数理解析研究所講究録 1365 巻 2004 年 166-175

ON SOME CONJECTURES AND PROBLEMS IN ANALYTICAL FIXED POINT THEORY, REVISITED

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ABSTRACT. We discuss the current stare of research related to the Schauder conjecture and other problems in analytical fixed point theory. We revise and update the contents of the previous version [P1].

In our previous work [P1], we discussed the state of research related to the Schauder conjecture and other problems in analytical fixed point theory. Since Cauty [C] obtained the affirmative answer to the conjecture, we have to revise and update the contents of [P1]. In the present paper, we discuss the current status of conjectures and problems in [P1]. (The arabic numbers attached to them and to theorems are same to those in [P1].)

The following is the well-known Schauder conjecture raised in 1935; see *The Scottish Book* [15], Problem 54.

Conjecture 1. (Schauder) *Every nonempty compact convex subset X of a (metrizable) t.v.s. E has the fixed point property; that is, every continuous function $f : X \to X$ has a fixed point $x_0 \in X$ such that $x_0 = f(x_0)$.*

The following famous long-standing conjecture is known to be the compact *AR* problem:

Conjecture 2. *Every compact convex subset X of a metrizable t.v.s. is an AR.*

2000 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. The Schauder conjecture, convexly totally bounded, Kakutani map, (weakly) admissible set.

This is also resolved affirmatively [P2]: Recently, such X is known to be admissible (in the sense of Klee) and, in 1960, Klee [13] showed that any admissible compact convex subset of a metrizable t.v.s. is an AR.

A multimap (or a map) is said to be *compact* if its range is contained in a compact subset of its codomain.

Now Conjecture 1 and the following hold by the recent work of Cauty [C] when E is Hausdorff.

Conjecture 3. *For every nonempty convex subset X of a t.v.s. E, a compact continuous function $f : X \to X$ has a fixed point.*

Note that Schauder showed in 1930 that Conjecture 3 holds for a normed vector space E and Hukuhara in 1950 for a locally convex Hausdorff t.v.s.

We give some known results related to partial solutions of Conjecture 3.

A subset B of a t.v.s. E is said to be *convexly totally bounded* (c.t.b. for short), by Idzik [9], if for every neighborhood V of the origin 0 of E there exist a finite subset $\{x_i : i \in I\} \subset B$ and a finite family of convex sets $\{C_i : i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i : i \in I\}$.

The following is well-known:

Theorem 1. (Idzik [9]) *Let X be a convex subset of a Hausdorff t.v.s. E and $\Phi : X \multimap X$ a Kakutani map (that is, an upper semicontinuous multimap with nonempty compact convex values). If $\overline{\Phi(X)}$ is a compact c.t.b. subset of X, then there exists an $x \in X$ such that $x \in \Phi(x)$.*

In view of Theorem 1, Idzik [9] raised the problem: Is every compact convex subset of a t.v.s. convexly totally bounded? A positive answer to this problem would resolve the Schauder conjecture. However, Idzik's problem was resolved negatively by De Pascale, Trombetta, and Weber [6]. They showed that, for $0 \leq p < 1$, the space L^p contains compact convex subsets which are not c.t.b.

In a recent work, Dobrowolski [D2] showed that, for a metrizable t.v.s. E, Theorem 1 holds without assuming the c.t.b. of X.

A *polytope* P in a t.v.s. E is a nonempty compact convex subset of E contained in a finite dimensional subspace of E. Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [13]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Note that every nonempty convex subset of a locally convex t.v.s., l^p, L^p, the Hardy space H^p for $0 < p < 1$, and many other t.v.s. are admissible; see [1, 8, 22, 23] and references therein.

Let X be a nonempty subset of a t.v.s. E and Y a topological space. The "better" admissible class $\mathfrak{B}$ of maps is defined as follows [22, 23]:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that for any polytope P in X and any continuous function $f : F(P) \to P$, $f \circ (F|_P) : P \multimap P$ has a fixed point.

The following is due to the author [22, 23]:

Theorem 2. [22] *Let X be an admissible convex subset of a Hausdorff t.v.s. E and $\Phi \in \mathfrak{B}(X,X)$. If Φ is closed and compact, then Φ has a fixed point.*

In view of the single-valued case of Theorem 2, the following was raised:

Problem 1. (Klee [13]) *Is any convex subset of a Hausdorff t.v.s. admissible?*

A positive answer to this would resolve the Schauder Conjecture 1 and Conjecture 3. An example of a nonconvex, compact, and nonadmissible subset of the Hilbert space l^2 is known; see Hadžić [8].

Problem 2. (Idzik) *Is a c.t.b. compact convex subset admissible?*

In [6], examples of admissible sets which are not c.t.b. were given. Recently, we showed that any convex and c.t.b. (where each C_i is open) subset in a Hausdorff t.v.s. is admissible; see Park [P2].

We have another result related to admissible sets:

Theorem 3. [20] *Let E and F be Hausdorff t.v.s. and X a subset of E which is homeomorphic to an admissible convex subset of F. Then any compact map $\Phi \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

Here, $\mathfrak{A}_c^\kappa$ is a subclass of $\mathfrak{B}$ due to the author.

Recently, Nguyen To Nhu [16] defined the notion of *weakly admissible* compact convex subsets of a metrizable t.v.s. and showed that such subsets have the fixed point property. He raised several problems related to Problem 1.

Arandelović [1] introduced the notion of weak admissibility on arbitrary Hausdorff t.v.s. and gave a non-metrizable extension of Nhu's result as follows:

Let E be a Hausdorff t.v.s., $\mathcal{V}$ a fundamental system of open neighborhoods of 0 in E and $X \subset E$ a nonempty closed convex subset of E. We say that X is *weakly admissible* if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \cdots, X_n$ of X with $X = \mathrm{co}(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L$, $i = 1, 2, \cdots, n$, where L is a finite dimensional subspace of E, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \cdots, n$.

Theorem 4. (Nhu [16], Arandelović [1]) *Let X be a weakly admissible, compact convex subset of a Hausdorff t.v.s. Then X has the fixed point property.*

Because of the Cauty theorem, the weak admissibility of X in Theorem 4 can now be eliminated.

In view of Theorem 4, the following were raised:

Problem 3. (Nhu [16]) *Is every compact convex subset of a Hausdorff t.v.s. weakly admissible?*

Problem 4. (Nhu [16]) *Is every weakly admissible compact convex subset of a Hausdorff t.v.s. admissible?*

Problems 3 and 4 were originally raised for a metrizable t.v.s. A partial solution is that every compact convex subset of a separable metric t.v.s. E is admissible. This is shown in [P2] by using results of Kalton, Peck, and Roberts [KPR] and of Dobrowolski [D1]. However, for not-metrizable t.v.s., Problems 3 and 4 are still open.

Problem 5. (Arandelović [1]) *Is a c.t.b. compact convex set weakly admissible?*

As we noted for Problem 2, we showed that any c.t.b. (by open sets) convex subset in a Hausdorff t.v.s. is admissible; see Park [P2].

Theorem 4 is generalized as follows:

Theorem 5. (Okon [19]) *Let X be a weakly admissible compact convex subset of a Hausdorff t.v.s. Then every Kakutani map $F : X \multimap X$ has a fixed point.*

Recently, Dobrowolski [D2] showed that Theorem 5 holds without assuming the weak admissibility of the domain X.

Here, we raise the following:

Problem 6. *Does Theorem 2 hold for a weakly admissible convex subset X? More generally, can we eliminate the admissibility of X in Theorem 2?*

The following is the Browder fixed point theorem in 1968:

Theorem 6. (Browder [4]) *Let K be a nonempty compact convex subset of a t.v.s. Let T be a map of K into 2^K, where for each $x \in K$, $T(x)$ is a nonempty convex [resp. open] subset of K. Suppose further that for each $y \in K$, $T^-(y) = \{x \in K : y \in T(x)\}$ is open [resp. nonempty convex] in K. Then T has a fixed point $x_0 \in K$, that is, $x_0 \in T(x_0)$.*

Later, this is known to be equivalent to the Brouwer fixed point theorem, the Sperner lemma, and the Knaster–Kuratowski–Mazurkiewicz (KKM) principle. Browder [4] applied his theorem to a systematic treatment of the interconnections between fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. For further developments on generalizations and applications of the Browder theorem, we refer to [21-26].

It is natural to ask if Theorem 6 holds when the compactness of the domain K of the multimap T in Theorem 6 is replaced by the compactness of T; that is, $T(K)$ is contained in a compact subset of K.

For the case when the fiber $T^-(y)$ is nonempty for each $y \in K$, if T is compact, then $K = T(K) \subset \overline{T(K)} \subset K$ and hence K is compact. Therefore, in this case, we do not have any problem.

For any subset X of a t.v.s., a map $T : X \multimap X$ is called a *Browder map* if it has nonempty convex values and open fibers.

In 1990, Ben-El-Mechaiekh raised the following:

Conjecture 4. (Ben-El-Mechaiekh [2,3]) *For a nonempty convex subset X of a t.v.s. E, a compact Browder map $T : X \multimap X$ has a fixed point.*

Of course, if X itself is compact, then Conjecture 4 reduces to the Browder Theorem 6. Hence, we assume that T is not surjective in Conjecture 4.

For a locally convex Hausdorff t.v.s. E, Ben-El-Mechaiekh [3] showed that Conjecture 4 holds. Moreover, he obtained the following:

Theorem 7. (Ben-El-Mechaiekh [2]) *Let X be a nonempty convex subset of a Hausdorff t.v.s. E, and $T : X \multimap X$ a Browder map. If T is compact, then T^n has a fixed point for $n \geq 2$.*

Some detailed discussions on partial solutions of Conjecture 4 were given in [21]. It is noted by Komiya [14] that any noncompact convex subset of a locally convex metrizable t.v.s. lacks the fixed point property for Browder maps.

We note that a multimap $T : X \multimap X$ satisfying the hypothesis of Conjecture 4 has the (convexly) almost fixed point property as follows:

Theorem 8. [25] *Let X be a nonempty convex subset of a t.v.s. E and $T : X \multimap X$ a compact Browder map. Then for any convex neighborhood V of the origin 0 of E, there exists a point $x_V \in X$ such that $T(x_V) \cap (x_V + V) \neq \emptyset$.*

It should be noted that the compactness of T might be replaced by the total boundedness of $T(X)$.

Note that, in a sense, Conjecture 3 implies Conjecture 4 as follows:

Theorem 9. [21] *Let E be a Hausdorff t.v.s. whose nonempty convex subsets have the fixed point property for compact continuous self-functions. Let X be a nonempty convex subset of E and $T : X \multimap X$ a Browder map. If T is compact, then T has a fixed point.*

Now, in virtue of the Cauty theorem, Theorem 9 becomes as follows:

Theorem 9′. *Let X be a convex subset of a Hausdorff t.v.s. and $T : X \multimap X$ a Browder map. If T is compact, then T has a fixed point.*

Note that this new result resolves not only Conjecture 4 affirmatively when E is Hausdorff, but also improves all of Theorems 7, 8, and the following known partial solutions of Conjecture 4.

Theorem 10. [21] *Let X be a nonempty convex subset of a Hausdorff t.v.s. E and $T : X \multimap X$ a Browder map. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point.*

Theorem 11. [21] *Let X be an admissible convex subset of a Hausdorff t.v.s. E and $T : X \multimap X$ a Browder map. If T is compact, then T has a fixed point.*

We give a more general form of Theorem 11 as follows:

Theorem 12. [20] *Let E and F be Hausdorff t.v.s. and X a subset of E which is homeomorphic to an admissible convex subset of F. If $T : X \multimap X$ is a compact Browder map, then T has a fixed point.*

In virtue of Theorem 9′, we can delete "c.t.b." from Theorem 10 and "admissible" from Theorem 11. However, it is not known whether the admissibility in Theorem 12 can be eliminated or not.

For a set D, let $\langle D \rangle$ denote the set of nonempty finite subsets of D.

Let X be a subset in a vector space and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any $N \in \langle D \rangle$ are contained in X and X has a topology that induces the Euclidean topology on such convex hulls; see Park [26]. If $X = D$ is convex, then $X := (X, X)$ becomes a convex space in the sense of Lassonde.

Recently, we obtained the following generalization of the Browder fixed point theorem:

Theorem 13. [26] *Let (X, D) be a convex space and $S : D \multimap X$, $T : X \multimap X$ maps. Suppose that*

(1) *$S(z)$ is open [resp. closed] for each $z \in D$;*
(2) *$\operatorname{co} S^-(y) \subset T^-(y)$ for each $y \in X$; and*
(3) *$X = S(M)$ for some $M \in \langle D \rangle$.*

Then T has a fixed point.

In the remainder of this paper, we improve the last part of [P1].

Theorem 14. [26] *Let (X, D) be a convex space and $A : X \multimap D$ a map. If there exist $z_1, z_2, \cdots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n G_i$, then the map $\operatorname{co} A : X \multimap X$ has a fixed point.*

Note that Theorem 14 reduces to the Browder theorem whenever $X = D$ is compact and each $A^-(z)$ is open.

¿From Theorem 14, we immediately have the following:

Theorem 15. *Let X be a convex space and $A : X \multimap X$ a map having open [resp. closed] fibers. If $A(X)$ is covered by a finite number of fibers of A, then either the map $\operatorname{co} A : X \multimap X$ has a fixed point or $A^-(y) = \emptyset$ for some $y \in X$.*

Proof. Suppose that $A^-(y) \neq \emptyset$ for all $y \in X$. Then there exists an $x \in A^-(y)$ or $y \in A(x)$. Therefore, $X = A(X)$ and X is covered by a finite number of open [resp. closed] fibers of A. Now, by Theorem 14, $\operatorname{co} A$ has a fixed point.

From Theorem 14, we have the following:

Theorem 16. *Let (X, D) be a compact convex space and $P : X \multimap D$ a map having open fibers such that $x \notin \operatorname{co} P(x)$ for all $x \in X$. Then $P(x) = \emptyset$ for some $x \in X$.*

Proof. Suppose $P(x) \neq \emptyset$ for all $x \in X$. Then X is covered by $\{P^{-}(z) : z \in D\}$. Since X is compact, it is covered by a finite number of open fibers of P. Then, by Theorem 14, $\mathrm{co}\, P$ has a fixed point, a contradiction.

A point $x_0 \in X$ is called a *maximal element* of a map $T : X \multimap X$ if $T(x_0) = \emptyset$.

Corollary. (Yannelis-Prabhakar [30]) *Let X be a nonempty compact convex subset of a t.v.s. E and $P : X \multimap X$ a map having open fibers such that $x \notin \mathrm{co}P(x)$ for all $x \in X$. Then P has a maximal element.*

Motivated by Theorems 14, 15, and the problem of Ben-El-Mechaiekh (see Theorem 9′), we raised the following conjecture in [P1]:

Conjecture 5. *Let X be a nonempty convex subset of a t.v.s. E and $A : X \multimap X$ a map having open [resp. closed] fibers such that $A(X) \subsetneq X$. If $A(X)$ is covered by a finite number of fibers of A, then the map $\mathrm{co}\, A : X \multimap X$ has a fixed point.*

In view of Theorem 15, Conjecture 5 was incorrectly raised.

In the following references, [1]-[30] are same as in [P1], except [26], which should be replaced by the present one.

Acknowledgement. Parts of this paper were presented at Inter. Conf. on Fixed Point Theory and Applications, Univ. of Valencia, Spain (July 14-19, 2003); the 3rd Inter. Conf. on Nonlinear Anal. and Convex Anal., Tokyo Inst. of Technology, Japan (Aug. 25-31, 2003); and RIMS Workshop on Nonlinear Anal. and Convex Anal., Univ. of Kyoto, Japan (Sept. 15-18, 2003). The author would like to express his gratitude to the efforts of their organizers.

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