}_s \rangle$ is a condition of $\mathbb{Q}$ and a common extension of r and s. $\square$

2.3 The end of the proof of the theorem

To get $\tilde{f}$ and $\mathcal{X}$, we take any countable elementary submodel M of $H(\theta)$ containing $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, $H(\aleph_3)$, etc. Let $M_0 := M \cap H(\aleph_3)$ and pick a $(M_0, \mathbb{P}')$-generic condition $f \in \mathbb{P}'$. We notice that $\mathcal{P}(\omega) \cap M_0 = \mathcal{P}(\omega) \cap M$. Now we have the following lemma:

Lemma 2.6 ([23], Lemma 2.9). *Under $\mathfrak{m}(\mathbb{P}') = \mathfrak{c} = \aleph_2$ (in particular under* PFA*),*

$\Vdash_{\mathbb{P}'}$ " $\check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(\mathbf{1})$ *is not a union of countably many K_1-homogeneous subsets* ".

More explicitly, for any $\mathbb{P}'$-names $\dot{X}_n$ for K_1-homogeneous subsets, $n \in \omega$ and $f \in \mathbb{P}'$, there exist $f' \leq_{\mathbb{P}'} f$ and $\langle a, b \rangle \in \mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$ such that

$$f' \Vdash \text{“} \check{b} \in \dot{\mathcal{B}}^+ \ \& \ \langle \check{a}, \check{b} \rangle \notin \dot{X}_n \textit{ for every } n \in \omega \text{ ”}.$$

Therefore, there are $x \in \mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$ and $g \leq_{\mathbb{P}'} f$ such that

$$g \Vdash_{\mathbb{P}'} \text{“} \check{x} \notin \bigcup \{Y \in \check{M}_0[\dot{G}]; Y \subseteq \check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(\mathbf{1}) \ \& \ Y \star Y \subseteq K_1\} \ \& \ \check{b_x} \in \dot{\mathcal{B}}^+.$$

Let $p := \langle g, \{x\}, \{M_0\} \rangle$ which is a condition of $\mathbb{Q}$ and we can show that p is $(M, \mathbb{Q})$-generic by the same argument as in the proof of Lemma 2.4. The following lemma indicates the density argument of $\mathbb{Q}$.

Lemma 2.7.

$$p \Vdash_{\mathbb{Q}} \text{“} \dot{\mathcal{X}} := \bigcup \{X_q; q \in \dot{G}\} \textit{ is uncountable } K_0\textit{-homogeneous ”}.$$

Proof. It is trivial that $\Vdash_{\mathbb{Q}}$ " $\dot{\mathcal{X}}$ is K_0-homogeneous ". From now on we show that $\Vdash_{\mathbb{Q}}$" $\dot{\mathcal{X}}$ is uncountable ".

Assume not, then there is $q \leq_{\mathbb{Q}} p$ so that $q \Vdash_{\mathbb{Q}}$ " $\dot{\mathcal{X}}$ is countable ". Now $q \Vdash_{\mathbb{Q}}$" $\check{x} \in \dot{\mathcal{X}}$ ". Since $\Vdash_{\mathbb{Q}}$" $\dot{G} \in \check{M}_0[\dot{G}]$ ", $\Vdash_{\mathbb{Q}}$" $\dot{\mathcal{X}} \in \check{M}_0[\dot{G}]$ ". Since $q \Vdash_{\mathbb{Q}}$" $\dot{\mathcal{X}}$ is countable ", $q \Vdash_{\mathbb{Q}}$" $\dot{\mathcal{X}} \subseteq \check{M}_0[\dot{G}]$ ". So $q \Vdash_{\mathbb{Q}}$" $\check{x} \in \dot{\mathcal{X}} \subseteq \check{M}_0[\dot{G}]$ ". Now q is $(M, \mathbb{Q})$-generic, $q \Vdash_{\mathbb{Q}}$" $\check{M}[\dot{G}] \cap \mathbf{V} = \check{M}$, so $\check{M}_0[\dot{G}] \cap \mathcal{P}(\omega) \cap \mathbf{V} = \check{M}[\dot{G}] \cap \mathcal{P}(\omega) \cap \mathbf{V} = \check{M} \cap \mathcal{P}(\omega) = \check{M}_0 \cap \mathcal{P}(\omega)$ ". Thus $q \Vdash_{\mathbb{Q}}$" $\check{x} \notin \check{M}_0[\dot{G}]$ ", because of $x \notin M_0$, which is a contradiction. $\square$

Applying PFA to $\mathbb{Q}$, we can get a filter $G \subset \mathbb{Q}$ such that $\mathcal{X}' = \bigcup\{X_p; p \in G\}$ is uncountable and K_0-homogeneous. Let $\{x_\alpha; \alpha < \omega_1\}$ list $\mathcal{X}'$. For each $\alpha < \omega_1$, we choose $p_\alpha \in G$ with $x_\alpha \in X_{p_\alpha}$. Then we take a fusion $\tilde{f}$ of $\langle f_{p_\alpha}; \alpha < \omega_1 \rangle$, i.e. take $X \in \mathcal{U}^*$, a natural number n and an uncountable subset A of ω_1 such that $\mathrm{dom}(f_{p_\alpha}) \subseteq X \cup n$ for all $\alpha \in A$ and take a condition $\tilde{f} \in 2^{X \cup n}$ of $\mathbb{P}'$ with $\tilde{f} \supseteq \bigcup_{\alpha \in A} f_{p_\alpha}$. Then $\tilde{f}$ is an extension of f_{p_α} in $\mathbb{P}$ for all $\alpha \in A$, so $\tilde{f} \Vdash_{\mathbb{P}}$" $\check{b}_{x_\alpha} \in \dot{\mathcal{B}}^+$ ". Put $\mathcal{X} := \{x_\alpha; \alpha \in A\}$, then these are as desired so we finish the proof of the theorem.

3 Appendix: An iteration of the method of models as side conditions

3.1 Redefinition of the freezing forcing

In this section, we prove the following theorem, i.e. we can eliminate any large cardinal property of Theorem 1.1.

Theorem 3.1. *It is consistent with* ZFC *that Martin's Axiom holds and there are* $(\mathfrak{c},\mathfrak{c})$*-gaps but no* $(\omega_1,\mathfrak{c})$*-gaps.*

The key-point of the proof of Theorem 3.1 is same as the proof of Theorem 1.1. To prove Theorem 3.1, we use a countable support iteration instead of PFA. The problem is that in general $\mathbb{Q}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$ collapses $\aleph_2$, so we cannot force by an iteration of $\mathbb{Q}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$. To overcome this problem, we redefine the *freezing forcing* using the following objects:

Definition 3.2. *1. For a model N of* ZFC *(i.e. a model of sufficiently large fragments of* ZFC*), denote the transitive collapse of N by $\overline{N}$ and denote the unique isomorphism from N onto $\overline{N}$ by π_N, i.e. for $x \in N$, $\pi_N(x) := \{\pi_N(y); y \in N \ \& \ y \in x\}$. (This is defined by the $\in$-recursion.)*

2. For $\mathcal{A}$, $\dot{\mathcal{B}}$ and $\mathcal{U}$, let

$$\mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U}) := \left\{\overline{N}; N \prec H(\mathfrak{c}^+) \ \& \ N \text{ is countable} \ \& \ \mathcal{A},\dot{\mathcal{B}},\mathcal{U} \in N\right\}.$$

3. For $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ and $M \in \mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$, let

$$\mathfrak{M}_M := \left\{N \prec H(\mathfrak{c}^+);\ N \text{ is countable} \ \& \ \mathcal{A},\dot{\mathcal{B}},\mathcal{U} \in N \& \ \overline{N} = M\right\}.$$

We note that

- for $x \in \mathcal{P}(\omega) \cap N$, $\pi_N(x) = x$, so $\mathcal{P}(\omega) \cap \overline{N} = \mathcal{P}(\omega) \cap N$,
- for $N, N' \in \mathfrak{M}_M$, N and N' are isomorphic and ${\pi_{N'}}^{-1} \circ \pi_N$ is an isomorphism from N onto N',
- for a countable elementary submodel N of $H(\mathfrak{c}^+)$, N is an element of $H(\mathfrak{c}^+)$, so $\mathfrak{M}_M \subseteq H(\mathfrak{c}^+)$ for each $M \in \mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$.

The following partial order $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$ is the new freezing forcing notion designed for the iteration with countable support. This is similar to the forcing notion due to Todorčević ([18]).

Definition 3.3. *For $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ and $f \in \bigcup_{X\in\mathcal{U}^*} 2^X$, define $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$ whose conditions p are triples $\langle f_p, X_p, \mathcal{N}_p\rangle$ such that*

(a') *f_p is a member of $\bigcup_{X\in\mathcal{U}^*} 2^X$ with $f_p \supseteq f$,*

(b) *X_p is a finite K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$,*
(recall that for $f \in \mathbb{P}(\mathcal{U})$, $\mathcal{B}^+(f) = \{b \subseteq \omega; \exists g \leq_{\mathbb{P}(\mathcal{U})} f(g \Vdash_{\mathbb{P}(\mathcal{U})} \text{“} \check{b} \in \dot{\mathcal{B}}^+ \text{”})\}$)

(c') *$\mathcal{N}_p$ is a function so that*

(c1) $\mathrm{dom}(\mathcal{N}_p)$ *is a finite $\in$-chain of elements of $\mathfrak{T}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$,*

(c2) *for each $M \in \mathrm{dom}(\mathcal{N}_p)$, $\mathcal{N}_p(M)$ is a finite subset of $\mathfrak{M}_M$,*

(c3) *for all M, $M' \in \mathrm{dom}(\mathcal{N}_p)$ with $M \in M'$ and $N \in \mathcal{N}_p(M)$, there exists $N' \in \mathcal{N}(M')$ with $N \in N'$ and $N \prec N'$,*

(d) *for any $x\ (= \langle a_x, b_x \rangle) \in X_p$, $f_p \Vdash_{\mathbb{P}(\mathcal{U})}$ “$\check{b}_x \in \dot{\mathcal{B}}^+$”,*

(e') *for any $x, y \in X_p$ with $x \neq y$ there exists $M \in \mathrm{dom}(\mathcal{N}_p)$ so that $|M \cap \{x, y\}| = 1$,*
(define $x \lhd y :\iff \exists M \in \mathrm{dom}(\mathcal{N}_p)(x \in M \,\&\, y \notin M)$),

(f') *for all $N \in \bigcup \mathrm{ran}(\mathcal{N}_p)$, f_p is $(N, \mathbb{P}')$-generic, and*

(g') *for every $x \in X_p$, $M \in \mathrm{dom}(\mathcal{N}_p)$ with $x \notin M$ and $N \in \mathcal{N}_p(M)$,*

$$f_p \Vdash_{\mathbb{P}'(\mathcal{U})} \text{“} \forall Y \in \check{N}[\dot{G}](Y \subseteq \check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(\mathbf{1}) \ \&\ Y \star Y \subseteq K_1 \Rightarrow \check{x} \notin Y) \text{”}.$$

For conditions $p, q \in \mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$,
$p \leq_{\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})} q :\iff f_p \supseteq f_q$ *(i.e. $f_p \leq_{\mathbb{P}'} f_q$)* $\&\ X_p \supseteq X_q \ \&\ \mathrm{dom}(\mathcal{N}_p) \supseteq \mathrm{dom}(\mathcal{N}_q) \ \&\ \forall M \in \mathrm{dom}(\mathcal{N}_q)(\mathcal{N}_q(M) \subseteq \mathcal{N}_p(M))$.

By an argument similar to the one of Lemma 2.4, we show the following lemma.

Lemma 3.4. *For $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ and $f \in \bigcup_{X \in \mathcal{U}^*} 2^X$, $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ is proper.*

Proof. Let θ be a large enough regular cardinal, $H \prec H(\theta)$ a countable elementary submodel containing all relevant objects, e.g. $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, $H(\mathfrak{c}^+)$ etc, and $p = \langle f_p, X_p, \mathcal{N}_p \rangle \in H$ a condition of $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)(= \mathbb{Q}')$.

Since $\mathbb{P}'(\mathcal{U})(= \mathbb{P}')$ is proper, there is an extension $f_q \leq_{\mathbb{P}'} f_p$ such that f_q is $(H \cap H(\mathfrak{c}^+), \mathbb{P}')$-generic. Let $M_0 := \overline{H \cap H(\mathfrak{c}^+)}$ and

$$q := \left\langle f_q, X_p, \mathcal{N}_p \cup \left\{ \left\langle M_0, \{H \cap H(\mathfrak{c}^+)\} \right\rangle \right\} \right\rangle.$$

Then q is a condition of $\mathbb{Q}'$. We show that q is $(H, \mathbb{Q}')$-generic.

Let $\mathcal{D} \in H$ be dense open in $\mathbb{Q}'$ and $r \leq_{\mathbb{Q}'} q$. We may assume that r is in $\mathcal{D}$. Let $X_r \setminus H = \{x_i; i \leq n\}$ where $x_i \lhd x_{i+1}$ for $i < n$ and take rational open intervals $U_i \subseteq \mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$ such that $x_i \in U_i$ for $i \leq n$, $U_i \cap U_j = \emptyset$ and $U_i \star U_j \subseteq K_0$ for every $i, j \leq n$ with $i \neq j$. Let $M_i \in \mathrm{dom}(\mathcal{N}_r)$ for $1 \leq i \leq n$ be such that $x_{i-1} \in M_i$ and $x_i \notin M_i$. And let $N_0 := H \cap H(\mathfrak{c}^+)$ and recursively pick $N_i \in \mathcal{N}_i(M_i)$ with $N_{i-1} \in N_i$ for $1 \leq i \leq n$. Let G be $\mathbb{P}'$-generic over $H(\theta)$ with $f_r \in G$. By the same argument as in the proof of claim 2.5, it is proved that in $H(\theta)[G]$, there are rational open intervals $V_i^0, V_i^1 \subseteq U_i$, $y_i \in V_i^1 \cap \mathbf{V}$ for $i \leq n$ and $s \in \mathcal{D} \cap \mathbf{V}$ such that

1. $x_i \in V_i^0$ for all $i \leq n$,
2. $V_i^0 \cap V_i^1 = \emptyset$ and $V_i^0 \star V_i^1 \subseteq K_0$ for all $i \leq n$,
3. $f_s \in G$,
4. $X_s = (X_r \cap H) \cup \{y_i; i \leq n\}$ and for any $x \in X_r \cap H$ and $i < j \leq n$, and $x \lhd y_i \lhd y_j$,
5. $\mathrm{dom}(\mathcal{N}_s)$ is an end extension of $\mathrm{dom}(\mathcal{N}_r) \cap H$ and for all $M \in \mathrm{dom}(\mathcal{N}_r) \cap H$, $\mathcal{N}_r(M) \cap H \subseteq \mathcal{N}_s(M)$.

By $H \prec H(\theta)$ and the genericity of f_r, we can find $g \in G$ which decides all values of $V_i^0, V_i^1 \subseteq U_i$, $y_i \in V_i^1 \cap H$ for $i \leq n$ and $s \in \mathcal{D} \cap H$ and is a common extension of f_r and f_s in $\mathbb{P}'$. It's enough to find a common extension of r and s.

To find it, let $\{L_i; i < l\}$ enumerate $\mathcal{N}_r(M_0)$ with $L_0 := H \cap H(\mathfrak{c}^+) = N_0$ and $\varphi_i := {\pi_{L_i}}^{-1}$ for $i < l$. We notice that for each $M \in \mathrm{dom}(\mathcal{N}_s) \setminus \mathrm{dom}(\mathcal{N}_r)$,

- $\varphi_i(M) \in L_i$ (because $M \in \overline{H \cap H(\mathfrak{c}^+)} = M_0$),
- $\overline{\varphi_i(M)} = M$ (because M is transitive), and
- $\varphi_i(M) \prec H(\mathfrak{c}^+)$ (because $\varphi_i(M)$ and N are isomorphic and $N \prec H(\mathfrak{c}^+)$ for $N \in \mathcal{N}_s(M)$).

We define a function $\mathcal{N}'$ with domain $\mathrm{dom}(\mathcal{N}_r) \cup \mathrm{dom}(\mathcal{N}_s)$ by:

$$\mathcal{N}'(M) := \begin{cases} \mathcal{N}_r(M) \cup \mathcal{N}_s(M) & \text{if } M \in \mathrm{dom}(\mathcal{N}_r) \cap H \\ \mathcal{N}_s(M) \cup \{\varphi_i(M); i < l\} & \text{if } M \in \mathrm{dom}(\mathcal{N}_s) \setminus \mathrm{dom}(\mathcal{N}_r) \\ \mathcal{N}_r(M) & \text{if } M \in \mathrm{dom}(\mathcal{N}_r) \setminus H \end{cases}$$

for every $M \in \mathrm{dom}(\mathcal{N}')$. Then it can be checked that $\langle g, X_r \cup X_s, \mathcal{N}' \rangle$ is a common extension of r and s if it is a condition of $\mathbb{Q}'$. To check $\langle g, X_r \cup X_s, \mathcal{N}' \rangle \in \mathbb{Q}'$, the only non-trivial requirement is that $\mathcal{N}'$ satisfies (c3), in particular the case that $M \in \mathrm{dom}(\mathcal{N}_r) \cap H$, $M' \in \mathrm{dom}(\mathcal{N}_s) \setminus \mathrm{dom}(\mathcal{N}_r)$ with $M \in M'$ and $N \in \mathcal{N}_r(M) \setminus H$. Then we can find $L_i \in \mathcal{N}_r(M_0)$ with $N \in L_i$, because of $r \in \mathbb{Q}'$ and $M \in M_0$. Then N is in $\varphi_i(M')$ and an elementary submodel of $\varphi_i(M')$, since $M \prec M' \prec M_0$ and $\varphi_i \restriction M = {\pi_M}^{-1} \subseteq {\pi_{M'}}^{-1} = \varphi_i \restriction M'$. □

If $\Vdash_{\mathbb{P}'(\mathcal{U})}$ " $\check{\mathcal{A}} \otimes \check{\mathcal{B}}^+(\mathbf{1})$ is not countably separated ", (by the argument similar to Lemma 2.7) $\mathcal{X}_G := \bigcup\{X_p; p \in G\}$ is uncountable. The biggest difference between $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ and $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ is that $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ has a good chain condition. The following lemma says that it preserves cardinalities under CH.

Lemma 3.5. *For $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ and $f \in \bigcup_{X \in \mathcal{U}^*} 2^X$, $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ has the $\mathfrak{c}^+$-c.c.*

Proof. For conditions p and q in $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$, if $f_p = f_q$, $X_p = X_q$ and $\mathrm{dom}(\mathcal{N}_p) = \mathrm{dom}(\mathcal{N}_q)$, then $\langle f_p, X_p, \mathcal{N}'' \rangle$ is a common extension of p and q, where $\mathcal{N}''$ has the domain $\mathrm{dom}(\mathcal{N}_p)$ and $\mathcal{N}''(M) = \mathcal{N}_p(M) \cup \mathcal{N}_q(M)$. Therefore $\{p \in \mathbb{Q}'; f_p = f \ \&\ X_p = X \ \&\ \mathrm{dom}(\mathcal{N}_p) = \mathfrak{N}\}$ is centered for every $f \in \mathbb{P}$, $X \in [\mathcal{A} \otimes \mathcal{B}(\mathbf{1})]^{<\omega}$ and a finite $\in$-chain $\mathfrak{N}$ of countable transitive elementary submodels. □

3.2 Proof of Theorem 3.1

To prove Theorem 3.1, we assume that the ground model $\mathbf{V}$ is $\mathbf{L}$. Let S_0 and S_1 be stationary on ω_2 with $S_0 \cap S_1 = \emptyset$ and $S_0 \cup S_1 = \mathrm{Cof}(\omega_1) \cap \omega_2$, where $\mathrm{Cof}(\omega_1) = \{\alpha \in \mathbf{On}; \mathrm{cf}(\alpha) = \omega_1\}$. Then $\mathbf{V}$ satisfies $\diamondsuit_{\omega_2}(S_1)$. Let $\{D_\alpha; \alpha \in S_1\}$ be a diamond sequence, i.e. for any subset E of ω_2, $\{\alpha \in S_1; E \cap \alpha = D_\alpha\}$ is stationary. We define a countable support iteration $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha; \alpha < \omega_2 \right\rangle$ (and pick a $\mathbb{P}_\alpha$-generic filter $G \restriction \alpha$ over $\mathbf{V}$ for $\alpha < \omega_2$ recursively) as follows:

Stage 2α with $2\alpha \notin \mathrm{Cof}(\omega_1)$ Construct an ultrafilter base $\langle X_\alpha; \alpha < \omega_2 \rangle$ (e.g. using a σ-centered Mathias forcing).

Stage $2\alpha + 1$ Construct to force MA by a book-keeping argument.

Stage $\alpha \in S_0$ Let $\mathbb{Q}_\alpha := \mathbb{P}'(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$, where $\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle)$ is the ultrafilter generated by $\langle X_\xi; \xi < \alpha \rangle$. (We notice that

$$\Vdash_{\mathbb{P}_\alpha} \text{“ } \mathcal{U}(\langle X_\xi; \xi < \alpha \rangle) \text{ is an ultrafilter ”}$$

if $\alpha < \omega_2$ has the cofinality ω_1.)

Stage $\alpha \in S_1$ If D_α codes some $\left\langle \dot{f}, \dot{\mathcal{A}}, \dot{\mathcal{B}} \right\rangle$, where

- $\dot{f}$ is a $\mathbb{P}_\alpha$-name for a condition of $\dot{\mathbb{P}}(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$,
- $\dot{\mathcal{A}}$ is a $\mathbb{P}_\alpha$-name for a family of infinite subsets of ω, and
- $\dot{\mathcal{B}}$ is a $\mathbb{P}_\alpha * \dot{\mathbb{P}}(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$-name for a family of infinite subsets of ω,

such that

$$\mathbf{V}[G \restriction \alpha] \models \text{“ } \dot{f}[G \restriction \alpha] \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))} \text{“ } (\dot{\mathcal{A}}[G \restriction \alpha], \dot{\mathcal{B}}[G \restriction \alpha]) \text{ forms an } (\omega_1, \alpha)\text{-gap ” ”},$$

then let $\mathbb{Q}_\alpha := \mathbb{Q}'(\dot{\mathcal{A}}[G \restriction \alpha], \dot{\mathcal{B}}[G \restriction \alpha], \mathcal{U}(\langle X_\xi; \xi < \alpha \rangle), \dot{f}[G \restriction \alpha])$. Otherwise, let $\mathbb{Q}_\alpha := \{\mathbf{1}\}$.

We write $G \restriction \omega_2$ by G.

We note that $\mathbb{P}_{\omega_2}$ is proper because proper-ness is closed under countable support iterations. So the following lemma indicates that it does not collapse cardinals. To show it, we use the following definition (see [16], [17] or [18]).

Definition 3.6. (Shelah) *For a forcing notion $\mathbb{X}$, $\mathbb{X}$ satisfies the $\aleph_2$-properness isomorphism condition ($\aleph_2$-pic) if for all (some) large enough regular cardinal θ, $\alpha < \beta < \omega_2$, countable elementary submodels N_α, N_β of $H(\theta)$ and a function $\pi : N_\alpha \to N_\beta$ satisfying that*

- *$\alpha \in N_\alpha$, $\beta \in N_\beta$, $N_\alpha \cap \omega_2 \subseteq \beta$, $N_\alpha \cap \alpha = N_\beta \cap \beta$, $\mathbb{X} \in N_\alpha \cap N_\beta$,*
- *π is an isomorphism, $\pi(\alpha) = \pi(\beta)$, and $\pi \restriction (N_\alpha \cap N_\beta)$ is identity,*

if $p \in \mathbb{X} \cap N_\alpha$, then there exists an $(N_\alpha, \mathbb{X})$-generic condition q which is a common extension of p and $\pi(p)$ such that

$$q \Vdash_{\mathbb{X}} \text{“ } \pi''(\dot{G} \cap \check{N}_\alpha) = \dot{G} \cap \check{N}_\beta \text{ ”}.$$

Lemma 3.7. *$\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c.*

Proof. Shelah has shown the following facts about the $\aleph_2$-pic (see [17]):

- Under CH, any $\aleph_2$-pic forcing notion has the $\aleph_2$-chain condition and preserves $\aleph_1$.
- Under CH, $\aleph_2$-pic-ness is closed under countable support iterations.
- If a forcing notion is proper and has size $\leq \aleph_1$, it has the $\aleph_2$-pic.

Therefore it suffices to show that all $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ have the $\aleph_2$-pic. (I refer to the proof of Lemma 6 in [18] for the argument below.)

Let θ be a large enough regular cardinal, and $\alpha < \beta < \omega_2$, countable elementary submodels N_α, N_β of $H(\theta)$ and a function $\pi : N_\alpha \to N_\beta$ satisfy the assumptions of $\aleph_2$-pic. And let $p \in \mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f) \cap N_\alpha$. Because of $N_\alpha \cap \mathcal{P}(\omega) = N_\beta \cap \mathcal{P}(\omega)$ and $\overline{N_\alpha} = \overline{N_\beta}$, it is proved that $\pi(p)$ is a condition of $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$, $f_p = f_{\pi(p)}$, $X_p = X_{\pi(p)}$, and $\mathrm{dom}(\mathcal{N}_p) = \mathrm{dom}(\mathcal{N}_{\pi(p)})$. So $\langle f_p, X_p, \mathcal{N}' \rangle$ is a common extension of p and $\pi(p)$, where $\mathrm{dom}(\mathcal{N}') = \mathrm{dom}(\mathcal{N}_p)$ and for $M \in \mathrm{dom}(\mathcal{N}_p)$, $\mathcal{N}'(M) = \mathcal{N}_p(M) \cup \mathcal{N}_{\pi(p)}(M)$. We put

$$q := \left\langle f_p, X_p, \mathcal{N}' \cup \left\{ \left\langle \overline{N_\alpha \cap H(\mathfrak{c}^+)}, \{N_\alpha \cap H(\mathfrak{c}^+), N_\beta \cap H(\mathfrak{c}^+)\} \right\rangle \right\} \right\rangle.$$

As before, we can prove that q is also a condition of $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ and an $(N_\alpha, \mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f))$-generic. So it is true that $q \Vdash_{\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)}$ “ $\pi''(\dot{G} \cap \check{N}_\alpha) = \dot{G} \cap \check{N}_\beta$ ” because the compatibility in $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ is simply decided by f_p, X_p and $\mathrm{dom}(\mathcal{N}_p)$ for any $p \in \mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$. $\square$

In $\mathbf{V}[G]$, $\mathfrak{c} = \aleph_2$ and MA holds. By the standard Löwenheim-Skolem argument (see also [9]), since we iterate $\mathbb{P}'(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$ stationary many times, it follows that $\mathfrak{m}(\mathbb{P}'(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle))) = \aleph_2$ in $\mathbf{V}[G]$, hence $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle))$ is ω_2-Baire. So it suffices to show that $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle))$ adds no (ω_1, ω_2)-gaps.

Assume not, i.e. in $\mathbf{V}$ there are $\mathbb{P}_{\omega_2}$-names $\dot{f}$, $\dot{\mathcal{A}}$ and a $\mathbb{P}_{\omega_2} * \dot{\mathbb{P}}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle))$-name $\dot{\mathcal{B}}$ such that $\dot{f}[G](= f) \in \mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle))$, $\dot{\mathcal{A}}[G](= \mathcal{A}) \subseteq \mathcal{P}(\omega)$ and

$$f \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle))} \text{“ } (\check{\mathcal{A}}, \dot{\mathcal{B}}[G]) \text{ forms an } (\omega_1, \omega_2)\text{-gap ”}.$$

We may consider $\dot{\mathcal{B}}[G]$ as a $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$-name for a function from ω_2 into $\mathcal{P}(\omega)$, i.e.

$$f \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{“} \forall\alpha < \beta < \omega_2(\dot{\mathcal{B}}[G](\alpha) \subseteq^* \dot{\mathcal{B}}[G](\beta)) \\ \& \ \forall a \in \check{\mathcal{A}} \forall \alpha < \omega_2 (a \perp \dot{\mathcal{B}}[G](\alpha)) \text{”}.$$

We note that $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$ does not add new reals.

Claim 3.8. $\mathcal{C}\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha < \omega_2\right\rangle\right), \dot{f}\right) := \Big\{\alpha \in \mathrm{Cof}(\omega_1) \cap \omega_2;$

$$\mathbf{V}[G \restriction \alpha] \models \text{“} \ f \in \mathbb{P}(\mathcal{U}(\langle X_\xi; \xi < \alpha\rangle)), \\ \mathcal{A} \subseteq \mathbf{V}[G\restriction\alpha], \textit{ and} \\ f \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle))} \text{“} (\check{\mathcal{A}}, \dot{\mathcal{B}}[G]\restriction\alpha) \textit{ forms an } (\omega_1, \alpha)\textit{-gap} \text{” ”}\Big\}$$

is ω_1-club.

Proof. “ $\mathbf{V}[G\restriction\alpha] \models$“ $f \in \mathbb{P}(\mathcal{U}(\langle X_\xi; \xi < \alpha\rangle))$ & $\mathcal{A} \subseteq \mathbf{V}[G\restriction\alpha]$ ” ” is upward closed with respect to α, and ω_1-closed-ness is trivial because for $\alpha \in \mathrm{Cof}(\omega_1) \cap \omega_2$,

$$\mathbf{V}[G\restriction\alpha] \cap 2^\omega = \bigcup_{\xi<\alpha} \mathbf{V}[G\restriction\xi] \cap 2^\omega.$$

So we check that it is unbounded.

We note that in $\mathbf{V}[G]$ for all $x \in \mathcal{A}^\perp$ and $g \leq_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} f$, there are $y_{x,g} \in \mathcal{P}(\omega)$, $r_{x,g} \leq_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} g$ and $\xi_{x,g} < \beta_{x,g} < \omega_2$ such that

- $x \not\subseteq^* y_{x,g}$,
- $y_{x,g} \in \mathbf{V}[G\restriction\beta_{x,g}]$, and
- $r_{x,g} \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{“} \dot{\mathcal{B}}[G](\check{\xi}_{x,g}) = \check{y}_{x,g} \text{”}$.

Taking $\alpha < \omega_2$, we recursively construct $\langle\gamma_\xi; \xi < \omega_1\rangle \subseteq \mathrm{Cof}(\omega_1) \cap \omega_2$ such that

- $\alpha \leq \gamma_0$ and $\gamma_\xi \leq \gamma_\eta$ for $\xi \leq \eta < \omega_1$,
- $\mathbf{V}[G\restriction\gamma_{\xi+1}] \models$“ $\forall x \in \mathcal{A}^\perp \cap \mathbf{V}[G\restriction\gamma_\xi] \forall g \in \mathbb{P}(\mathcal{U}(\langle X_\zeta; \zeta < \gamma_\xi\rangle))$ with $g \leq_{\mathbb{P}(\mathcal{U}(\langle X_\zeta;\zeta<\gamma_\xi\rangle))} f$ $(r_{x,g} \in \mathbb{P}(\mathcal{U}(\langle X_\zeta; \zeta < \gamma_{\xi+1}\rangle))$ & $\xi_{x,g} < \gamma_{\xi+1}$ ”, and
- if η is limit, then let $\gamma_\eta := \sup_{\xi<\eta} \gamma_\xi$.

Then $\sup_{\xi<\omega_1} \gamma_\xi$ is in $\mathcal{C}\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha < \omega_2\right\rangle\right), \dot{f}\right)$. □

Since $\mathfrak{m}(\mathbb{P}'(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))) = \aleph_2$, by Lemma 2.6, in $\mathbf{V}[G]$

$$f \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{“} \check{\mathcal{A}} \otimes \left(\dot{\mathcal{B}}[G]^+(f)\right)^\vee \\ \text{is not a union of countably many } K_1\text{-homogeneous subsets ”}.$$

By Proposition 1.5, this is equivalent to

$$f \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{“}\left(\check{\mathcal{A}}, \left(\dot{\mathcal{B}}[G]^+(f)\right)^{\vee}\right) \text{ is not countably separated ”}$$

(in $\mathbf{V}[G]$), i.e. for all $\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))$-names $\vec{c} = \langle \dot{c}_n; n<\omega\rangle \in (\mathcal{A}^{\perp})^{\omega}$ and $g \leq_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} f$, there are $y_{\vec{c},g}, z_{\vec{c},g} \in \mathcal{P}(\omega)$, $r_{\vec{c},g} \leq_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} g$ and $\xi_{\vec{c},g} < \beta_{\vec{c},g} < \omega_2$ such that

- $\langle z_{\vec{c},g}, y_{\vec{c},g}\rangle \in \mathcal{A} \otimes \dot{\mathcal{B}}[G]^+(f) \cap \mathbf{V}[G \restriction \beta_{\vec{c},g}]$, and
- $r_{\vec{c},g} \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))}$ “ $\dot{\mathcal{B}}[G](\check{\xi}_{\vec{c},g}) = \check{y}_{\vec{c},g}$ & $\forall n<\omega(\check{z}_{\vec{c},g} \not\perp \dot{c}_n \vee \check{y}_{\vec{c},g} \not\subseteq^* \dot{c}_n)$ ”.

So by an argument similar to the one of the previous claim,

$$\begin{aligned} C'\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha<\omega_2\right\rangle\right), \dot{f}\right) := \Big\{\alpha \in \mathrm{Cof}(\omega_1) \cap \omega_2;& \\ \mathbf{V}[G\restriction\alpha] \models \text{“ } & f \in \mathbb{P}(\mathcal{U}(\langle X_\xi; \xi<\alpha\rangle)), \\ & \mathcal{A} \subseteq \mathbf{V}[G\restriction\alpha], \text{ and} \\ & f \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle))} \text{“ } \left(\check{\mathcal{A}}, \left(((\dot{\mathcal{B}}\restriction\alpha)[G\restriction\alpha])^+(f)\right)^{\vee}\right) \\ & \text{is not countably separated ” ”}\Big\} \end{aligned}$$

is ω_1-club. Thus by the diamond sequence, there exists

$$\alpha \in C\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha<\omega_2\right\rangle\right), \dot{f}\right) \cap C'\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha<\omega_2\right\rangle\right), \dot{f}\right)$$

such that D_α codes $\left\langle \dot{f}, \dot{\mathcal{A}}, \dot{\mathcal{B}}\restriction\alpha\right\rangle$. So $\mathbb{Q}_\alpha = \mathbb{Q}'(\mathcal{A}, (\dot{\mathcal{B}}\restriction\alpha)[G\restriction\alpha], \mathcal{U}(\langle X_\xi;\xi<\alpha\rangle), f)$ and $G(\alpha)$ is $\mathbb{Q}_\alpha$-generic over $\mathbf{V}[G\restriction\alpha]$. Then $\mathcal{X}' := \bigcup\{X_p; p \in G(\alpha)\}$ is uncountable K_0-homogeneous. We note that for all $p \in G(\alpha)$, f_p is in $\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))$. So by the same argument at the end of the proof of Main Lemma ??, there are a fusion $\tilde{f}$ of $\langle f_p; p \in G(\alpha)\rangle$ and uncountable K_0-homogeneous $\mathcal{X} \subseteq \mathcal{A} \otimes \dot{\mathcal{B}}[G]^+(f)$ such that $\tilde{f} \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))}$ “ $\check{b} \in \dot{\mathcal{B}}^+$ ” for all $\langle a,b\rangle \in \mathcal{X}$, which is a contradiction and completes the proof of Theorem 3.1.

Acknowledgement During this work, Stevo Todorčević supervised me. I would like to thank him for his guidance for introducing me to this work on forcing notions with models as side conditions and for many useful comments on the present work. And I would also like to thank: Jörg Brendle, Ilijas Farah, Tetsuya Ishiu and Juris Steprāns for discussions with me. More!!

References

[1] T. Bartoszyński and H. Judah. *Set Theory: On the structure of the real line,* A.K.Peters, Wellesley, Massachusetts, 1995.

[2] J. Baumgartner. *Iterated forcing,* Surveys in set theory (edited by A.R.D. Mathias), Cambridge University Press, Cambridge 1983, pp. 1-59.

[3] J. Baumgartner. *Applications of the Proper Forcing Axiom,* Handbook of Set-Theoretic Topology, chapter 21, pp. 913-959.

[4] A. Dow, P. Simon and J. E. Vaughan. *Strong homology and the proper forcing axiom*, Proceedings of American Mathematical Society, vol. 106, pp. 821-828, 1989.

[5] I. Farah, preprint, 2002.

[6] S. Grigorieff. *Combinatorics on ideals and forcing*, Annals of Mathematical Logic 3, pp. 363-394, 1971.

[7] F. Hausdorff. *Die Graduierung nach dem Endverlauf*, Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften; Mathematisch-Physische Klasse 31, pp. 296-334, 1909.

[8] T. Jech. *Multiple Forcing,* volume 88 of Cambridge Tracts in Mathematics. Cambridge university Press, 1986.

[9] H. Judah, A. Miller and S. Shelah. *Sacks forcing, Laver forcing, and Martin's Axiom,* Archive for Mathematical Logic, vol.31, pp. 145-162, 1992.

[10] P. Koszmider. *Models as side conditions,* Set theory (Curaco, 1995; Barcelona, 1996), 99–107, Kluwer Acad. Publ., Dordrecht, 1998.

[11] K. Kunen. *(κ, λ^*)-gaps under MA*, handwritten note, 1976.

[12] K. Kunen. *Set Theory: An Introduction to Independence Proofs*, volume 102 of *Studies in Logic*, North Holland, 1980.

[13] R. Laver. *Linear orders in $(\omega)^\omega$ under eventual dominance*, Logic Colloquium '78, North-Holland, pp. 299-302, 1979.

[14] F. Rothberger. *Sur les familles indenombrables de suites de nombres naturels et les problemes concernant la propriete C*, Mathematical Proceedings of the Cambridge Philosophical Society, vol.37, pp. 109-126, 1941.

[15] M. Scheepers. *Gaps in ω^ω*, In *Set Theory of the Reals*, volume 6 of *Israel Mathematical Conference Proceedings*, pp. 439-561, 1993.

[16] S. Shelah. *Proper Forcing*, Lecture Notes in Mathematics 940, Springer, 1982.

[17] S. Shelah. *Proper and improper forcing*, 2nd edition, Springer, 1998.

[18] S. Todorčević. *Directed sets and cofinal types*, Transactions of American Mathematical Society, vol. 290, no. 2, pp. 711–723, 1985.

[19] S. Todorčević. *Partition Problems in Topology*, volume 84 of *Contemporary mathematics*, American Mathematical Society, Providence, Rhode Island, 1989.

[20] S. Todorčević and I. Farah. *Some Applications of the Method of Forcing*, Mathematical Institute, Belgrade and Yenisei, Moscow, 1995.

[21] S. Todorčević. *The first derived limit and compactly F_σ sets*, Journal of the Mathematical Society of Japan, vol. 50. no.4 , pp. 831-836, 1998.

[22] B. Veličković. OCA *and automorphisms of* $\mathcal{P}(\omega)/\text{fin}$, Topology and its Applications, vol. 49, no. 1, pp.1-13, 1993.

[23] T. Yorioka *Distinguishing types of gaps in* $\mathcal{P}(\omega)/$ fin, Journal of Symbolic Logic, to appear.

[24] J. Zapletal. *Strongly almost disjoint functions*, Israel Journal of Mathematics , 97 (1997), 101–111.