A New Formulation for Stochastic Linear Complementarity Problems[1]

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The stochastic variational inequality problem is to find a vector $x \in R^n$ such that

$$x \in S, \quad F(x,\omega)^T(y-x) \geq 0 \quad \forall y \in S, \tag{1}$$

where $S \subseteq R^n$ is a nonempty closed convex set, $F : R^n \times \Omega \to R^n$ is a vector-valued function, and $(\Omega, \mathcal{F}, P)$ is a probability space with $\Omega \subseteq R^m$. When S is the nonnegative orthant $R^n_+ := \{x \in R^n \,|\, x \geq 0\}$, this problem is rewritten as the stochastic complementarity problem

$$F(x,\omega) \geq 0, \quad x \geq 0, \quad F(x,\omega)^T x = 0. \tag{2}$$

In general, there is no x satisfying (1) or (2) for *all* $\omega \in \Omega$. An existing approach is to consider the following deterministic formulations of (1) and (2), respectively:

$$x \in S, \quad F_\infty(x)^T(y-x) \geq 0 \quad \forall y \in S, \tag{3}$$

and

$$F_\infty(x) \geq 0, \quad x \geq 0, \quad F_\infty(x)^T x = 0, \tag{4}$$

where $F_\infty(x) := \mathbf{E}[F(x,\omega)]$ is the expectation function of the random function $F(x,\omega)$. Note that these problems are in general different from those which are obtained by simply replacing the random variable ω by its expected value $\mathbf{E}[\omega]$ in (1) or (2). Since the expectation function $F_\infty(x)$ is usually still difficult to evaluate exactly, one may construct a sequence of functions $\{F_k(x)\}$ that converges in a certain sense to $F_\infty(x)$, and solve a

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sequence of problems (3) or (4) in which $F_\infty(x)$ is replaced by $F_k(x)$. In practice, approximating functions $F_k(x)$ may be constructed by using discrete distributions $\{(\omega^i, p_i),\ i = 1, \ldots, k\}$ as

$$F_k(x) := \sum_{i=1}^{k} F(x, \omega^i) p_i,$$

where p_i is the probability of sample ω^i.

Convergence properties of such approximation problems have been studied in [8] by extending the earlier results for stochastic optimization and deterministic variational inequality problems.

The deterministic complementarity problem has played an important role in studying equilibrium systems that arise in mathematical programming, operations research and game theory. There are numerous publications on complementarity problems. In particular, Cottle, Pang and Stone [4] and Facchinei and Pang [5] give comprehensive treatment of theory and methods in complementarity problems. Ferris and Pang [6] present a survey of applications in engineering and economics. On the other hand, in many practical applications, complenmentarity problems often involve uncertain data. However, references for stochastic complementarity problems are relatively scarce [1], compared with stochastic optimization problems for which abundant results are available in the literature; see [9, 10] in particular for simulation-based approaches in stochastic optimization.

In [2], confining ourselves to the stochastic linear complementarity problem (SLCP)

$$M(\omega)x + q(\omega) \geq 0, \quad x \geq 0, \quad (M(\omega)x + q(\omega))^T x = 0, \tag{5}$$

where $M(\omega) \in R^{n\times n}$ and $q(\omega) \in R^n$ are random matrices and vectors, we propose a new deterministic formulation that is based on the concept of *expected residual minimization.*

To this end, we will use a function $\phi : R^2 \to R$, called an NCP function, which has the property

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

Two popular NCP functions are the "min" function

$$\phi(a, b) = \min(a, b)$$

and the Fischer-Burmeister (FB) function

$$\phi(a,b) = a + b - \sqrt{a^2 + b^2}.$$

All NCP functions including the "min" function and FB function are equivalent in the sense that they can reformulate any complementarity problem as a system of nonlinear equations having the same solution set. In the last decade, NCP functions have been used as a powerful tool for dealing with linear complementarity problems [3].

With an NCP function ϕ, we may consider the following problem which is to find a vector $x \in R^n_+$ that minimizes an *expected residual* for the SLCP (5):

$$\min_{x \in R^n_+} \mathbf{E}[\|\Phi(x,\omega)\|^2], \tag{6}$$

where

$$\Phi(x,\omega) := \begin{pmatrix} \phi((M(\omega)x + q(\omega))_1, x_1) \\ \vdots \\ \phi((M(\omega)x + q(\omega))_n, x_n) \end{pmatrix}.$$

We call problem (6) an *expected residual minimization* (ERM) problem associated with the SLCP (5). Throughout, we assume that $M(\omega)$ and $q(\omega)$ are continuous functions of ω and the norm $\|\cdot\|$ is the Euclidean norm $\|\cdot\|_2$. Now let us note that, if Ω has only one realization, then the ERM problem (6) reduces to the standard LCP and the solubility of (6) does not depend on the choice of NCP functions. However, the following example shows that we do not have such equivalence if Ω has more than one realization.

Example 1. Let $n = 1$, $m = 1$, $\Omega = \{\omega^1, \omega^2\} = \{0, 1\}$, $p_1 = p_2 = 1/2$, $M(\omega) = \omega(1-\omega)$ and $q(\omega) = 1 - 2\omega$. Then we have $M(\omega^1) = M(\omega^2) = 0$, $q(\omega^1) = 1$, $q(\omega^2) = -1$ and

$$\mathbf{E}[\|\Phi(x,\omega)\|^2] = \frac{1}{2}\sum_{i=1}^{2} \|\Phi(x,\omega^i)\|^2.$$

The objective function of the ERM problem (6) defined by the "min" function is

$$\frac{1}{2}[(\min(1,x))^2 + (\min(-1,x))^2] = \begin{cases} x^2 & x \le -1 \\ \frac{1}{2}(x^2+1) & -1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

and the problem has a unique solution $x^* = 0$. However, problem (6) defined by the FB function has no solution as the objective function

$$\frac{1}{2}[(1 + x - \sqrt{1 + x^2})^2 + (-1 + x - \sqrt{1 + x^2})^2]$$

is monotonically decreasing on $[0, \infty)$.

In order to find a solution of an ERM problem (6) numerically, it is necessary to study the objective function of (6) defined by an NCP function. There are a number of NCP functions. In [2], we focus on the "min" function and the FB function. We use $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ to distinguish the function $\Phi(x, \omega)$ defined by the "min" function and the FB function, respectively. We use $\Phi(x, \omega)$ to represent both $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ when we discuss their common properties.

We consider the following ERM problem:

$$\min_{x \geq 0} f(x) := \int_{\Omega} \|\Phi(x, \omega)\|^2 \rho(\omega) d\omega, \tag{7}$$

where $\rho : \Omega \to R_+$ is a continuous probability density function satisfying

$$\int_{\Omega} \rho(\omega) d\omega = 1 \quad \text{and} \quad \int_{\Omega} \|\omega\|^2 \rho(\omega) d\omega < \infty. \tag{8}$$

Obviously, if $M(\omega) \equiv M$ and $q(\omega) \equiv q$, then (7) reduces to the standard linear complementarity problem.

In [2], we show that a sufficient condition for the existence of minimizers of the ERM problem (7) and its discrete approximations is that there is an observation ω^i such that the coefficient matrix $M(\omega^i)$ is an R_0 matrix. Moreover, we prove that every accumulation point of minimizers of discrete approximation problems is a solution of the ERM problem (7). Especially, for a class of SLCPs with a fixed coefficient matrix $M(\omega) \equiv M$, we show that M being an R_0 matrix is a necessary and sufficient condition for the boundedness of the solution sets of the ERM problem and its discrete approximations with any $q(\omega)$. We show that a class of SLCPs with a fixed coefficient matrix, the ERM problem with the "min" function is smooth and can be solved without using discrete approximation. We present numerical results to compare the formulations (4) and (6), as well as the formulations as a stochastic program with recourse and a stochastic program with joint probabilistic constraints, for solving a stochastic linear programming problem in oil refinery plants [9].

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