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Recurrent dimensions of quasi-periodic orbits with multiple frequencies: Extended common multiples and Diophantine conditions

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1. Introduction

We consider a finite set of irrational numbers $\{\tau_1, \tau_2, ..., \tau_n\}$, which are rationally independent. Diophanitine conditions for these numbers, which are very well known for the relations to the KAM theorem, are as follows:

There exist constants $\gamma, d : \gamma > 0,\ d > n$, which satisfy

$$|(\tau_1 m_1 + \tau_2 m_2 + \cdots + \tau_n m_n) - l| \geq \frac{\gamma}{|m|^d}$$

for every integers $m = (m_1, m_2, ..., m_n) \in \mathbf{Z}^n,\ l \in \mathbf{Z}$ where $|\cdot|$ denotes a usual Euclidean norm.

In our previous paper [5] we treat the case $n = 2$ and consider the Diophantine sequences of $\{n_j/m_j, r_j/l_j\}$ for $\{\tau_1, \tau_2\}$, respectively. Then we define the extended sets of positive integers, denoted by $[M]_k^\alpha, [L]_s^\beta$, which are given by using certain finite sequences in $\{m_j\}, \{l_j\}$ as bases, where k and s point out the largest subscripts of the finite sequences and $\alpha, \beta : 0 \leq \alpha, \beta < 1$ are the parameters given by their lengths of the finite sequences in $\{m_j\}, \{l_j\}$, respectively. We consider a sequence of positive integers in the intersection of the two sets: $T_j \in [M]_{k_j}^{\alpha_j} \cap [L]_{s_j}^{\beta_j}$, which we call extended common multiples (abr. ECM). We introduce δ_0-ECM condition (or pairs) where

$$\delta_0 := \liminf_j \max\{\alpha_j, \beta_j\} < 1$$

and, also we introduce a parametrizing Diophanitne condition, which we call d_0-(D) condition where d_0 is the infimum of the constants d in the usual Diophanitne condition. Under some restrictive condition for the partial quotients of continued fraction expansions (Hypotheses (A), (A') in [5]) we have shown the relations between the δ_0-ECM condition and d_0-(D) condition. In this paper we treat the general case $n \geq 2$ and show the relation between the two conditions without assuming Hypotheses (A) or (A').

Our plan of this paper is as follows. In section 2 we introduce the definition of Extended Common Multiples. In section 3, introducing the definitions of δ_0-ECM condition and d_0-(D) condition, we show the inequality relations between these two parameters δ_0 and d_0. In section 4 we estimate the recurrent dimensions of quasi-periodic orbits with n irrational frequencies of (KL) class.

2. Extended common multiples

Let us call an irrational number τ a Khinchin-Lévy class number or (KL) class number if, for the denominators $\{m_j\}$ of the Diophantine approximation of τ, there exist constants $C_1, C_2 > 1$, which satisfy

$$C_1^j \leq m_j \leq C_2^j, \qquad \forall j \geq j_0 \tag{2.1}$$

for some $j_0 \in \mathbf{N}$.

Remark 2.1. In [1] Khinchin proved that almost all irrational numbers satisfy (2.1) and furthermore, he had shown that there exists a constant γ_0, which satisfies

$$\lim_{k\to\infty} (m_k)^{\frac{1}{k}} = \gamma_0$$

for almost all irrational numbers. By Lévy this constant was estimated:

$$\gamma_0 = e^{\frac{\pi^2}{12\log 2}} \sim 3.27582...$$

Let $\{\tau_1, \tau_2, ..., \tau_n\}$ be rationally independent irrational numbers, $\tau_i = [a_{i,1}, a_{i,2}, ..., a_{i,j}, ..]$ be the continued fraction expansion and $\{n_{i,1}/m_{i,1}, n_{i,2}/m_{i,2}, ..., n_{i,j}/m_{i,j}, ...\}$ be the Diophantine sequence of τ_i for each $i \in \{1, ..., n\}$. We assume that $\tau_i, i = 1, ..., n$ are (KL) class numbers:

There exist constants $C_{i,1}, C_{i,2} > 0$, which satisfy

$$C_{i,1}^j \leq m_{i,j} \leq C_{i,2}^j, \qquad \forall j \geq j_{i,0} \tag{2.2}$$

for some $j_{i,0} \in \mathbf{N}, i = 1, ..., n$.

In view of Remark 2.1 we use the following notations:

$$E_1 = \min_i C_{i,1}, \quad E_2 = \max_i C_{i,2}.$$

We define the following sets of positive integers by using $\{m_{i,j}\}$ as the bases. For each $i \in \{1, ..., n\}$, let $0 \leq \alpha_i < 1$ and $k_i \in \mathbf{N}$, then we put

$$\begin{aligned}[M_i]_{k_i}^{\alpha_i} := \{m \in \mathbf{N} : m = p_{i,k_i} m_{i,k_i} + p_{i,k_i-1} m_{i,k_i-1} + \cdots + p_{i,u_i} m_{i,u_i}, \\ k_i \geq u_i \geq 1 : \frac{k_i - u_i}{k_i} = \alpha_i, \quad p_{i,j} \in \mathbf{N}_0, j = u_i, u_i+1, ..., k_i : \\ p_{i,k_i}, p_{i,u_i} \geq 1, \quad p_{i,j} < \frac{m_{i,j+1}}{m_{i,j}}, \quad j = u_i, u_i+1, ..., k_i\}.\end{aligned}$$

Furthermore, we define

$$\begin{aligned}[M_i]_{k_i}^{(d)} := \{m \in \mathbf{N} : m = p_{i,k_i} m_{i,k_i} + p_{i,k_i-1} m_{i,k_i-1} + \cdots + p_{i,d} m_{i,d}, \\ p_{i,j} \in \mathbf{N}_0, j = d, d+1, ..., k_i : \\ 1 \leq p_{i,k_i} < \frac{m_{i,k_i+1}}{m_{i,k_i}}, \quad 0 \leq p_{i,j} < \frac{m_{i,j+1}}{m_{i,j}}, \quad j = d, d+1, ..., k_i - 1\}\end{aligned}$$

and define

$$[M_i]^{(d)} := \bigcup_{k_i=d}^{\infty} [M_i]_{k_i}^{(d)}$$

for $d = 0, 1, 2, \ldots,\ i = 1, \ldots, n$. Since $m_{i,0} = l_{i,0} = 1$, we note that

$$\mathbf{N} = [M_i]^{(0)}.$$

Furthermore, since we have

$$m_{i,1} \geq 2 \text{ if } 0 < \tau_i < \frac{1}{2}$$

and we have

$$m_{i,1} = 1,\ m_{i,2} \geq 2 \text{ if } \frac{1}{2} < \tau_i < 1,$$

we consider the intersection of the sets $\bigcap_{i=1}^{n} [M_i]^{(d_i)}$ as follows:

(2.3) $$d_i = 1 \text{ if } 0 < \tau_i < \frac{1}{2} \text{ and } d_i = 2 \text{ if } \frac{1}{2} < \tau_i < 1.$$

For each positive integer m we can consider the unique expression;

$$m = p_{i,k_i} m_{i,k_i} + p_{i,k_i-1} m_{i,k_i-1} + \cdots + p_{i,u_i} m_{i,u_i}$$

by introducing the lexicographical order as follows.

Hereafter we use the simplified notations $p_k = p_{i,k_i}$, $m_k = m_{i,k_i}$ in the case not confused. Assume that some number m has two expressions such that

$$\begin{aligned} m &= p_{k_1} m_{k_1} + p_{k_1-1} m_{k_1-1} + \cdots + p_{u_1} m_{u_1} := [m1] \\ &= p_{k_2} m_{k_2} + p_{k_2-1} m_{k_2-1} + \cdots + p_{u_2} m_{u_2} := [m2]. \end{aligned}$$

Define $[m1] \leq [m2]$ if $k_1 < k_2$, or otherwise if $k_1 = k_2$ and $p_{k_1} < p_{k_2}$, or otherwise if $k_1 = k_2$ and

$$p_{k_1} = p_{k_2},\ p_{k_1-1} = p_{k_2-1}, \cdots,\ p_{k_1-j+1} = p_{k_2-j+1},\ p_{k_1-j} < p_{k_2-j}$$

for some $j \in \mathbf{N}$. Then we can take the largest expression for this order.

For example, note that $p_j \leq [m_{j+1}/m_j] = a_{j+1}$ and let

$$m = p_k m_k + a_k m_{k-1} + p_{k-2} m_{k-2} + \cdots + p_u m_u,\ p_k < a_{k+1},\ p_{k-2} \geq 1,$$

then we choose the expression

$$m = (p_k + 1) m_k + (p_{k-2} - 1) m_{k-2} + \cdots + p_u m_u.$$

For our purpose we should choose a suitable subsequence in $\bigcap_{i=1}^{n} [M_i]^{(d_i)}$ by the following construction method.

(T) For a positive integer m:

$$m = p_{i,k_i} m_{i,k_i} + \cdots + p_{i,u_i+1} m_{i,u_i+1} + p_{i,u_i} m_{i,u_i},$$

define $\zeta_i : \mathbf{N} \to \mathbf{N}$ by

$$\zeta_i(m) = u_i.$$

Define a sequence of positive integers $T_j \in \bigcap_{i=1}^{n}[M_i]^{(d_i)}$ as follows. Let

$$T_1 = \min\{m : m \in \bigcap_{i=1}^{n}[M_i]^{(d_i)}\}$$

and

$$T_2 = \min\{m \in \bigcap_{i=1}^{n}[M_i]^{(d_i)} : \min_i \zeta_i(m) > \min_i \zeta_i(T_1)\}.$$

Iteratively, let

$$T_{j+1} = \min\{m \in \bigcap_{i=1}^{n}[M_i]^{(d_i)} : \min_i \zeta_i(m) > \min_i \zeta_i(T_j)\}.$$

$[T_j]$ denotes the sequence $\{T_j\}$ in $\bigcap_{i=1}^{n}[M_i]^{(d_i)}$, which is constructed by the method **(T)** and then we call $[T_j]$ the sequence of extended common multiples (abr. ECM).

Let $\zeta_i(T_j) = u_{i,j}$, then we note that the sequence $\{\min_i u_{i,j}\}$ is strictly increasing and also, for each $T_j \in \bigcap_{i=1}^{n}[M_i]^{(d_i)}$, there exist sequences of parameters $\{\alpha_i^{(j)}\}, \{k_i^{(j)}\}, i = 1, ..., n$:

$$T_j \in [M_1]_{k_1^{(j)}}^{\alpha_1^{(j)}} \cap [M_2]_{k_2^{(j)}}^{\alpha_2^{(j)}} \cap \cdots \cap [M_n]_{k_n^{(j)}}^{\alpha_n^{(j)}}, \quad j = 1, 2, ... \tag{2.4}$$

3. ECM and Diophantine conditions

In this section, introducing the Diophantine conditions, which are given by parametrizing the famous Diophantine conditions in KAM theorem and considering a condition for the ECM sequence, we show some relations between the the Diophantine condition and the ECM condition.

Let $\{\tau_1, ..., \tau_n\} : 0 < \tau_1, ..., \tau_n < 1$, be rationally independent irrational numbers and let $[T_j] \subset \bigcap_{i=1}^{n}[M_i]^{(d_i)}$ be the ECM sequence constructed by **(T)** where $d_i \in \{1, 2\}$ with (2.3). In view of (2.4), we put

$$\delta_0 := \liminf_j \max_i \alpha_i^{(j)}.$$

Then we say that the n-tuples of irrationals $\{\tau_1, ..., \tau_n\}$ satisfies δ_0-(ECM) condition or we call it a δ_0-(ECM) class if $0 \le \delta_0 < 1$.

Usual definitions of the Diophanitine condition in KAM theorem are given as follows.

There exist constants $\gamma, d : \gamma > 0,\ d > n$, which satisfy

$$|(\tau_1 m_1 + \cdots + \tau_n m_n) - l| \ge \frac{\gamma}{|m|^d}$$

for every integers $m = (m_1, ..., m_n) \in \mathbf{Z}^n$, $l \in \mathbf{Z}$ where $|\cdot|$ denotes a usual Euclidean norm.

Here we say that $\{\tau_1, ..., \tau_n\}$ satisfies d_0-(D) condition or we call it a d_0-(D) class if there exists a constant $d_0 : d_0 \geq n$, such that, for each $d > d_0$, there exists $\gamma_d > 0$, which satisfies

$$(3.1) \qquad |(\tau_1 m_1 + \cdots + \tau_n m_n) - l| \geq \frac{\gamma_d}{|m|^d}$$

for every integers $m = (m_1, ..., m_n) \in \mathbf{Z}^n$, $l \in \mathbf{Z}$
and furthermore, for each $d : 0 < d < d_0$ and each $\gamma > 0$, there exist integers $m_\gamma = (m_{1,\gamma}, ..., m_{n,\gamma}) \in \mathbf{Z}^n$ and $l_\gamma \in \mathbf{Z}$, which satisfy

$$(3.2) \qquad |(\tau_1 m_{1,\gamma} + \cdots + \tau_n m_{n,\gamma}) - l_\gamma| < \frac{\gamma}{|m_\gamma|^d}.$$

By (3.2) the constant d_0 specifies the infimum value of d, which satisfies (3.1).

For the Liouville class numbers, we call $\{\tau_1, .., \tau_n\}$ a ∞-(D) class if, for every $d_0 > 0$, there exists $d : d > d_0$ such that for each $\gamma > 0$, there exist integers $m_\gamma = (m_{1,\gamma}, ..., m_{n,\gamma})$, l_γ, which satisfy

$$|(\tau_1 m_{1,\gamma} + \cdots + \tau_n m_{n,\gamma}) - l_\gamma| < \frac{\gamma}{|m_\gamma|^d}.$$

Theorem 3.1. *Let $\{\tau_1, ..., \tau_n\}$ be (KL) class irrational numbers, which satisfy (2.1). Then, for constants $d_0, \delta_0 : d_0 \geq n$, $0 \leq \delta_0 < 1$, if $\{\tau_1, ..., \tau_2\}$ satisfies d_0-(D) condition, then it is a δ_0-(ECM) class for some constant δ_0, which satisfies*

$$(3.3) \qquad \delta_0 \leq 1 - \frac{d_0}{n + (n-1)d_0} \cdot \frac{\log E_1}{\log E_2}$$

and on the contrary, if $\{\tau_1, ..., \tau_n\}$ satisfies δ_0-(ECM) condition, then it is a d_0-(D) class for some constant $d_0 : n \leq d_0 \leq \infty$, which satisfies

$$(3.4) \qquad d_0 \geq n - 1 + \frac{n(1-\delta_0)\log E_1}{\log E_2}.$$

Remark 3.2. It follows from Theorem 3.1 that, if $\{\tau_1, ..., \tau_n\}$ is d_0-(D) class, then it is δ_0-(ECM) class for

$$1 - \frac{d_0 - (n-1)}{n} \cdot \frac{\log E_2}{\log E_1} \leq \delta_0 \leq 1 - \frac{d_0}{n + (n-1)d_0} \cdot \frac{\log E_1}{\log E_2}$$

Since $E_1 \simeq E_2$, we obtain the relation

$$1 - \frac{d_0 - (n-1)}{n} \leq \delta_0 \leq 1 - \frac{d_0}{n + (n-1)d_0}.$$

It follows that the Liouville type condition ($d_0 \sim \infty$), which is of null measure in the Lesbeague sense, yields the δ_0-(ECM) condition: $\delta_0 \leq (n-2)/(n-1)$. That is, if $\delta_0 > (n-2)/(n-1)$, the set of irratrional numbers satisfies the Diophantine condition: $d_0 < \infty$. If $n = 2$ and $d_0 \sim \infty$, then we have $\delta_0 = 0$. The typical example of 0-(ECM) is the class of irrational pairs, which admit a common subsequence in the denominators of the Diophantine approximations such that

$$\{m_{1,k_j}\} \subset \{m_{1,k}\},\quad \{m_{2,s_j}\} \subset \{m_{2,s}\} : \{m_{1,k_j}\} = \{m_{2,s_j}\}.$$

On the other hand, since the class of n irrational numbers, which satisfy d_0-(D): $d_0 = n$, is of full measure, it follows that almost all irrational class $\{\tau_1, ..., \tau_n\}$ is δ_0-(ECM) class for $\delta_0 = 1 - \frac{1}{n}$.

The proof is given by the transference theorem (cf. [1]) for the case where $s = 1, m = n$ and $s = n, m = 1$.

Theorem 3.3 (Transference Theorem). *Define the linear forms $L_j, j = 1, ..., s$, $M_i, i = 1, ..., m$ by*

$$L_j(x) = \sum_{i=1}^{m} \vartheta_{ij} x_i, \quad M_i(u) = \sum_{j=1}^{s} \vartheta_{ji} u_j$$

where we consider the case $x_i, u_j \in \mathbf{Z}$, $\vartheta_{ij} \in \mathbf{R}$. Suppose that there are integers $x \neq 0$:

$$\|L_j(x)\| \leq C, \quad |x_i| \leq X,$$

for some constant C and X: $0 < C < 1 \leq X$. Then there are integers $u \neq 0$:

$$\|M_i(u)\| \leq D, \quad |u_j| \leq U,$$

where

$$\begin{aligned} D &= (l-1)X^{(1-s)/(l-1)} C^{s/(l-1)}, \\ U &= (l-1)X^{m/(l-1)} C^{(1-m)/(l-1)}, \\ l &= m + s, \end{aligned}$$

and $\|a\| = \min\{|a - z| : z \in \mathbf{Z}\}$ for $a \in \mathbf{R}$.

Proof of Theorem 3.1. Let $\{\tau_1, ..., \tau_n\}$ be d_0-(D) class, then for every $d : 0 < d < d_0$ and every $\gamma > 0$, there exist integers $m_\gamma = (m_{1,\gamma}, ..., m_{n,\gamma})$, l_γ, which satisfy

$$|(\tau_1 m_{1,\gamma} + \cdots + \tau_n m_{n,\gamma}) - l_\gamma| \leq \frac{\gamma}{|m_\gamma|^d}.$$

We put

$$X := \max\{|m_{1,\gamma}|, ..., |m_{n,\gamma}|\} \leq |m_\gamma|, \quad C := \gamma X^{-d}.$$

By applying Transference Theorem with $s = 1, m = n$ we can show that there exist postive integers M_γ, $l_{1,\gamma}, ..., l_{n,\gamma}$, which satisfy

(3.5) $\quad |\tau_1 M_\gamma - l_{1,\gamma}| \leq D,\ |\tau_2 M_\gamma - l_{2,\gamma}| \leq D, ..., |\tau_n M_\gamma - l_{n,\gamma}| \leq D,$

(3.6) $\quad M_\gamma \leq U$

where

$$D = nC^{\frac{1}{n}}, \quad U = nXC^{\frac{1}{n}-1}.$$

It follows that

$$
\begin{aligned}
(3.7)\qquad & |\tau_1 M_\gamma - l_{1,\gamma}| \le n\gamma^{\frac{1}{n}} X^{-\frac{d}{n}},\\
& \vdots\\
& |\tau_n M_\gamma - l_{n,\gamma}| \le n\gamma^{\frac{1}{n}} X^{-\frac{d}{n}},\\
& M_\gamma \le nX(\gamma X^{-d})^{\frac{1}{n}-1} = n\gamma^{\frac{1}{n}-1} X^{1-d(\frac{1}{n}-1)}.
\end{aligned}
$$

Since it follows that

$$X \ge (n^{-1}\gamma^{1-\frac{1}{n}} M_\gamma)^{\frac{n}{n+(n-1)d}},$$

we have

$$
\begin{aligned}
& |\tau_1 M_\gamma - l_{1,\gamma}|,\ \dots,\ |\tau_n M_\gamma - l_{n,\gamma}|\\
& \quad \le n\gamma^{\frac{1}{n}} (n^{-1}\gamma^{1-\frac{1}{n}} M_\gamma)^{-\frac{d}{n+(n-1)d}}\\
& \quad = n^{\frac{n(d+1)}{n+(n-1)d}} \gamma^{\frac{1}{n+(n-1)d}} M_\gamma^{-\frac{d}{n+(n-1)d}}.
\end{aligned}
$$

We note that as $\gamma \to 0$, then $M_\gamma \to \infty$. In fact, if M_γ is bounded, then we can take a convergent subsequence. Then, using (3.5) with $D \to 0$ as $\gamma \to 0$, we obtain a contradiction that τ_i is a rational number.

Let $\gamma = \gamma_j : \gamma_j \to 0$ as $j \to \infty$. Consider the expressions of M_{γ_j} by $\{m_{i,k_i^{(j)}}\}$

$$
\begin{aligned}
M_{\gamma_j} &\in \bigcap_{i=1}^{n} [M]_{k_i^{(j)}}^{\alpha_i^{(j)}},\\
M_{\gamma_j} &= p_{1,k_1^{(j)}} m_{1,k_1^{(j)}} + p_{1,k_1^{(j)}-1} m_{1,k_1^{(j)}-1} + \cdots + p_{1,u_1^{(j)}} m_{1,u_1^{(j)}} \in [M_1]_{k_1^{(j)}}^{\alpha_1^{(j)}}\\
&= p_{2,k_2^{(j)}} m_{2,k_2^{(j)}} + p_{2,k_2^{(j)}-1} m_{2,k_2^{(j)}-1} + \cdots + p_{2,u_1^{(j)}} m_{2,u_2^{(j)}} \in [M_2]_{k_2^{(j)}}^{\alpha_2^{(j)}}\\
&\ \ \vdots\\
&= p_{n,k_n^{(j)}} m_{n,k_n^{(j)}} + p_{n,k_n^{(j)}-1} m_{n,k_n^{(j)}-1} + \cdots + p_{n,u_n^{(j)}} m_{n,u_n^{(j)}} \in [M_n]_{k_n^{(j)}}^{\alpha_n^{(j)}}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(3.8)\quad & |\tau_1(p_{1,k_1^{(j)}} m_{1,k_1^{(j)}} + \cdots + p_{1,u_1^{(j)}} m_{1,u_1^{(j)}}) - (p_{1,k_1^{(j)}} n_{1,k_1^{(j)}} + \cdots + p_{1,u_1^{(j)}} n_{1,u_1^{(j)}})|\\
\le\ & \frac{m_{1,k_1^{(j)}+1}}{m_{1,k_1^{(j)}}} |\tau_1 m_{1,k_1^{(j)}} - n_{1,k_1^{(j)}}| + \cdots + \frac{m_{1,u_1^{(j)}+1}}{m_{1,u_1^{(j)}}} |\tau_1 m_{1,u_1^{(j)}} - n_{1,u_1^{(j)}}|\\
\le\ & \frac{1}{m_{1,k_1^{(j)}}} + \cdots + \frac{1}{m_{1,u_1^{(j)}}}\\
\le\ & \frac{1}{C_{1,1}^{k_1^{(j)}}} \cdot \frac{(C_{1,1}^{k_1^{(j)}-u_1^{(j)}+1} - 1)}{C_{1,1} - 1}\\
\le\ & \frac{E_1}{E_1 - 1} E_1^{-(1-\alpha_1^{(j)})k_1^{(j)}} << 1.
\end{aligned}
$$

It follows that

$$(3.9)\quad |\tau_1(p_{1,k_1^{(j)}}m_{1,k_1^{(j)}}+\cdots+p_{1,u_1^{(j)}}m_{1,u_1^{(j)}})-(p_{1,k_1^{(j)}}n_{1,k_1^{(j)}}+\cdots+p_{1,u_1^{(j)}}n_{1,u_1^{(j)}})|$$
$$=|\tau_1 M_{\gamma_j}-l_{1,\gamma_j}|$$
$$\leq c_{\gamma_j}M_{\gamma_j}^{-\frac{d}{n+(n-1)d}}\leq c_{\gamma_j}m_{k_1^{(j)}}^{-\frac{d}{n+(n-1)d}}\leq c_{\gamma_j}(\frac{1}{E_1})^{\frac{d}{n+(n-1)d}k_1^{(j)}},$$

$$(3.10)\quad c_{\gamma_j}=n^{\frac{n(d+1)}{n+(n-1)d}}\gamma_j^{\frac{1}{n+(n-1)d}}$$

where the first equality holds, since the first and the second terms are less than one.

On the other hand, applying the argument in the proof of Theorem 4.4 in [5], we have

$$(3.11)\, |\tau_1(p_{1,k_1^{(j)}}m_{1,k_1^{(j)}}+\cdots+p_{1,u_1^{(j)}}m_{1,u_1^{(j)}})-(p_{1,k_1^{(j)}}n_{1,k_1^{(j)}}+\cdots+p_{1,u_1^{(j)}}n_{1,u_1^{(j)}})|$$
$$\geq c(\frac{1}{m_{1,u_1^{(j)}}})\geq c(\frac{1}{E_2})^{(1-\alpha_1^{(j)})k_1^{(j)}}.$$

It follows from (3.9) and (3.11) that we have

$$c(\frac{1}{E_2})^{(1-\alpha_1^{(j)})k_1^{(j)}}\leq c_{\gamma_j}(\frac{1}{E_1})^{(\frac{d}{n+(n-1)d})k_1^{(j)}}.$$

Thus we have

$$(1-\alpha_1^{(j)})\log E_2\geq\frac{d}{n+(n-1)d}\log E_1+\frac{\log c-\log c_{\gamma_j}}{k_1^{(j)}}.$$

Since $c>c_{\gamma_j}$ for small γ_j, we obtain

$$(3.12)\qquad \alpha_1^{(j)}\leq 1-\frac{d}{n+(n-1)d}\cdot\frac{\log E_1}{\log E_2}$$

for every $d<d_0$. Similary, we have

$$(3.13)\qquad \alpha_i^{(j)}\leq 1-\frac{d}{n+(n-1)d}\cdot\frac{\log E_1}{\log E_2},\quad i=2,...,n$$

for every $d<d_0$. Thus we can obtain the first estimate

$$(3.14)\qquad \delta_0=\liminf_{j\to\infty}\max_i\alpha_i^{(j)}\leq 1-\frac{d_0}{n+(n-1)d_0}\cdot\frac{\log E_1}{\log E_2}<1.$$

Next, let $\{\tau_1,...,\tau_n\}$ be δ_0-(ECM) class: $0\leq\delta_0<1$. That is, there exists a sequence $[T_j]$ of ECM, constructed by **(T)**, which satisfies

$$[T_j]\subset\bigcap_{i=1}^{n}[M_i]^{(d_i)},$$
$$T_j\in[M_1]_{k_1^{(j)}}^{\alpha_1^{(j)}}\cap[M_2]_{k_2^{(j)}}^{\alpha_2^{(j)}}\cap\cdots\cap[M_n]_{k_n^{(j)}}^{\alpha_n^{(j)}},\quad j=1,2,...$$

for the sequences of real numbers $\alpha_i^{(j)}:0\leq\alpha_i^{(j)}<1,\ \ j=1,2,...$ such that

$$\delta_0=\liminf_j\max_i\alpha_i^{(j)}<1.$$

Let

$$T_j = p_{1,k_1^{(j)}} m_{1,k_1^{(j)}} + p_{1,k_1^{(j)}-1} m_{1,k_1^{(j)}-1} + \cdots + p_{1,u_1^{(j)}} m_{1,u_1^{(j)}}$$

and

$$N_{1,j} = p_{1,k_1^{(j)}} n_{1,k_1^{(j)}} + p_{1,k_1^{(j)}-1} n_{1,k_1^{(j)}-1} + \cdots + p_{1,u_1^{(j)}} n_{1,u_1^{(j)}}.$$

It follows from (3.8) that we can estimate

$$|\tau_1 T_j - N_{1,j}| \le \frac{E_1}{E_1 - 1} E_1^{-(1-\alpha_1^{(j)})k_1^{(j)}}$$

and, similarly

$$\begin{aligned} |\tau_i T_j - N_{i,j}| &\le \frac{1}{m_{i,k_1^{(j)}}} + \cdots + \frac{1}{m_{i,u_i^{(j)}}} \\ &\le \frac{E_1}{E_1 - 1} E_1^{-(1-\alpha_i^{(j)})k_i^{(j)}}, \quad i = 2, ..., n. \end{aligned}$$

Thus we have

$$|\tau_i T_j - N_{i,j}| \le C, \quad i = 1, ..., n$$

where we can put

$$C := \frac{E_1}{E_1 - 1} E_1^{-(1-\max_i \alpha_i^{(j)}) \min_i k_i^{(j)}}$$

and also, since we have

$$T_j \le m_{i,k_i^{(j)}+1},$$

we can put

$$X := E_2 E_2^{\min_i k_i^{(j)}} \ge T_j.$$

By applying Transference Theorem for $s = n, m = 1$, we can show that there exists positive integers $\mu_j = (\mu_{1,j}, ..., \mu_{n,j})$, l_j, which satisfy

$$|(\tau_1 \mu_{1,j} + \cdots + \tau_n \mu_{n,j}) - l_j| \le D, \quad \max_i \mu_{i,j} \le U \tag{3.15}$$

where we have

$$D = nCX^{\frac{1}{n}-1}, \quad U = nX^{\frac{1}{n}}.$$

Since

$$\frac{1}{\sqrt{n}} |\mu_j| \le \max_i \mu_{i,j} \le U,$$

we have

$$X \ge (\frac{1}{n\sqrt{n}})^n |\mu_j|^n. \tag{3.16}$$

And also, we have

$$
\begin{aligned}
C &= cE_2^{-\frac{\log E_1}{\log E_2}\cdot(1-\max_i \alpha_i^{(j)})\min_i k_i^{(j)}} \\
&= cE_2^{\frac{(1-\max_i \alpha_i^{(j)})\log E_1}{\log E_2}} X^{-\frac{(1-\max_i \alpha_i^{(j)})\log E_1}{\log E_2}}.
\end{aligned}
$$

Thus we have

$$D = nX^{\frac{1}{n}-1} cE_2^{\frac{(1-\max_i \alpha_i^{(j)})\log E_1}{\log E_2}} X^{-\frac{(1-\max_i \alpha_i^{(j)})\log E_1 - 1}{\log E_2}}.$$

For a small $\varepsilon_1 > 0$ it follow from (3.15) and (3.16) that we have

$$
\begin{aligned}
&|(\tau_1\mu_{1,j} + \cdots + \tau_n\mu_{n,j}) - l_j| \\
&\le cE_2^{\frac{\log E_1(1-\max_i \alpha_i^{(j)})}{\log E_2}} \cdot \frac{1}{|\mu_j|^{\varepsilon_1}} \cdot \frac{1}{|\mu_j|^{n-1+\frac{n\log E_1(1-\max_i \alpha_i^{(j)})}{\log E_2}-\varepsilon_1}}.
\end{aligned}
$$

Note that for each small $\varepsilon_2 > 0$ we can admit a large number j, that is, T_j:

$$\max_i \alpha_i^{(j)} \le \delta_0 + \varepsilon_2 < 1.$$

Since for every small $\gamma > 0$, there exists a large number j_1:

$$\gamma > cE_2^{\frac{\log E_1}{\log E_2}} \cdot \frac{1}{|\mu_j|^{\varepsilon_1}}, \quad \forall j \ge j_1,$$

we can show that there exist integers $\mu_j = (\mu_{1,j}, \dots, \mu_{n,j})$, l_j, which satisfy

$$|(\tau_1\mu_{1,j} + \cdots + \tau_n\mu_{n,j}) - l_j| \le \frac{\gamma}{|\mu_j|^{n-1+\frac{n(1-\delta_0-\varepsilon_2)\log E_1}{\log E_2}-\varepsilon_1}}.$$

Since all class of irrational numbers satisfy d_0-(D) condition for some $d_0 : d_0 \ge n$, or $d_0 = \infty$ and $\varepsilon_1, \varepsilon_2$ can be given arbitrarily small, we can conclude that the δ_0-(ECM) class of irrational numbers satisfies d_0-(D) condition for some d_0:

$$d_0 \ge n - 1 + \frac{n(1-\delta_0)\log E_1}{\log E_2}.$$

□

4. Recurrent dimensions of quasi-periodic orbits

In this section, considering a quasi-periodic orbit in a Banach space X with n-irrational frequencies:

$$\Sigma = \{\varphi(l) \in X : \varphi(l) = f(\tau_1 l, \tau_2 l, \dots, \tau_n l),\ l \in \mathbf{N}_0\},$$

we estimate the recurrent dimensions of Σ (see [4] or [5] for the definitions). Here, let $f : \mathbf{R}^n \to X$ be a nonlinear function, which satisfies the following Hölder conditions:

(H1) There exist constants $K_1 > 0$ and $\vartheta_1 :\ 0 < \vartheta_1 \le 1$, which satisfy

$$\|f(t_1, \dots, t_n) - f(s_1, \dots, s_n)\| \le K_1 \sum_{i=1}^{n} |t_i - s_i|^{\vartheta_1}, \quad t_i, s_i \in \mathbf{R} : \sum_i |t_i - s_i| \le \varepsilon_0$$

for a small constant $\varepsilon_0 > 0$.

(H2) There exist constants $K_2 > 0$ and $\vartheta_2 :\ 0 < \vartheta_2 \leq 1$, which satisfy

$$\|f(t_1, ..., t_n) - f(s_1, ..., s_n)\| \geq K_2 \sum_{i=1}^{n} |t_i - s_i|^{\vartheta_2}, \quad t_i, s_i \in \mathbf{R} :\ |t_i - s_i| \leq \frac{1}{2}.$$

Then, by applying the arguments in the proof of Theorem 3.3 and Theorem 4.4 in [5] to the n frequencies cases, we can estimate the upper and lower recurrent dimensions as follows.

Theorem 4.1. *Under Hypothesis* **(H1)**, *let* $\{\tau_1, ..., \tau_n\}$ *be (KL) class numbers and for the sequence* $[T_j]$ *of ECM, constructed by the method* **(T)**, *such that*

$$T_j \in [M_1]^{\alpha_1^{(j)}}_{k_1^{(j)}} \cap [M_2]^{\alpha_2^{(j)}}_{k_2^{(j)}} \cap \cdots \cap [M_n]^{\alpha_n^{(j)}}_{k_n^{(j)}}, \quad j = 1, 2, ...,$$

assume that the sequences of real numbers $\alpha_i^{(j)} :\ 0 \leq \alpha_i^{(j)} < 1, \quad j = 1, 2, ...,$ *satisfies*

$$\delta_0 := \liminf_j \max_i \alpha_i^{(j)} < 1.$$

Then we have

$$\underline{d}_r(\Sigma) \leq \frac{\log E_2}{(1 - \delta_0)\vartheta_1 \log E_1}. \tag{4.1}$$

For the lower estimate we need the following Hypotheses on the partial quotients of continued fraction expansions of τ_i: $\tau_i = [a_{i,1}, a_{i,2}, ..., a_{i,j}, ...]$ where we consider the case $0 < \tau_i < 1/2$, $i = 1, ..., 2$ for simplicity.

(A) $\quad a_{i,2} \geq 2 \quad$ or $\quad a_{i,2} = 1 \quad$ and $\quad a_{i,3} = 1, \quad i = 1, ..., n.$

Theorem 4.2. *Under Hypotheses* **(H2)** *and* **(A)**, *let the irrational frequencies* $\tau_i : 0 < \tau_i < 1/2,\ i = 1, ..., n$ *be (KL) class numbers. We assume that the infinite sequence of ECM* $[T_j]$, *constructed by the method* **(T)**, *satisfies*

$$\delta_1 := \limsup_j \max_i \alpha_i^{(j)} < 1.$$

Then we have

$$\overline{d}_r(\Sigma) \geq \frac{\log E_1}{(1 - \delta_1)\vartheta_2 \log E_2}. \tag{4.2}$$

Using Theorem 4.1 and Theorem 4.2, we can also estimate the gaps of the recurrent dimensions, defined by

$$g_r(\Sigma) := \overline{d}_r(\Sigma) - \underline{d}_r(\Sigma).$$

Corollary 4.3. *Under Hypotheses* **(H1)**, **(H2)** *and* **(A)**, *let the irrational frequencies* $\tau_i : 0 < \tau_i < 1/2,\ i = 1, ..., n$ *be (KL) class numbers. Assume the same Hypotheses as those of Theorem 4.1 and Theorem 4.2 for ECM* $[T_j] \subset \bigcap_i [M_i]^{(d_i)}$, *given by* **(T)**, *with the parameters* δ_0, δ_1. *Then we have*

$$g_r(\Sigma) \geq \frac{\log E_1}{(1 - \delta_1)\vartheta_2 \log E_2} - \frac{\log E_2}{(1 - \delta_0)\vartheta_1 \log E_1}. \tag{4.3}$$

Remark 4.4. Since we can take the limit supremum in (3.14), we have

$$\delta_1 = \limsup_{j \to \infty} \max_i \alpha_i^{(j)} \leq 1 - \frac{d_0}{n + (n-1)d_0} \cdot \frac{\log E_1}{\log E_2}.$$

Thus, considering $E_1 \simeq E_2$, the gaps of the recurrent dimensions become positive if the difference between δ_1 and δ_0 is positive and $\vartheta_1 \simeq \vartheta_2$. However, for the case where the Diophantine condition is satisfied and $d_0 = n$, we have

$$\delta_0 = \delta_1 = 1 - \frac{1}{n}.$$

The gaps between δ_1 and δ_0 can be positive in the null measure case where $d_0 > n$.

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