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Geometric Properties of Generalized Fractional Integral Operator

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Abstract

Let $\mathcal{A}$ be the class of normalized analytic functions in the unit disk Δ and define the class

$$\mathcal{P}(\beta)=\{f\in\mathcal{A}:\exists\,\varphi\in\mathbf{R}\ \text{ such that } \operatorname{Re}[e^{i\varphi}(f'(z)-\beta)]>0,\ z\in\Delta\}.$$

In this paper we find conditions on the number β and the nonnegative weight function $\lambda(t)$ such that the integral transform

$$V_{\lambda}(f)(z)=\int_{0}^{1}\lambda(t)\frac{f(tz)}{t}\,dt$$

is convex of order γ $(0\leq\gamma\leq 1/2)$ when $f\in\mathcal{P}(\beta)$. Some interesting further consequences are also considered.

Key Words. Gaussian hypergeometric function, integral transform, convex function, starlike function, fractional integral

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in the open unit disk $\Delta=\{z\in\mathbb{C}:\ |z|<1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order γ and convex of order γ in Δ, respectively. In particular, the classes $\mathcal{S}^*(0)=\mathcal{S}^*$ and $\mathcal{K}(0)=\mathcal{K}$ are the familiar ones of starlike and convex functions in Δ, respectively. We note that for $0\leq\gamma<1$,

$$f(z)\in\mathcal{K}(\gamma)\Longleftrightarrow zf'(z)\in\mathcal{S}^*(\gamma)$$

and $f\in\mathcal{S}^*(\gamma)$ if and only if $\operatorname{Re}(zf(z)/f(z))>\gamma$ for $z\in\Delta$.

Let a, b and c be complex numbers with $c \neq 0, -1, -2, \ldots$. Then the *Gaussian/classical hypergeometric function* ${}_2F_1(a,b;c;z) \equiv F(a,b;c;z)$ is defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases}$$

The hypergeometric function $F(a,b;c;z)$ is analytic in Δ and if a or b is a negative integer, then it reduces to a polynomial. For functions $f_j(z)$ $(j=1,2)$ of the forms

$$f_j(z) := \sum_{n=1}^{\infty} a_{j,n}\, z^n \quad (a_{j,1} := 1; j = 1,2),$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) := \sum_{n=1}^{\infty} a_{1,n}\, a_{2,n}\, z^n \qquad (a_{j,1} := 1; j = 1,2).$$

For $f \in \mathcal{A}$, the following special case gives rise to a natural convolution operator $H_{a,b,c}$ defined by

$$H_{a,b,c}(f)(z) := zF(a,b;c;z) * f(z).$$

Note that this is a three-parameter family of operators and contains as special cases several of the known linear integral or differential operators studied by a number of authors. In fact, this operator was considered first time in this form by Hoholov [7] and has been studied extensively by Ponnusamy [11], Ponnusamy and Rønning [14] and many others [2, 8, 5]. For example, by letting $H_{1,b,c}(f) \equiv \mathcal{L}(b,c)(f)$, we get the operator $\mathcal{L}(b,c)(f)$ discussed by Carlson and Shaffer [4]. Clearly, $\mathcal{L}(b,c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c,b)$ is the inverse of $\mathcal{L}(b,c)$, provided that $b \neq 0, -1, -2, \ldots$. Furthermore, $\mathcal{L}(b,b)$ is the unit operator and

$$\mathcal{L}(b,c) = \mathcal{L}(b,e)\mathcal{L}(e,c) = \mathcal{L}(e,c)L(b,e) \quad (c, e \neq 0, -1, -2, \ldots). \tag{1.1}$$

Also, we note that $\mathcal{L}(b,b)f(z) = f(z)$, $\mathcal{L}(2,1)f(z) = zf'(z)$,

$$\mathcal{K}(\gamma) = \mathcal{L}(1,2)\mathcal{S}^*(\gamma) \quad (0 \leq \gamma < 1),$$

$$\mathcal{S}^*(\gamma) = \mathcal{L}(2,1)\mathcal{K}(\gamma) \quad (0 \leq \gamma < 1) \tag{1.2}$$

and the Ruscheweyh derivatives [16] of $f(z)$ are $\mathcal{L}(n+1,1)f(z)$, $n \in \mathbb{N} \cup \{0\}$. For $\beta < 1$, we define

$$\mathcal{P}(\beta) = \{f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \text{ such that } \mathrm{Re}\,[e^{i\varphi}(f'(z) - \beta)] > 0,\ z \in \Delta\}.$$

Throughout this paper we let $\lambda : [0,1] \to \mathbb{R}$ be a nonnegative function with the normalization $\int_0^1 \lambda(t)\, dt = 1$. For certain specific subclasses of $f \in \mathcal{A}$, many authors considered the geometric properties of the integral transform of the form

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt.$$

More recently, starlikeness of this general operator $V_\lambda(f)$ was discussed by Fournier and Ruscheweyh [6] by assuming that $f \in \mathcal{P}(\beta)$. The method of proof is the duality principle developed mainly by Ruscheweyh [17]. This result was later extended by Ponnusamy and Rønning [15] by means of finding conditions such that $V_\lambda(f)$ carries $\mathcal{P}(\beta)$ into starlike functions of order γ, $0 \leq \gamma \leq 1/2$ and was further generalized in [3].

In this paper, we find conditions on β, γ and the function $\lambda(t)$ such that $V_\lambda(f)$ carries $\mathcal{P}(\beta)$ into $\mathcal{K}(\gamma)$. As a consequence of this investigation, a number of new results are established. The following lemma is the key for the proof of our main results.

1.3. Lemma. *Let $\Lambda(t)$ be a real valued monotone decreasing function on $[0,1]$ satisfying $\Lambda(1)=0$, $t\Lambda(t) \to 0$ for $t \to 0^+$ and*

$$-\frac{t\Lambda'(t)}{(1+t)(1-t)^{1+2\gamma}} = \frac{\lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on $(0,1)$ where

$$\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s} ds.$$

If $\beta = \beta(\lambda,\gamma)$ is given by

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t) \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\, dt$$

then $V_\lambda(\mathcal{P}_\beta) \subset K(\gamma)$, $0 \leq \gamma \leq \frac{1}{2}$, where $V_\lambda(f)$ is defined above.

Proof. Proof of this lemma quickly follows from the work of Ponnusamy and Rønning [15] and therefore, we omit the details. □

2. Main Results

In order to apply Lemma 1.3 with $\gamma \in [0, \frac{1}{2}]$ it suffices to show that

$$u(t) = -\frac{t\Lambda'(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on the interval $(0,1)$ where $\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s}\, ds$. Taking the logarithmic derivative of $u(t)$ and using the fact that $\Lambda'(t) = -\frac{\lambda(t)}{t}$, we have

$$\frac{u'(t)}{u(t)} = \frac{\lambda'(t)}{\lambda(t)} + \frac{2\,(\gamma + (1+\gamma)t)}{1-t^2}$$

and therefore, $u(t)$ is decreasing on $(0,1)$ if and only if

$$(1-t^2)\lambda'(t) + 2\,(\gamma + (1+\gamma)t)\,\lambda(t) \leq 0. \tag{2.1}$$

From now on, we define

$$\varphi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n \quad (b_n \geq 0) \tag{2.2}$$

and

$$\lambda(t) = Ct^{b-1}(1-t)^{c-a-b}\varphi(1-t) \tag{2.3}$$

where C is a normalized constant so that $\int_0^1 \lambda(t)\,dt = 1$. For $f \in \mathcal{A}$, Balasubramanian *et al.* [2] defined the operator $P_{a,b,c}$ by

$$P_{a,b,c}(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt,$$

where $\lambda(t)$ is given by (2.3). Special choices of $\varphi(1-t)$ led to various interesting geometric properties concerning certain well-known operators. Observe that $\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s}\,ds$ is monotone decreasing on $[0,1]$, $\lim_{t\to 0+} t\Lambda(t) = 0$ and (2.1) is equivalent to

$$(c-a-3-2\gamma)t^2 + (c-a-b-2\gamma)t + 1 - b \geq -t(1-t^2)\frac{\varphi'(1-t)}{\varphi(1-t)}$$

and this inequality may be rewritten in a convenient form as

$$D(t^2+t) + (1-b)(1-t^2) + t(1-t) \geq -t(1-t^2)\frac{\varphi'(1-t)}{\varphi(1-t)} \tag{2.4}$$

where $D = c-a-b-1-2\gamma$. In view of (2.2), $\varphi(1-t) > 0$ and $\varphi'(1-t) \geq 0$ on $(0,1)$, so that the right hand side of the inequality (2.4) is nonpositive for all $t \in (0,1)$. If we assume that $0 \leq \gamma \leq 1/2$, $a > 0$, $0 < b \leq 1$ and $c \geq a+b+2\gamma+1$, then the left hand side of the inequality (2.4) clearly is nonnegative for all $t \in (0,1)$. Thus, the inequality (2.4) holds for all $t \in (0,1)$. In conclusion, from Lemma 1.3, we have the following theorem and techniques as in the proofs of [5, Theorem 1] and [13, 15, 8] show that the value β in Theorem 2.5 is sharp.

2.5. Theorem. *Let $0 \leq \gamma \leq 1/2$, $a > 0$, $0 < b \leq 1$ and $c \geq a+b+2\gamma+1$, and let $\lambda(t)$ be given by (2.3). Define $\beta = \beta(a,b,c,\gamma)$ by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t)\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\,dt.$$

If $f(z) \in \mathcal{P}(\beta)$, then $P_{a,b,c}(f)(z) \in \mathcal{K}(\gamma)$. The value of β is sharp.

2.6. Corollary. *Let $0 \leq \gamma \leq 1/2$, $0 < a \leq 1$, $0 < b \leq 1$ and $c \geq a+b+2\gamma+1$. Suppose that $\varphi(1-t)$ and C are defined by*

$$\varphi(1-t) = F(c-a, 1-a; c-a-b+1; 1-t) \tag{2.7}$$

and

$$C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}, \tag{2.8}$$

respectively. Define $\beta = \beta(a,b,c,\gamma)$ *by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -C\int_0^1 (1-t)^{c-a-b}t^{b-1}\left(\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right)\varphi(1-t)\,dt. \tag{2.9}$$

If $f(z) \in \mathcal{P}(\beta)$*, then* $H_{a,b,c}(f)(z)$ *defined by*

$$H_{a,b,c}(f)(z) := C\int_0^1 (1-t)^{c-a-b}t^{b-2}\varphi(1-t)f(tz)\,dt.$$

belongs to $\mathcal{K}(\gamma)$*. The value of* β *is sharp.*

Proof. The integral representation for $H_{a,b,c}(f)(z)$ has been obtained in [2, 8]. By (2.7) and (2.8), it follows that the corresponding operator $P_{a,b,c}(f)(z)$ equals $H_{a,b,c}(f)(z)$. Note that the assumption implies that $0 < a \le 1$ and $c - a > 0$ and $c - a - b + 1 > 0$ from which the nonnegativity of $\varphi(1-t)$ on $(0,1)$ is clear. Now, the desired result follows from Theorem 2.5. □

Setting $a = 1$ in Corollary 2.6, we obtain

2.10. Corollary. *Let* $0 \le \gamma \le 1/2$, $0 < b \le 1$ *and* $c \ge b + 2\gamma + 2$*. Also let*

$$\beta(1,b,c,\gamma) = 1 - \frac{1-\gamma}{2[1 - F(2,b;c;-1) - \gamma(1 - F(1,b;c;-1))]}. \tag{2.11}$$

If $\beta(1,b,c,\gamma) \le \beta < 1$ *and* $f(z) \in \mathcal{P}(\beta)$*, then* $\mathcal{L}(b,c)f(z) \in \mathcal{K}(\gamma)$*.*

Proof. Putting $a = 1$ in (2.9) it follows that

$$\begin{aligned}
\frac{\beta - \frac{1}{2}}{1-\beta} &= -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\,dt \\
&= \frac{\Gamma(c)}{(1-\gamma)\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left\{\frac{\gamma}{1+t} - \frac{1}{(1+t)^2}\right\}dt \\
&= \frac{1}{1-\gamma}[\gamma F(1,b;c;-1) - F(2,b;c;-1)]
\end{aligned}$$

where the last step follows from the Euler integral representation. Solving the last equation gives the number $\beta(1,b,c,\gamma)$ given by (2.11). The desired conclusion follows from Corollary 2.6. □

2.12. Theorem. *Let* $-1 < a \le 2$, $0 \le \gamma \le 1/2$ *and* $p \ge 2(1+\gamma)$*. Suppose that* $\beta = \beta(a,p,\gamma)$ *is given by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)}\int_0^1 t^a(\log(1/t))^{p-1}\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\,dt.$$

Then, for $f \in \mathcal{P}(\beta)$, the Hadamard product function $\Phi_p(a;z) * f(z)$ defined by

$$\Phi_p(a;z) * f(z) = \left(\sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n\right) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log 1/t)^{p-1} t^{a-1} f(tz) dt$$

belongs to $\mathcal{K}(\gamma)$. The value of β is sharp.

Proof. To obtain this theorem, we choose $\phi(1-t)$ and $\lambda(t)$ in Theorem 2.5 as

$$\phi(1-t) = \left(\frac{\log(1/t)}{1-t}\right)^{p-1} = \left(\frac{-\log(1-(1-t))}{1-t}\right)^{p-1},$$

and

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (1-t)^{p-1} \phi(1-t),$$

respectively. The desired conclusion follows from Theorem 2.5 and the hypotheses. □

Our final application concerns the integral operator studied by Ponnusamy [12], Ponnusamy and Rønning [13] and later by Balasubramanian, Ponnusamy and Vuorinen [2]. Define

$$\lambda(t) = \begin{cases} (a+1)(b+1)\left(\dfrac{t^a(1-t^{b-a})}{b-a}\right) & \text{for } b \neq a,\ a > -1,\ b > -1, \\ (a+1)^2 t^a \log(1/t) & \text{for } b = a,\ a > -1. \end{cases} \tag{2.13}$$

With this $\lambda(t)$, we have an integral transform

$$G_f(a,b;z) := \left(\sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n\right) * f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt.$$

In view of symmetry between a and b, without loss of generality, we assume that $b > a$ in the case $b \neq a$. Note that in the limiting case $b \to \infty$ $(b \neq a)$, $G_f(a,b;z)$ reduces to a well-known Bernadi operator given by

$$G_f(a,\infty;z) := \left(\sum_{n=1}^{\infty} \frac{1+a}{n+a} z^n\right) * f(z) = \frac{1+a}{z^a} \int_0^z t^{a-1} f(t)\, dt \equiv \mathcal{L}(a+1,a+2)f(z).$$

2.14. Theorem. *Let $b > -1$, $a > -1$ be such that any one of the following conditions holds:*

(i) $-1 < a \leq 0$ and $a = b$

(ii) $-1 < a \leq 0$ and $b > a$ *with* $-1 < b \leq 2$.

Suppose that $\lambda(t)$ is defined by (2.13) and β is given by

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \frac{\lambda(t)}{(1+t)^2}\, dt.$$

If $f \in \mathcal{P}(\beta)$, then the function $G_f(a,b;z)$ is convex in Δ. The value of β is sharp.

Proof. Clearly, as in the proof of Theorem 2.5, it suffices to verify the inequality (2.1) for the $\lambda(t)$ defined by (2.13). Now, for the $\lambda(t)$ given by (2.13), we have

$$\lambda'(t) = \begin{cases} \dfrac{(a+1)(b+1)}{b-a} t^{a-1} \left(a - bt^{b-a}\right) & \text{for } b > a > -1, \\ (a+1)^2 \left(-1 + a\log(1/t)\right) t^{a-1} & \text{for } b = a > -1. \end{cases}$$

Case (i): Let $b = a > -1$. If we substitute the $\lambda(t)$ and the $t\lambda'(t)$ expression in (2.1), the inequality (2.1) is seen to be equivalent to

$$-a\left(1-t^2\right)\log(1/t) + 1 - t^2 - 2t^2\log(1/t) \geq 0, \quad t \in (0,1). \tag{2.15}$$

Clearly, as $-1 < a \leq 0$, this inequality holds if it holds for $a = 0$. Substituting $a = 0$, this becomes

$$1 - t^2 - 2t^2\log(1/t) \geq 0, \quad t \in (0,1),$$

which, for $t = e^{-x}$, is equivalent to

$$e^{2x} \geq 1 + 2x, \quad x \geq 0.$$

Since this inequality holds for all $x \geq 0$, the inequality (2.15) holds for all $t \in (0,1)$ and the desired conclusion holds in this case.

Case (ii): Let $b > a > -1$. If we substitute the $\lambda(t)$ given by (2.13) and the corresponding $t\lambda'(t)$ expression in (2.1), the inequality (2.1) is seen to be equivalent to

$$\left(1-t^2\right)\left(at^{a-1} - bt^{b-1}\right) + 2\left(t^{a+1} - t^{b+1}\right) \leq 0 \tag{2.16}$$

which may be rewritten as

$$\psi_t(a) - \psi_t(b) \leq 0, \quad t \in (0,1),$$

where

$$\psi_t(a) = a\left(1-t^2\right)t^{a-1} + 2t^{a+1}.$$

For each fixed $t \in (0,1)$, we first claim that $\psi_t(a)$ is an increasing function of a. Differentiating $\psi_t(a)$ with respect to a, we find that

$$\psi_t'(a) = t^{a-1}\left[1 - t^2 - 2t^2\log(1/t) - a\left(1-t^2\right)\log(1/t)\right].$$

Using the previous case, namely the inequality (2.15), it follows that $\psi_t'(a) \geq 0$ for all $a \in (-1,0)$ and for $t \in (0,1)$. In particular, for $b > a$ with $b \in (-1,0)$ and $a \in (-1,0)$, the inequality (2.16) holds.

When $b > a$ with $0 \leq b \leq 2$ and $a \in (-1,0]$, we have

$$\psi_t(a) \leq \psi_t(0) = 2t \quad \text{for } t \in (0,1).$$

Now, we claim that for $b > a$ with $0 \leq b \leq 2$ and $a \in (-1, 0]$, the inequality

$$2t \leq \psi_t(b) = b\left(1 - t^2\right)t^{b-1} + 2t^{b+1}$$

holds for all $t \in (0,1)$. To verify this inequality, we rewrite it as

$$2\left(t^{-b} - 1\right) \leq b\left(t^{-2} - 1\right) \quad \text{for } t \in (0,1)$$

which, for $t = 1 - x$, is equivalent to the inequality

$$2\left((1-x)^{-b} - 1\right) \leq b\left((1-x)^{-2} - 1\right) \quad \text{for } x \in (0,1). \tag{2.17}$$

Since

$$2(b)_n \leq b(2)_n \quad \text{for all } n \geq 1,$$

a comparison of the coefficients of x^n on both sides of the inequality (2.17) implies that (2.17) clearly holds. Thus, for $0 \leq b \leq 2$ and $a \in (-1, 0]$ with $b > a$, we have

$$\psi_t(a) \leq 2t \leq \psi_t(b) \quad \text{for } t \in (0,1)$$

and the proof is now complete. □

3. The Fractional Integral Operator

There are a number of definitions for fractional calculus operators in the literature. We use here the following definition due to Saigo [18] (see also [10, 19]).

For $\lambda > 0$, μ, $\nu \in \mathbb{R}$, the fractional integral operator $\mathcal{I}^{\lambda,\mu,\nu}$ is defined by

$$\mathcal{I}^{\lambda,\mu,\nu} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z - \zeta)^{\lambda-1} F(\lambda + \mu, -\nu; \lambda; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta,$$

where $f(z)$ is taken to be an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = \mathcal{O}(|z|^\epsilon) \quad (z \to 0)$$

for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multivaluedness of $(z - \zeta)^{\lambda-1}$ is removed by requiring that $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

In [10], Owa *et al.* considered the normalized fractional integral operator by defining $\mathcal{J}^{\lambda,\mu,\nu}$ by

$$\mathcal{J}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(2-\mu+\nu)} z^\mu \mathcal{I}^{\lambda,\mu,\nu} f(z), \quad \min\{\lambda + \nu, -\mu + \nu, -\mu\} > -2.$$

Clearly, $\mathcal{J}^{\lambda,\mu,\nu}$ maps $\mathcal{A}$ onto itself and for $f \in \mathcal{A}$

$$\mathcal{J}^{\lambda,\mu,\nu} f(z) = \mathcal{L}(2, 2-\mu)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z). \tag{3.1}$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha,\gamma)$ if

$$(f * s_\alpha)(z) \in \mathcal{S}^*(\gamma) \quad (0 \leq \alpha < 1;\ 0 \leq \gamma < 1).$$

Here $s_\alpha(z) = z/(1-z)^{-2(1-\alpha)}$ $(0 \leq \alpha < 1)$ denotes the well-known extremal function for the class $\mathcal{S}^*(\alpha)$. Note that

$$\mathcal{R}(\alpha,\gamma) = \mathcal{L}(1, 2-2\alpha)\mathcal{S}^*(\gamma) \tag{3.2}$$

and $\mathcal{R}(\alpha,\alpha) \equiv \mathcal{R}(\alpha)$ is the subclass of $\mathcal{A}$ consisting of ***prestarlike functions of order*** α which was introduced by Suffridge [21]. In [20], it is shown that $\mathcal{R}(\alpha) \subset \mathcal{S}$ if and only if $\alpha \leq 1/2$.

Our result in this section is to obtain a univalence criterion for the operator $\mathcal{J}^{\lambda,\mu,\nu}$.

3.3. Theorem. *Let $0 \leq \gamma \leq 1/2$, $0 \leq \mu < 2$, $\lambda \geq 2(1+\gamma)-\mu$ and $\mu - 2 < \nu \leq \mu - 1$. Define $\beta = \beta(\lambda,\mu,\nu,\gamma)$ by*

$$\beta = 1 - \frac{1-\gamma}{2\left[1 - F(2, 2-\mu+\nu; 2+\lambda+\nu; -1) - \gamma\left(1 - F(1, 2-\mu+\nu; 2+\lambda+\nu; -1)\right)\right]}.$$

If $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{J}^{\lambda,\mu,\nu} f(z) \in \mathcal{R}(\mu/2,\gamma)$.

Proof. Making use of (1.1) and (3.1), we note that

$$\begin{aligned} \mathcal{J}^{\lambda,\mu,\nu} f(z) &= \mathcal{L}(2, 2-\mu)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z) \\ &= \mathcal{L}(1, 2-\mu)\mathcal{L}(2,1)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z). \end{aligned} \tag{3.4}$$

By using Corollary 2.10, we obtain

$$\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu) f(z) \in \mathcal{K}(\gamma).$$

Since $0 \leq \mu < 2$, from (1.2), (3.2) and (3.4), we have $\mathcal{J}^{\lambda,\mu,\nu} f(z) \in \mathcal{R}(\mu/2,\gamma)$ and we complete the proof. □

Taking $\mu = 2\gamma$ in Theorem 3.3, we get

3.5. Corollary. *Let $0 \leq \gamma \leq 1/2$, $\lambda \geq 2$ and $2(\gamma - 1) < \nu \leq 2\gamma - 1$. Define $\beta = \beta(\lambda,\nu,\gamma)$ by*

$$\beta = 1 - \frac{1-\gamma}{2\left[1 - F(2, 2-2\gamma+\nu; 2+\lambda+\nu; -1) - \gamma\left(1 - F(1, 2-2\gamma+\nu; 2+\lambda+\nu; -1)\right)\right]}.$$

If $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{J}_{0,z}^{\lambda,2\gamma,\nu} f(z) \in \mathcal{R}(\gamma) \subset \mathcal{S}$.

Proof. If we put $\mu = 2\gamma$ in Theorem 3.3, then

$$\mathcal{J}_{0,z}^{\lambda,2\gamma,\nu} f(z) \in \mathcal{R}(\gamma,\gamma) = \mathcal{R}(\gamma).$$

Since $\gamma \leq 1/2$, we have $\mathcal{R}(\gamma) \subset \mathcal{S}$ and therefore, the proof is completed. □

3.6. Remark. In [2], Balasubramanian *et al.* found the conditions on the number β and the function $\lambda(t)$ such that $P_{a,b,c}(f)(z) \in \mathcal{S}^*(\gamma)$ $(0 \le \gamma \le 1/2)$. Since

$$\mathcal{J}^{\lambda,\mu,\nu}f(z) = P_{1-\mu,2,\lambda-\nu+2}(f)(z)$$

with

$$\phi(1-t) = F(\lambda+\mu, -\nu; \lambda; 1-t)$$

and

$$C = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)},$$

it is easy to find that the condition on β and $\lambda(t)$ such that $\mathcal{J}^{\lambda,\mu,\nu}f(z) \in \mathcal{S}^*(\gamma)$.

Finally, by using Lemma 1.3 again, we investigate convexity of the operator $\mathcal{J}^{\lambda,\mu,\nu}$.

3.7. Theorem. *Let $0 \le \gamma \le 1/2$, $0 < \lambda \le 1+2\gamma$, $2 < \mu < 3$ and $\nu > \mu - 2$. Define $\beta = \beta(\lambda, \mu, \nu, \gamma)$ by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} \int_0^1 \frac{t(1-t)^{\lambda-1}(1-\gamma(1+t))}{(1-\gamma)(1+t)^2} F(\lambda+\mu, -\nu; \lambda; 1-t)\, dt.$$

If $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{J}^{\lambda,\mu,\nu}f(z) \in \mathcal{K}(\gamma)$. The value of β is sharp.

Proof. Let $0 \le \gamma \le 1/2$, $0 < \lambda \le 1+2\gamma$, $2 < \mu < 3$, $\nu > \mu - 2$, and let

$$\lambda(t) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} t(1-t)^{\lambda-1} F(\lambda+\mu, -\nu; \lambda; 1-t). \tag{3.8}$$

Then we can easily see that $\int_0^1 \lambda(t)dt = 1$, $\Lambda(t) = \int_t^1 \lambda(s)ds/s$ is monotone decreasing on $[0,1]$ and $\lim_{t\to 0+} t\Lambda(t) = 0$. Also we find that the function $u(t) = \lambda(t)/(1+t)(1-t)^{1+2\gamma}$ is decreasing on $(0,1)$, where $\lambda(t)$ is given by (3.8). Hence, $t\Lambda'(t)/(1+t)(1-t)^{1+2\gamma} = -u(t)$ is increasing on $(0,1)$. From Lemma 1.3, we obtain the desired result. □

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