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GEOMETRIC PROPERTIES OF SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS

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Abstract

The main object of this paper is to investigate several geometric properties of the solutions of the second-order linear differential equation:

$$w''(z) + a(z)\, w'(z) + b(z)\, w(z) = 0,$$

where the functions $a(z)$ and $b(z)$ are analytic in the open unit disk $\mathbb{U}$. Relevant connections of the results presented in this paper with those given earlier by (for example) M.S. Robertson, S.S. Miller, and H. Saitoh are also considered.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}$, $\mathcal{S}^*$, and $\mathcal{S}^*(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, *univalent*, *starlike with respect to the origin*, and *starlike of order* α in $\mathbb{U}$ $(0 \leqq \alpha < 1)$. Thus, by definition, we have (see, for details, [2] and [8]; see also [7] and [11])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}; 0 \leqq \alpha < 1) \right\} \tag{1.2}$$

and

$$\mathcal{S}^* := \mathcal{S}^*(\alpha)|_{\alpha=0} = \mathcal{S}^*(0). \tag{1.3}$$

For functions $f \in \mathcal{A}$ with $f'(z) \neq 0$ $(z \in \mathbb{U})$, we define the Schwarzian derivative of $f(z)$ by

$$S(f,z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1.4}$$

$$(f \in \mathcal{A};\ f'(z) \neq 0\ (z \in \mathbb{U})).$$

We begin by recalling the following result of Miller [4].

Theorem A (Miller [4]). *Let the function* $p(z)$ *be analytic in* $\mathbb{U}$ *with*

$$|zp(z)| < 1 \qquad (z \in \mathbb{U}).$$

Also let $v(z)$ *denote the unique solution of the following initial-value problem:*

$$v''(z) + p(z)v(z) = 0 \qquad (v(0) = 0;\ v'(0) = 1) \tag{1.5}$$

in $\mathbb{U}$. *Then*

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < 1 \qquad (z \in \mathbb{U}) \tag{1.6}$$

and $v(z)$ *is a starlike conformal map of the unit disk* $\mathbb{U}$.

Theorem A is related rather closely to some earlier results of Robertson [9] and Nehari [6], which we recall here as Theorem B and Theorem C below.

Theorem B (Robertson [9]). *Let* $zp(z)$ *be analytic in* $\mathbb{U}$ *and*

$$\Re\{z^2 p(z)\} \leqq \frac{\pi^2}{4}|z|^2 \qquad (z \in \mathbb{U}). \tag{1.7}$$

Then the unique solution $W = W(z)$ of the following initial-value problem:

$$W''(z) + p(z)W(z) = 0 \qquad (W(0) = 0;\ W'(0) = 1) \tag{1.8}$$

is univalent and starlike in $\mathbb{U}$. *The constant* $\frac{\pi^2}{4}$ *in the inequality* (1.7) *is the best possible.*

Theorem C (Nehari [6]). *If* $f \in \mathcal{A}$ *satisfies the following inequality involving its Schwarzian derivative defined by* (1.4):

$$|S(f,z)| \leqq \frac{\pi^2}{2} \qquad (z \in \mathbb{U}), \tag{1.9}$$

then $f \in \mathcal{S}$. *The result is sharp for the function* $f(z)$ *given by*

$$f(z) = \frac{e^{i\pi z} - 1}{i\pi} \qquad \left(i := \sqrt{-1}\right). \tag{1.10}$$

Remark 1. By setting

$$p(z) = \frac{1}{2}S(f,z) \qquad (z \in \mathbb{U}) \tag{1.11}$$

and using (1.9), we obtain the inequality (1.7). Obviously, therefore, the hypothesis in Theorem C is stronger than that in Theorem B.

In the present paper, we aim at investigating several *geometric* properties of the solutions of the following initial-value problem which involves a general family of second-order linear differential equations:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0 \tag{1.12}$$
$$(w(0) = 0;\ w'(0) = 1),$$

where the functions $a(z)$ and $b(z)$ are analytic in $\mathbb{U}$ (see [3]). We also show how our results are related to those of (for example) Robertson [9], Miller [4], and Saitoh [10].

2. A Class of Bounded Functions

Let $\mathcal{B}_J$ denote the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

analytic in $\mathbb{U}$, for which

$$|w(z)| < J \qquad (z \in \mathbb{U};\ J > 0). \tag{2.2}$$

If $g(z) \in \mathcal{B}_J$, then we can show (by using the Schwarz lemma [1]) that the function $w(z)$ defined by

$$w(z) := z^{-\frac{1}{2}} \int_0^z g(t)\, t^{-\frac{1}{2}}\, dt \tag{2.3}$$

is also in the class $\mathcal{B}_J$. Thus, in terms of derivatives, we have

$$\left|\frac{1}{2}w(z)+zw'(z)\right|<J \quad (z\in\mathbb{U}) \Longrightarrow |w(z)|<J \quad (z\in\mathbb{U}). \tag{2.4}$$

Furthermore, by letting

$$h(u,v):=\frac{1}{2}u+v, \tag{2.5}$$

we can rewrite (2.4) in the form:

$$|h(w(z),zw'(z))|<J \quad (z\in\mathbb{U}) \Longrightarrow |w(z)|<J \quad (z\in\mathbb{U}). \tag{2.6}$$

In this section, we show that the implication (2.6) holds true for functions $h(u,v)$ in the class $\mathcal{H}_J$ given by Definition 1 below (see also [5]).

Definition 1. Let $\mathcal{H}_J$ be the class of *complex* functions $h(u,v)$ satisfying *each* of the following conditions:

(i) $h(u,v)$ is continuous in a domain $\mathbb{D}\subset\mathbb{C}\times\mathbb{C}$;
(ii) $(0,0)\in\mathbb{D}$ and $|h(0,0)|<J \quad (J>0)$;
(iii) $\left|h\left(Je^{i\theta},Ke^{i\theta}\right)\right|\geqq J$ whenever

$$\left(Je^{i\theta},Ke^{i\theta}\right)\in\mathbb{D} \qquad (\theta\in\mathbb{R};\ K\geqq J>0).$$

Example 1. It is easily seen that the function

$$h(u,v)=\alpha u+v \qquad (\Re(\alpha)\geqq 0;\ \mathbb{D}=\mathbb{C}\times\mathbb{C}) \tag{2.7}$$

is in the class $\mathcal{H}_J$.

Definition 2. Let $h\in\mathcal{H}_J$ with the corresponding domain $\mathbb{D}$. We denote by $\mathcal{B}_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in $\mathbb{U}$ and satisfy each of the following conditions:

(i) $(w(z),zw'(z))\in\mathbb{D}$;
(ii) $|h(w(z),zw'(z))|<J \quad (z\in\mathbb{U};\ J>0)$.

The function class $\mathcal{B}_J(h)$ is not empty. Indeed, for any given function $h\in\mathcal{H}_J$, we have

$$w(z)=c_1z\in\mathcal{B}_J(h) \tag{2.8}$$

for sufficiently small $|c_1|$ depending on h.

Theorem D (Saitoh [10]). *For any* $h\in\mathcal{H}_J$,

$$\mathcal{B}_J(h)\subset\mathcal{B}_J \qquad (h\in\mathcal{H}_J;\ J>0).$$

Remark 2. Theorem D shows that, if $h \in \mathcal{H}_J$ (with the corresponding domain $\mathbb{D}$) and if $w(z)$, given by (2.1), is analytic in $\mathbb{U}$ and

$$(w(z), zw'(z)) \in \mathbb{D},$$

then the implication (2.4) holds true.

Theorem D leads us immediately to the following result, which was also given by Saitoh [10].

Theorem E (Saitoh [10]). *Let $h \in \mathcal{H}_J$ and let the function $b(z)$ be analytic in $\mathbb{U}$ with*

$$|b(z)| < J \qquad (z \in \mathbb{U};\ J > 0).$$

If the initial-value problem:

$$h(w(z), zw'(z)) = b(z) \qquad (w(0) = 0) \tag{2.9}$$

has a solution $w(z)$ analytic in $\mathbb{U}$, then

$$|w(z)| < J \qquad (z \in \mathbb{U};\ J > 0).$$

3. Main Results and Their Consequences

One of our *main* results is contained in the following theorem.

Theorem 1. *Let the functions $a(z)$ and $b(z)$ be analytic in $\mathbb{U}$ with*

$$\left| z^2 \left\{ b(z) - \frac{1}{2} a'(z) - \frac{1}{4} [a(z)]^2 \right\} \right| < J \qquad (z \in \mathbb{U};\ J > 0) \tag{3.1}$$

and

$$\Re\{za(z)\} > -2J \qquad (z \in \mathbb{U};\ J > 0). \tag{3.2}$$

Also let $w(z)$ denote the solution of the initial-value problem (1.12) *in $\mathbb{U}$. Then*

$$1 - J - \frac{1}{2}\Re\{za(z)\} < \Re\left(\frac{zw'(z)}{w(z)}\right) < 1 + J - \frac{1}{2}\Re\{za(z)\} \tag{3.3}$$

$$(z \in \mathbb{U};\ J > 0).$$

Proof. First of all, by means of the transformation:

$$w(z) = \exp\left(-\frac{1}{2}\int_0^z a(t)\,dt\right) \cdot v(z), \tag{3.4}$$

we can rewrite the initial-value problem (1.12) in the *normal* form:

$$v''(z) + \left\{ b(z) - \frac{1}{2} a'(z) - \frac{1}{4} [a(z)]^2 \right\} v(z) = 0 \tag{3.5}$$

$$(v(0) = 0;\ v'(0) = 1).$$

If we now put

$$u(z) = \frac{zv'(z)}{v(z)} - 1 \qquad (z \in \mathbb{U}), \tag{3.6}$$

then $u(z)$ is analytic in $\mathbb{U}$ with $u(0) = 0$, and (3.5) becomes

$$[u(z)]^2 + u(z) + zu'(z) = -z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right\} \qquad (u(0) = 0) \tag{3.7}$$

or, equivalently,

$$h(u(z), zu'(z)) = -z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right\} \qquad (u(0) = 0), \tag{3.8}$$

where, for convenience,

$$h(\xi, \eta) := \xi^2 + \xi + \eta. \tag{3.9}$$

It is easily observed from (3.1), (3.2), and (3.8) that $h(\xi, \eta) \in \mathcal{H}_J$, that is, that

(i) $h(\xi, \eta)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;

(ii) $(0,0) \in \mathbb{D}$ and $|h(0,0)| = 0 < J \quad (J > 0)$;

(iii) For $\left(Je^{i\theta}, Ke^{i\theta}\right) \in \mathbb{D} \quad (\theta \in \mathbb{R};\ K \geqq J > 0)$,

$$\begin{aligned} \left|h\left(Je^{i\theta}, Ke^{i\theta}\right)\right| &= \left|J^2 e^{2i\theta} + Je^{i\theta} + Ke^{i\theta}\right| \\ &= \left|J^2 e^{i\theta} + J + K\right| \geqq J. \end{aligned}$$

Thus, by applying Theorem E, we find from the hypothesis (3.1) of Theorem 1 that

$$|u(z)| < J \qquad (z \in \mathbb{U};\ J > 0),$$

which, in view of the relationship (3.6), yields

$$1 - J < \Re\left(\frac{zv'(z)}{v(z)}\right) < 1 + J \qquad (z \in \mathbb{U};\ J > 0). \tag{3.10}$$

Next, by logarithmically differentiating (3.4) in its *equivalent* form:

$$v(z) = \exp\left(\frac{1}{2}\int_0^z a(t)\,dt\right) \cdot w(z),$$

we have

$$\frac{zv'(z)}{v(z)} = \frac{zw'(z)}{w(z)} + \frac{1}{2}za(z), \tag{3.11}$$

so that (3.10) becomes

$$1 - J < \Re\left(\frac{zw'(z)}{w(z)}\right) + \frac{1}{2}\Re\{za(z)\} < 1 + J \tag{3.12}$$

$$(z \in \mathbb{U};\ J > 0),$$

which obviously yields the assertion (3.3) of Theorem 1.

Remark 3. If, in Theorem 1, we have

$$\Re\{za(z)\} \leqq 2(1-J) \qquad (z \in \mathbb{U};\ J>0), \tag{3.13}$$

so that

$$0 \leqq 1 - J - \frac{1}{2}\Re\{za(z)\} < 1 \qquad (z \in \mathbb{U};\ J>0), \tag{3.14}$$

then the assertion (3.3) immediately yields

$$w(z) \in \mathcal{S}^*\left(1 - J - \frac{1}{2}\Re\{za(z)\}\right)$$

in conjunction with the definition (1.2).

Example 2. If we let

$$a(z) = -2Jz \quad \text{and} \quad b(z) = J^2z^2 \qquad (J>0) \tag{3.15}$$

in Theorem 1, then the solution of the initial-value problem:

$$w''(z) - 2Jzw'(z) + J^2z^2w(z) = 0 \tag{3.16}$$
$$(w(0)=0;\ w'(0)=1)$$

is given by

$$w(z) = \frac{1}{\sqrt{J}}\exp\left(\frac{1}{2}Jz^2\right)\cdot\sin\left(z\sqrt{J}\right). \tag{3.17}$$

In this case, if we further assume that

$$0 < J \leqq \frac{1}{2},$$

then

$$w(z) \in \mathcal{S}^*(1-2J) \qquad \left(0 < J \leqq \frac{1}{2}\right),$$

so that, in particular, we have

$$J = \frac{1}{2}:\ w(z) = \sqrt{2}\exp\left(\frac{1}{4}z^2\right)\cdot\sin\left(\frac{z}{\sqrt{2}}\right) \in \mathcal{S}^*,$$
$$J = \frac{1}{3}:\ w(z) = \sqrt{3}\exp\left(\frac{1}{6}z^2\right)\cdot\sin\left(\frac{z}{\sqrt{3}}\right) \in \mathcal{S}^*\left(\frac{1}{3}\right),$$
$$J = \frac{1}{4}:\ w(z) = 2\exp\left(\frac{1}{8}z^2\right)\cdot\sin\left(\frac{z}{2}\right) \in \mathcal{S}^*\left(\frac{1}{2}\right),$$

and so on.

Example 3. For

$$a(z) = -2Jz \quad \text{and} \quad b(z) = \lambda \qquad (J>0;\ \lambda \in \mathbb{C}), \tag{3.18}$$

the initial-value problem (1.12) becomes

$$w''(z) - 2Jzw'(z) + \lambda w(z) = 0 \qquad (3.19)$$

$$(w(0) = 0;\ w'(0) = 1),$$

which, under the transformation:

$$w(z) = \exp\left(\frac{1}{2}Jz^2\right) \cdot v(z), \qquad (3.20)$$

assumes the *normal* form:

$$v''(z) + \left(\lambda + J - J^2z^2\right) v(z) = 0 \qquad (3.21)$$

$$(v(0) = 0;\ v'(0) = 1).$$

Remark 4. In their special case when $J = \frac{1}{2}$, the differential equations in (3.19) and (3.21) can be identified with such classical differential equations as Hermite's equation and Weber's equation, respectively (*cf.*, *e.g.*, [1] and [12]).

Next we prove the following result for the solution of the initial-value problem (3.21).

Theorem 2. *If*

$$\left|\lambda + J - J^2z^2\right| < T \qquad (z \in \mathbb{U};\ J, T > 0), \qquad (3.22)$$

then the solution $v(z)$ *of the initial-value problem* (3.21) *satisfies the inequality*:

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < T \qquad (z \in \mathbb{U};\ T > 0). \qquad (3.23)$$

Proof. Just as in our demonstration of Theorem 1, the function $u(z)$, given by (3.6), is analytic in $\mathbb{U}$, $u(0) = 0$, and [*cf.* Equation (3.7)]

$$h(u(z), zu'(z)) = -z^2\left(\lambda + J - J^2z^2\right) \qquad (u(0) = 0), \qquad (3.24)$$

where $h(\xi, \eta)$ is defined, as before, by (3.9).

Now it is easily seen from (3.22) and (3.24) that $h(\xi, \eta) \in \mathcal{H}_T$, that is, that

(i) $h(\xi, \eta)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;
(ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| = 0 < T \quad (T > 0)$;
(iii) For $\left(Te^{i\theta}, Ke^{i\theta}\right) \in \mathbb{D} \quad (\theta \in \mathbb{R};\ K \geqq T > 0)$,

$$\left|h\left(Te^{i\theta}, Ke^{i\theta}\right)\right| \geqq T.$$

By applying Theorem E, we thus find from the hypothesis (3.22) of Theorem 2 that

$$|u(z)| < T \qquad (z \in \mathbb{U};\ T > 0),$$

which, in view of the relationship (3.6) again, leads us at once to the assertion (3.23) of Theorem 2.

Remark 5. If $0 < T \leqq 1$, then Theorem 2 yields the following geometric property:

$$v(z) \in \mathcal{S}^*(1-T) \qquad (0 < T \leqq 1)$$

for the solution $v(z)$ of the initial-value problem (3.21).

By putting $J = \frac{1}{2}$ and $T = 1$ in Theorem 2, we obtain the following known result.

Corollary (Saitoh [10]). *If*

$$\left|\lambda + \frac{1}{2} - \frac{1}{4}z^2\right| < 1 \qquad (z \in \mathbb{U}), \tag{3.25}$$

then the solution $v(z)$ *of Weber's differential equation:*

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{1}{4}z^2\right) v(z) = 0 \tag{3.26}$$

$$(v(0) = 0;\ v'(0) = 1)$$

is starlike in $\mathbb{U}$.

Remark 6. The solutions of Weber's differential equation in (3.26) are expressed as the parabolic cylinder (or Weber's) function $D_\lambda(z)$ defined by (*cf.*, *e.g.*, [1, p. 39 *et seq.*])

$$D_\lambda(z) := 2^{\lambda/2}\sqrt{\pi}\exp\left(-\frac{1}{4}z^2\right)\left[\frac{1}{\Gamma\left(\frac{1-\lambda}{2}\right)}\ {}_1F_1\left(-\frac{\lambda}{2};\frac{1}{2};\frac{1}{2}z^2\right)\right.$$
$$\left. -\frac{z\sqrt{2}}{\Gamma\left(\frac{1-\lambda}{2}\right)}\ {}_1F_1\left(\frac{1-\lambda}{2};\frac{3}{2};\frac{1}{2}z^2\right)\right], \tag{3.27}$$

where ${}_1F_1(\alpha;\gamma;z)$ denotes the confluent hypergeometric function (see, for details, [1] and [12]).

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