Generalization Properties of Integral Means by H.Silverman

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Abstract

For analytic and univalent functions \( f(z) \) with negative coefficients in the open unit disk \( U \), H. Silverman (Houston J. math. 28(1997)) has given some interesting results for integral means of \( f(z) \). In the present paper, we discuss generalization properties of integral means of \( f(z) \) given by H. Silverman. We also show some examples of our theorems.

1 Introduction

Let \( \mathcal{A} \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

that are analytic in the open unit disk \( U = \{z \in \mathbb{C} : |z| < 1\} \). Let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of all univalent functions \( f(z) \) in \( U \). Also let \( \mathcal{S}^* \) and \( \mathcal{K} \) denote the subclasses of \( \mathcal{S} \) consisting of functions \( f(z) \) which are starlike and convex in \( U \), respectively.

The class \( \mathcal{T} \) is defined as the subclass of \( \mathcal{S} \) consisting of all functions \( f(z) \) which are given by

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z_n \quad (a_n \geq 0).
\]

Further, we denote by \( \mathcal{T}^* = \mathcal{S}^* \cap \mathcal{T} \) and \( \mathcal{C} = \mathcal{K} \cap \mathcal{T} \). It is well-known by Silverman [3] that

\textbf{2000 Mathematics Subject Classification:} Primary 30C45.

\textbf{Key Words and Phrases:} Integral means, univalent function, starlike function, convex function.
Remark 1.1. A function $f(z) \in \mathcal{T}^*$ if and only if

$$
\sum_{n=2}^{\infty} na_n \leq 1.
$$

A function $f(z) \in \mathcal{C}$ if and only if

$$
\sum_{n=2}^{\infty} n^2 a_n \leq 1.
$$

For $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathcal{U}$ if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathcal{U}$), and $f(z) = g(\omega(z))$. We denote this subordination by

$$
f(z) \prec g(z). \quad (cf. \text{Duren}[1])
$$

For subordinations, Littlewood [2] has given the following integral mean.

**Theorem A.** If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $f(z) \prec g(z)$, then, for $\lambda > 0$ and $|z| = r \ (0 < r < 1)$,

$$
\int_{0}^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_{0}^{2\pi} |g(re^{i\theta})|^\lambda d\theta.
$$

Furthermore, Silverman [3] has shown that

**Remark 1.2.** $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n} \ (n \geq 2)$ are extreme points of the class $\mathcal{T}^*$ (or $\mathcal{T}$). $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n^2} \ (n \geq 2)$ are extreme points of the class $\mathcal{C}$.

Applying Theorem A with extreme points of $\mathcal{T}$, Silverman [4] has proved the following results.

**Theorem B.** Suppose that $f(z) \in \mathcal{T}$, $\lambda > 0$ and $f_2(z) = z - \frac{z^2}{2}$. Then, for $z = re^{i\theta} \ (0 < r < 1)$,

$$
\int_{0}^{2\pi} |f(z)|^\lambda d\theta \leq \int_{0}^{2\pi} |f_2(z)|^\lambda d\theta.
$$

**Theorem C.** If $f(z) \in \mathcal{T}$, $\lambda > 0$, and $f_2(z) = z - \frac{z^2}{2}$, then, for $z = re^{i\theta} \ (0 < r < 1)$,

$$
\int_{0}^{2\pi} |f'(z)|^\lambda d\theta \leq \int_{0}^{2\pi} |f_2'(z)|^\lambda d\theta.
$$

In the present paper, we consider the generalization properties for Theorem B and Theorem C with $f(z) \in \mathcal{T}^*$ and $f(z) \in \mathcal{C}$. 


2 Generalization properties

Our first result for the generalization properties is contained in

**Theorem 2.1.** Let \( f(z) \in \mathcal{T}^* \), \( \lambda > 0 \), and \( f_k(z) = z - \frac{z^k}{k} \) \((k \geq 2)\). If \( f(z) \) satisfies

\[
\sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0
\]

for \( k \geq 3 \), then, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[
\int_{0}^{2\pi} |f(z)|^\lambda d\theta \leq \int_{0}^{2\pi} |f(z)|^\lambda d\theta.
\]

**Proof.** For \( f(z) \in \mathcal{T}^* \), we have to show that

\[
\int_{0}^{2\pi} |1 - \sum_{n=2}^{\infty} a_n z^{n-1}|^\lambda d\theta \leq \int_{0}^{2\pi} |1 - \frac{z^{k-1}}{k}|^\lambda d\theta.
\]

By Theorem A, it suffices to prove that

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k}.
\]

Let us define the function \( \omega(z) \) by

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k} \omega(z)^{k-1}.
\]

It follows from (2.3) that

\[
|\omega(z)|^{k-1} = |k \sum_{n=2}^{\infty} a_n z^{n-1}| \leq |z| \left( \sum_{n=2}^{\infty} k a_n \right).
\]

Thus, we only show that

\[
\sum_{n=2}^{\infty} k a_n \leq \sum_{n=2}^{\infty} n a_n,
\]

or

\[
\sum_{n=2}^{\infty} a_n \leq \frac{1}{k} \left( \sum_{n=2}^{\infty} n a_n \right).
\]
Indeed, we see that

\[
\frac{1}{k} \sum_{n=2}^{\infty} na_n = \left(1 - \frac{k-2}{k}\right) a_2 + \left(1 - \frac{k-3}{k}\right) a_3 + \cdots \\
+ \left(1 - \frac{2}{k}\right) a_{k-2} + \left(1 - \frac{1}{k}\right) a_{k-1} + a_k + \left(1 + \frac{1}{k}\right) a_{k+1} \\
+ \left(1 + \frac{2}{k}\right) a_{k+2} + \cdots + \left(1 + \frac{k+1}{k}\right) a_{2k+1} + \left(1 + \frac{k+2}{k}\right) a_{2k+2} \\
+ \cdots \\
= \frac{k-2}{k} (a_{2k-2} - a_2) + \frac{k-3}{k} (a_{2k-3} - a_3) + \cdots \\
+ \frac{2}{k} (a_{k+2} - a_{k-2}) + \frac{1}{k} (a_{k+1} - a_{k-1}) \\
+ \left(1 + \frac{k-1}{k}\right) a_{2k-1} + \left(1 + \frac{k}{k}\right) a_{2k} + \left(1 + \frac{k+1}{k}\right) a_{2k+1} \\
+ \cdots + \sum_{n=2}^{2k-2} a_n.
\]

Nothing that

\[
1 + \frac{k+j}{k} \geq 1 + \frac{2+j}{k} \quad (j = -1, 0, 1, \cdots),
\]

we obtain

\[
(2.4) \quad \frac{1}{k} \left( \sum_{n=2}^{\infty} na_n \right) \geq \frac{k-2}{k} (a_{2k-2} - a_2) + \frac{k-3}{k} (a_{2k-3} - a_3) \\
+ \cdots + \frac{2}{k} (a_{k+2} - a_{k-2}) + \frac{1}{k} (a_{k+1} - a_{k-1}) \\
+ \left(1 + \frac{1}{k}\right) a_{2k-1} + \left(1 + \frac{2}{k}\right) a_{2k} + \cdots \\
+ \left(1 + \frac{k-3}{k}\right) a_{3k-5} + \left(1 + \frac{k-2}{k}\right) a_{3k-4} + \cdots \\
+ \sum_{n=2}^{2k-2} a_n \\
\geq \frac{1}{k} (a_{2k-1} + a_{k+1} - a_{k-1}) + \frac{2}{k} (a_{2k} + a_{k+2} - a_{k-2}) \\
+ \cdots + \frac{k-2}{k} (a_{3k-4} + a_{2k-2} - a_2) + \sum_{n=2}^{\infty} a_n \\
= \sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) + \sum_{n=2}^{\infty} a_n \\
\geq \sum_{n=2}^{\infty} a_n
\]
with the following condition
\[
\sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0.
\]
Thus, we observe that the function \( \omega(z) \) defined by (2.3) satisfies \( \omega(z) \) is analytic in \( \mathbb{U} \) with \( \omega(0) = 0, |\omega(z)| < 1 \) \((z \in \mathbb{U})\). This completes the proof of the theorem.

**Remark 2.1.** Taking \( k = 2 \) in Theorem 2.1, we have Theorem B by Silverman [4].

**Example 2.1.** Let us define

\[ f(z) = z - \frac{37}{1200}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 - \frac{1}{100}z^5 \tag{2.5} \]

and

\[ f_3(z) = z - \frac{1}{3}z^3 \tag{2.6} \]

with \( k = 3 \) in Theorem 2.1. Since \( f(z) \) satisfies

\[ \sum_{n=2}^{\infty} na_n = \frac{217}{600} < 1, \]

we have \( f(z) \in \mathcal{T}^* \). Furthermore, \( f(z) \) satisfies,

\[ \frac{1}{3} (a_6 + a_4 - a_2) = \frac{1}{3} \left( \frac{1}{100} + \frac{1}{48} - \frac{37}{1200} \right) = 0. \]

Thus, \( f(z) \) satisfies the conditions in Theorem 2.1 with \( k = 3 \). If we take \( \lambda = 2 \), then we have

\[ \int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left( 1 + \frac{1}{9}r^4 \right) < \frac{20}{9}\pi = 6.9813\ldots. \]

**Corollary 2.1.** Let \( f(z) \in \mathcal{T}^* \), \( 0 < \lambda \leq 2 \), and \( f_k(z) = z - \frac{z^k}{k} \) \((k \geq 2)\). If \( f(z) \) satisfies (2.1) for \( k \geq 3 \), then, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[ \int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left( 1 + \frac{1}{k^2} r^{2k(k-1)} \right)^{\frac{1}{\lambda}} < 2\pi \left( 1 + \frac{1}{k^2} \right)^{\frac{1}{\lambda}}. \tag{2.6} \]

**Proof.** It follows that

\[ \int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z^\lambda| \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta. \]
Applying Hölder inequality for $0 < \lambda < 2$, we obtain that

$$
\int_{0}^{2\pi} |z|^\lambda |1 - \frac{z^{k-1}}{k}|^{\lambda} d\theta \leq \left( \int_{0}^{2\pi} |z|^{\lambda} d\theta \right)^{\frac{2\lambda}{2\lambda - \lambda}} \left( \int_{0}^{2\pi} \left|1 - \frac{z^{k-1}}{k}\right|^{\lambda} d\theta \right)^{\frac{\lambda}{2}}
$$

$$
= \left( 2\pi r^{\frac{2\lambda}{2\lambda - \lambda}} \right)^{\frac{2\lambda}{2\lambda - \lambda}} \left( 2\pi \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)^{\frac{1}{2}} \right)^{\frac{\lambda}{2}}
$$

$$
< 2\pi \left(1 + \frac{1}{k^2}\right)^{\frac{1}{2}}.
$$

Further, it is clear for $\lambda = 2$. \qedhere

For the generalization of Theorem C by Silverman [4], we have

**Theorem 2.2.** Let $f(z) \in T^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,

(2.7) \[
\int_{0}^{2\pi} |f'(z)|^\lambda d\theta \leq \int_{0}^{2\pi} |f_k'(z)|^\lambda d\theta.
\]

**Proof.** For $f(z) \in T^*$, it is sufficient to show that

(2.8) \[
1 - \sum_{n=2}^{\infty} na_n z^{n-1} < 1 - z^{k-1}.
\]

Let us define the function $\omega(z)$ by

(2.9) \[
1 - \sum_{n=2}^{\infty} na_n z^{n-1} = 1 - \omega(z)^{k-1},
\]

or, by

$$
\omega(z)^{k-1} = \sum_{n=2}^{\infty} na_n z^{n-1}.
$$

Since $f(z)$ satisfies

$$
\sum_{n=2}^{\infty} na_n \leq 1,
$$

the function $\omega(z)$ is analytic in $U$, $\omega(0) = 0$, and $|\omega(z)| < 1$ $(z \in U)$. \qedhere

**Remark 2.2.** If we take $k = 2$ in Theorem 2.2, then we have Theorem C by Silverman [4].
Using Hölder inequality for Theorem 2.2, we have

**Corollary 2.2.** Let \( f(z) \in \mathcal{T}^* \), \( 0 < \lambda \leq 2 \), and \( f_k(z) = z - \frac{z^k}{k} \) \((k \geq 2)\). Then, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[
\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi(1 + r^{2(k-1)})^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}} \pi.
\]

## 3 Integral means for functions in the class \( C \)

In this section, we discuss the integral means for functions in the class \( C \).

**Theorem 3.1.** Let \( f(z) \in C \), \( \lambda > 0 \), and \( f_k(z) = z - \frac{z^k}{k^2} \) \((k \geq 2)\). If \( f(z) \) satisfies

\[(3.1) \quad \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j} - a_j) \geq 0 \]

for \( k \geq 3 \), then, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[(3.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.
\]

**Proof.** For the proof, we need to show that

\[(3.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} < 1 - \frac{z^{k-1}}{k^2} \]

by Theorem A. Define the function \( \omega(z) \) by

\[(3.4) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k^2} \omega(z)^{k-1},
\]
or by

\[(3.5) \quad \omega(z)^{k-1} = k^2 \left( \sum_{n=2}^{\infty} a_n z^{n-1} \right).
\]

Therefore, we have to show that

\[
\sum_{n=2}^{\infty} a_n \leq \frac{1}{k^2} \left( \sum_{n=2}^{\infty} n^2 a_n \right).
\]
Using the same technique as in the proof of Theorem 2.1, we see that

\[
\frac{1}{k^2} \left( \sum_{n=2}^{\infty} n^2 a_n \right) \geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n \\
\geq \sum_{n=2}^{\infty} a_n.
\]

Example 3.1. Consider the functions

(3.6) \[ f(z) = z - \frac{1}{40} z^2 - \frac{1}{18} z^3 - \frac{1}{40} z^4 \]

and

(3.7) \[ f_3(z) = z - \frac{1}{9} z^3 \]

with \( k = 3 \) in Theorem 3.1. Then we have that

\[ \sum_{n=2}^{\infty} n^2 a_n = \frac{4}{40} + \frac{9}{18} + \frac{16}{40} = 1 \]

which implies \( f(z) \in C \), and that

\[ \frac{5}{9} (a_4 - a_2) = 0. \]

Thus \( f(z) \) satisfies the conditions of Theorem 3.1. If we make \( \lambda = 2 \), then we see that

\[
\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left( 1 + \frac{1}{81} r^4 \right) < \frac{164}{81} \pi \approx 6.3607 \cdots.
\]

Corollary 3.1. Let \( f(z) \in C \), \( 0 < \lambda \leq 2 \), and \( f_k(z) = z - \frac{z^k}{k^2} \) (\( k \geq 2 \)). If \( f(z) \) satisfies the condition (3.1) for \( k \geq 3 \), then, for \( z = re^{i\theta} \) (\( 0 < r < 1 \)),

(3.8) \[
\int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left( 1 + \frac{1}{k^4}r^{2(k-1)} \right)^{\frac{3}{2}} < 2\pi \left( 1 + \frac{1}{k^4} \right)^{\frac{1}{2}}.
\]

Further, we may have

Theorem 3.2. Let \( f(z) \in C \), \( \lambda > 0 \), and \( f_k(z) = z - \frac{z^k}{k^2} \) (\( k \geq 2 \)). If \( f(z) \) satisfies
\[ (3.9) \quad \sum_{j=2}^{2k-2} j(k-j)a_j \leq 0, \]

then, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[ (3.10) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta. \]

**Example 3.2.** Take the functions

\[ (3.11) \quad f(z) = z - \frac{1}{24}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 \]

and

\[ (3.12) \quad f_3(z) = z - \frac{1}{9}z^3 \]

with \( k = 3 \) in Theorem 3.2. Since

\[ \sum_{n=2}^{\infty} n^2a_n = \frac{4}{24} + \frac{9}{18} + \frac{16}{48} = \frac{5}{6} < 1 \]

and

\[ 2(3-2)a_2 + 3(3-3)a_3 + 4(3-4)a_4 = \frac{1}{12} - \frac{1}{12} = 0, \]

\( f(z) \) satisfies the conditions in Theorem 3.2. If we take \( \lambda = 2 \), then we have

\[ \int_0^{2\pi} |f'(z)|^2 d\theta \leq 2\pi \left( 1 + \frac{1}{9}r^4 \right) < \frac{20}{9}\pi. \]

**Corollary 3.2.** Let \( f(z) \in \mathcal{C} \), \( 0 < \lambda \leq 2 \), and \( f_k(z) = z - \frac{z^k}{k^2} \) \((k \geq 2)\). If \( f(z) \) satisfies the condition \( (3.9) \) for \( k \geq 2 \), then, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[ \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi \left( 1 + \frac{1}{k}r^{2(k-1)} \right)^{\frac{\lambda}{2}} < 2\pi \left( 1 + \frac{1}{k} \right)^{\frac{\lambda}{2}}. \]

4 Appendix

For analytic functions \( h(z) \) and \( g(z) \), Hölder inequality gives that, for \( z = re^{i\theta} \) \((0 < r < 1)\),

\[ (4.1) \quad \int_0^{2\pi} |h(z)g(z)|d\theta \leq \left( \int_0^{2\pi} |h(z)|^p d\theta \right)^{\frac{1}{p}} \left( \int_0^{2\pi} |g(z)|^q d\theta \right)^{\frac{1}{q}} \]

with \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). It follows from \( (4.1) \) that
Letting \( p = \frac{\lambda}{2} \), \( q = \frac{\lambda}{\lambda - 2} \), \( \lambda > 2 \), we observe that the function \( f(z) \) in the class \( T \) satisfies

\[
\int_{0}^{2\pi} |f(z)|^\lambda d\theta \geq \frac{\left( \int_{0}^{2\pi} |f(z)|^2 d\theta \right)^{\frac{\lambda}{2}}}{\left( \int_{0}^{2\pi} |g(z)|^q d\theta \right)^{\frac{q}{2}}}.
\]

Further, when \( \lambda = 2 \), we see that, for \( z = re^{\iota\theta} \) \((0 < r < 1)\),

\[
\int_{0}^{2\pi} |f(z)|^2 d\theta = \pi \left( 1 + \sum_{n=2}^{\infty} a_n^2 r^{2n-1} \right).
\]

Therefore, we conclude that

**Theorem 4.1.** Let \( f(z) \in T \) and \( \lambda \geq 2 \). Then, for \( z = re^{\iota\theta} \) \((0 < r < 1)\),

\[
\int_{0}^{2\pi} |f(z)|^\lambda d\theta \geq 2\pi r^\lambda \left( 1 + \sum_{n=2}^{\infty} a_n^2 r^{2(n-1)} \right)^{\frac{1}{2}}.
\]
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