

## RADIUS OF STARLIKENESS AND CONVEXITY 2

RIKUO YAMAKAWA [山川陸夫] (芝浦工業大学)  
SHIBAURA INSTITUTE OF TECHNOLOGY, JAPAN

ABSTRACT. In this note, we will investigate radius of starlikeness and convexity for certain analytic functions on the unit disk.

### 1. INTRODUCTION AND NOTATIONS

Let  $U = \{|z| < 1\}$  and let  $B$  denote the family of functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  which are analytic in  $U$ , and which satisfy the condition  $|a_n| \leq n$  ( $n = 2, 3, 4, \dots$ ). For functions  $f \in B$ , we consider the radius of univalence, starlikeness and convexity i.e.,

$$R_U = \max\{r : f \in B \Rightarrow f \text{ is univalent in } |z| < r < 1\}$$

$$R_S = \max\{r : f \in B \Rightarrow f \text{ is starlike in } |z| < r < 1\}$$

$$R_C = \max\{r : f \in B \Rightarrow f \text{ is convex in } |z| < r < 1\}$$

For  $R_U$ , Yong Chan Kim and Mamoru Nunokawa obtained in [2] the following beautiful result:

**Theorem A.**  $R_U = 0.164878\dots$ , where  $R_U$  is the root of the equation  $2r^3 - 6r^2 + 7r - 1 = 0$ .

And for  $R_S$ , Shigeyoshi Owa and Nunokawa [1] showed

**Theorem B.**  $R_S > 0.08998\dots$ .

The present author has showed in [3]

$$R_S > 0.104\dots, \text{ and } R_C > 0.056\dots$$

In this note, we shall improve the above two results.

### 2. MAIN RESULTS

**Theorem 1.**  $R_S > 0.137788\dots$ .

**Theorem 2.**  $R_C > 0.075450\dots$ .

*Proof of theorem 1.* Noting  $\frac{f(z)}{z}$  is analytic in  $U$ , we will need two results. The first is the following lemma due to Owa and Nunokawa:

---

1991 *Mathematics Subject Classification.* Primary 30C45; Secondary 26A33, 30C80.

**Lemma 1.**

$$(1) \quad \begin{cases} \frac{1-4r+2r^2}{(1-r)^2} \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1-r)^2} & (0 \leq |z| = r < R_1) \\ 0 \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1-r)^2} & (R_1 \leq r < 1) \end{cases}$$

where  $R_1 = 1 - \frac{1}{\sqrt{2}} = 0.29289 \dots$

The second is Poisson-Jensen's Formula.

$$(2) \quad \log \frac{f(z)}{z} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(\zeta)}{\zeta} \right| \cdot \frac{\zeta + z}{\zeta - z} d\phi$$

where  $\zeta = \rho e^{i\phi}$ ,  $z = r e^{i\theta}$  and  $0 \leq r < \rho < 1$ . By logarithmic differentiation, we find from (2) that

$$(3) \quad R_e \frac{z f'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(\zeta)}{\zeta} \right| R_e \frac{2\zeta z}{(\zeta - z)^2} d\phi$$

where

$$R_e \frac{2\zeta z}{(\zeta - z)^2} = 2\rho r \frac{(\rho^2 + r^2) \cos(\phi - \theta) - 2\rho r}{\{\rho^2 + r^2 - 2\rho r \cos(\phi - \theta)\}^2}$$

For brevity, putting

$$(4) \quad \lambda = \frac{2\rho r}{\rho^2 + r^2}, \quad \alpha = \phi - \theta$$

and

$$(5) \quad \beta = \cos^{-1} \lambda, \quad g(\alpha) = \frac{\lambda(\cos \alpha - \lambda)}{(1 - \lambda \cos \alpha)^2}$$

we have from (3)

$$(6) \quad R_e \frac{z f'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_{-\beta}^{\beta} g(\alpha) \log \left| \frac{f(\zeta)}{\zeta} \right| d\phi + \frac{1}{2\pi} \int_{\beta}^{2\pi-\beta} g(\alpha) \log \left| \frac{f(\zeta)}{\zeta} \right| d\phi.$$

Since  $R_1 > R_U$ , we may assume  $r < R_1$ . And since

$$\begin{cases} g(\alpha) \geq 0 & (-\beta \leq \alpha \leq \beta) \\ g(\alpha) \leq 0 & (\beta \leq \alpha \leq 2\pi\beta), \end{cases}$$

we deduce from (1) and (5)

$$R_e \frac{zf'(z)}{f(z)} \geq 1 + \frac{1}{2\pi} \int_{-\beta}^{\beta} g(\alpha) \log \frac{1-4\rho+2\rho^2}{(1-\rho)^2} d\alpha \\ + \frac{1}{2\pi} \int_{\beta}^{2\pi-\beta} g(\alpha) \log \frac{1}{(1-\rho)^2} d\alpha$$

Evidently

$$\int g(\alpha) d\alpha = \int \frac{\lambda(\cos \alpha - \lambda)}{(1 - \lambda \cos \alpha)^2} d\alpha = \lambda \frac{\sin \alpha}{1 - \lambda \cos \alpha},$$

So we have

$$(7) R_e \frac{zf'(z)}{f(z)} \geq 1 - \frac{\lambda}{\pi} \left\{ \log \frac{(1-\rho)^2}{1-4\rho+2\rho^2} + \log \frac{1}{(1-\rho)^2} \right\} \cdot \frac{\sin \beta}{1 - \lambda \cos \beta}$$

It deduce

$$R_e \frac{zf'(z)}{f(z)} \geq 1 - \frac{\lambda}{\pi \sqrt{1-\lambda^2}} \cdot \log \frac{1}{1-4\rho+2\rho^2},$$

$$R_e \frac{zf'(z)}{f(z)} \geq 1 - \frac{2\rho r}{\pi(\rho^2 - r^2)} \cdot \log \frac{1}{1-4\rho+2\rho^2}.$$

We set

$$\phi(r, \rho) = 1 - \frac{2\rho r}{\pi(\rho^2 - r^2)} \cdot \log \frac{1}{1-4\rho+2\rho^2}.$$

According to

$$\phi(r, \rho_0) \nearrow \quad (\text{for fixed } \rho_0)$$

We deduce

$$\phi(r_0, \rho_0) \geq 0 \implies \phi(r, \rho_0) \geq 0 \quad (\text{for } r \leq r_0).$$

On the other hand

$$\phi(r_3, \rho_3) = 9.47656 \times 10^{-6} > 0$$

So we have

$$R_S \geq r_3 = 0.137788$$

*Proof of theorem 2.* The proof is almost similar. We begin with the following lemma:

**Lemma 2.**

$$(8) \quad \begin{cases} \frac{1-7r+6r^2-2r^3}{(1-r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} & (0 \leq r < R_3) \\ 0 \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} & (R_3 \leq r < 1) \end{cases}$$

where,  $R_3 = R_U = 0.164878 \dots$ . It's very easy to prove the lemma.

First,

$$|f'(z)| \leq \sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1+r}{(1-r)^3}.$$

Next,

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{n=2}^{\infty} n^2 r^{n-1} \\ &\geq 2 - \sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1-7r+6r^2-2r^3}{(1-r)^3}. \end{aligned}$$

From Poisson-Jensen's formula, we obtain

$$(9) \quad \log f'(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f'(\zeta)| \cdot \frac{\zeta+z}{\zeta-z} d\phi,$$

which implies that

$$(10) \quad \begin{aligned} R_e \frac{zf''(z)}{f'(z)} + 1 &= 1 + \frac{1}{2\pi} \int_{-\beta}^{\beta} g(\alpha) \log |f'(\zeta)| d\alpha \\ &+ \frac{1}{2\pi} \int_{\beta}^{2\pi-\beta} g(\alpha) \log |f'(\zeta)| d\alpha. \end{aligned}$$

By virtue of (8) and (10), we obtain that

$$(11) \quad \begin{aligned} R_e \frac{zf''(z)}{f'(z)} + 1 &\geq 1 - \frac{\lambda}{\pi} \left\{ \log \frac{(1-\rho)^3}{1-7\rho+6\rho^2-2\rho^3} \right. \\ &\left. + \log \frac{1+\rho}{(1-\rho)^3} \right\} \cdot \frac{\sin \beta}{1-\lambda \cos \beta}. \end{aligned}$$

$$(11) \quad R_e \frac{zf''(z)}{f'(z)} + 1 \geq 1 - \frac{2\rho r}{\pi(\rho^2 - r^2)} \cdot \log \frac{1+\rho}{1-7\rho+6\rho^2-2\rho^3}$$

We write the right hand side equation of (11) by  $\psi(r, \rho)$ . Finally, if we put  $r_4 = 0.075450$  and  $\rho_4 = 0.127$ , then  $\psi(r_4, \rho_4) = 2.23722 \times 10^{-5}$ . It deduce that

$$R_e \frac{zf''(z)}{f'(z)} + 1 > 0 \quad (0 \leq r_4).$$

So we have  $R_C > 0.075450$ .

## REFERENCES

- [1] S.Owa and M.Nunokawa, *On some inverse properties for univalent functions*, Kyoto Univ.Koukyuroku,1164(2000),73-76.
- [2] Y.C.Kim and M.Nunokawa, *On some radius results for certain analytic functions*, Kyungpook.Math.J.37(1997),61-65.
- [3] R.Yamakawa, *Radius of starlikeness and convexity*, Proceedings of the 2001 international symposium on complex analysis and its applications, 177-180.