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順序数の部分空間の有限積の mild normality

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概 要

The closure of an open set in a topological space is called a regular closed set. A space is called mildly normal (or κ-normal) if every pair of disjoint regular closed sets can be separated by disjoint open sets.

It is known that products of arbitrary many ordinals are mildly normal and products of two subspaces of ordinals are also mildly normal. We characterize the mild normality of products of finitely many subspaces of ω_1. Using this characterization, we show that there exist 3 subspaces of ω_1 whose product is not mildly normal.

A space is said to be *(sub)normal* if every disjoint pair of closed sets is separated by open (resp. G_δ-) sets. For a cover $\mathcal{U} = \langle U_i \mid i \in I\rangle$ of a space, a cover $\mathcal{V} = \langle V_i \mid i \in I\rangle$ satisfying that $V_i \subseteq U_i$ for every $i \in I$ is called a *shrinking* of $\mathcal{U}$. A space is said to be *(sub)shrinking* if every open cover has a closed (resp. F_σ-) shrinking. Obviously, every normal (resp. shrinking) space is subnormal (resp. subshrinking). It is easy to see that every (sub)shrinking space is (sub)normal. It is known that every Dowker space is normal, but not subshrinking, so normality does not imply the subshrinking property in general.

Let ω_1 denote the least uncountable ordinal number. $C \subseteq \omega_1$ is said to be *cofinal* in ω_1 if for every $\alpha < \omega_1$, there is $\gamma \in C$ such that $\alpha \leq \gamma$. $C \subseteq \omega_1$ is called a *club* set of ω_1 if it is closed and cofinal in ω_1. $S \subseteq \omega_1$ is said to be *stationary* in ω_1 if $S \cap C \neq \emptyset$ for every club set C of ω_1.

It is known that for $A, B \subseteq \omega_1$, $A \times B$ is normal iff it is shrinking iff either A or B is non-stationary, or $A \cap B$ is stationary in ω_1 [8]. Particularly, there are subspaces A and B of ω_1 such that $A \times B$ is not normal. On the other hand, it is proved in [7] that every subspace of ω_1^2 is subshrinking, so subnormal. Hence the subshrinking property does not imply normality in general. It was conjectured that every subspace of ω_1^n with $n < \omega$ is subnormal. But we proved in [3] the following theorem.

THEOREM 1. (*Hirata, Kemoto [3]*)

(1) ω_1^3 *has a subspace which is not subnormal.*

(2) *Every subspace of* $\{x \in \omega_1^n \mid \forall k_0, k_1 < n\ (x(k_0) < x(k_1))\}$ *with* $n < \omega$ *is subshrinking, so subnormal.*

The closure of an open set in a topological space is called a *regular closed* set. A space is called *mildly normal* (or κ-*normal*) if every pair of disjoint regular closed sets can be separated by disjoint open sets. Here we say that a space is *mildly subnormal* if every pair of disjoint regular closed sets can be separated by disjoint G_δ-sets. Obviously, every (sub)normal space is mildly (sub)normal. About mild normality, two theorems below were known.

THEOREM 2. (*Kalantan, Szeptycki 2002 [6]*)
If α_i *is an ordinal for every* $i \in I$*, then* $\Pi_{i \in I} \alpha_i$ *is mildly normal.*

THEOREM 3. (*Kalantan, Kemoto 2003 [5]*)

(1) *For every subspaces* A *and* B *of ordinals,* $A \times B$ *is mildly normal.*

(2) $(\omega + 1) \times \omega_1$ *has a subspace which is not mildly normal.*

By the second statement of the theorem above, we can see that the subshrinking property does not always imply mild normality. On the other hand, $\omega_1 \times (\omega_1 + 1)$ is mildly normal, but not subnormal. Hence mild normality does not imply subnormality in general.

In [5], it is asked whether the product of finitely many subspaces of ordinals are mildly normal. We gave the negative answer for this question. Moreover, we characterized the subshrinking property, subnormality, and mild (sub)normality of products of finitely many subspaces of ω_1 in terms of stationarity.

THEOREM 4. (*Hirata, Kemoto [4], Hirata [2]*)
Let $\mathcal{A} = \langle A_k \mid k \in N \rangle$ *be a finite family of non-empty subspaces of* ω_1 *and* $X = \Pi_{k \in N} A_k$*. Then the following conditions are equivalent.*

(a) X *is subshrinking.*

(b) X *is subnormal.*

(c) X *is mildly normal.*

(d) X is mildly subnormal.

(e) For every sequence $\langle k_i \mid i < l\rangle$ of distinct elements of N with $2 \leq l \leq |N|$, if $A_{k_{i-1}} \cap A_{k_i}$ is stationary in ω_1 for every $0 < i < l$, then $\bigcap_{i<l} A_{k_i}$ is stationary in ω_1.

COROLLARY 5. *There are subspaces A, B, C of ω_1 such that $A \times B \times C$ is neither subnormal nor mildly normal.*

Proof. Let S_0, S_1, S_2 be disjoint stationary sets of ω_1. Put $A = S_0 \cup S_1$, $B = S_1 \cup S_2$, and $C = S_2 \cup S_0$. $A \cap B = S_1$ and $B \cap C = S_2$ are stationary, but $A \cap B \cap C = \emptyset$. Hence $\langle A, B, C\rangle$ does not satisfy the last condition of the theorem. □

Proof. We give here a sketch of a part of the proof of a canonical case of (d) $\rightarrow$ (e). For the rest part of the proof, see [4] and [2].

Assume that $\langle A_k \mid k < l\rangle$, $2 \leq l < \omega$, is a family of subspaces of ω_1, $A_{k-1} \cap A_k$ is stationary in ω_1 for every $0 < k < l$, and $\bigcap_{k<l} A_k$ is not stationary in ω_1. We will prove that $X = \Pi_{k<l} A_k$ is not mildly subnormal.

Pick a club set C of ω_1 disjoint from $\bigcap_{k<l} A_k$. Let $\sigma_0 : l \longrightarrow m_0$ and $\sigma_1 : l \longrightarrow m_1$ with $m_0, m_1 \leq l$ be non-decreasing onto functions such that for each $0 < k < l$, $\sigma_0(k-1) < \sigma_0(k)$ iff k is even, and $\sigma_1(k-1) < \sigma_1(k)$ iff k is odd. And let τ_0 and τ_1 are bijections from l onto l such that for each $i = 0, 1$, $j < m_i$, and $k < l$, if $\sigma_i^{-1}``\{j\} = \{k\}$ then $\tau_i(k) = k$, and if $\sigma_i^{-1}``\{j\} = \{k-1, k\}$ with $0 < k < l$ then $\tau_i(k-1) = k$ and $\tau_i(k) = k-1$. (The table below expresses values of σ_i and τ_i in case $l = 5$.)

k	σ_0	τ_0	σ_1	τ_1
0	0	1	0	0
1	0	0	1	2
2	1	3	1	1
3	1	2	2	4
4	2	4	2	3

For $k_0, k_1 < l$, put

$$P(k_0, k_1) = \{x \in X \mid \exists \gamma \in C\ (x(k_0) \leq \gamma < x(k_1))\},$$

$$E(k_0, k_1) = \{x \in X \mid x(k_0) \leq x(k_1)\}.$$

Then $P(k_0, k_1)$ is open, $E(k_0, k_1)$ is closed in X, and $P(k_0, k_1) \subseteq E(k_0, k_1)$. For $i = 0, 1$, put

$$P_i = \bigcap\{P(k_0, k_1) \mid k_0, k_1 < l,\ \tau_i(k_0) < \tau_i(k_1)\},$$

$$E_i = \bigcap\{E(k_0, k_1) \mid k_0, k_1 < l,\ \tau_i(k_0) < \tau_i(k_1)\}.$$

Then P_i is open, E_i is closed in X, and $P_i \subseteq E_i$. Put $F_i = \mathrm{cl}_X P_i$. Then F_i is a regular closed set and $F_i \subseteq E_i$. It suffices to show that F_0 and F_1 are disjoint and cannot be separated by disjoint G_δ-stets.

Assume that $0 < k < l$ is odd. Then $\sigma_0(k-1) = \sigma_0(k)$ and $\sigma_1(k-1) < \sigma_1(k)$, so $\tau_0(k-1) = k > k-1 = \tau_0(k)$ and $\tau_1(k-1) \leq k-1 < k \leq \tau_1(k)$. Hence $E_0 \subseteq E(k, k-1)$ and $E_1 \subseteq E(k-1, k)$. In the same way, we have $E_0 \subseteq E(k-1, k)$ and $E_1 \subseteq E(k, k-1)$ for every even $0 < k < l$. Therefore $F_0 \cap F_1 \subseteq E_0 \cap E_1 \subseteq \bigcap_{0<k<l} E(k-1, k) \cap E(k, k-1) = \{\mathrm{const}_l(\alpha) \mid \alpha \in \bigcap_{k<l} A_k\}$ where $\mathrm{const}_l(\alpha)$ denotes the constant sequence of length l and of value α. Let $\alpha \in \bigcap_{k<l} A_k$ and $\delta = \sup(C \cap \alpha)$. (We consider that $\sup \emptyset = -1$.) Since C is closed in ω_1, $\delta \in C \cup \{-1\}$ holds. C is disjoint from $\bigcap_{k<l} A_k$, so $\delta < \alpha$ and $C \cap (\delta, \alpha] = \emptyset$. $X \cap (\delta, \alpha]^l$ is a neighborhood of $\mathrm{const}_l\, \alpha$ and disjoint from both $P(1,0) \supseteq P_0$. So $\mathrm{const}_l\, \alpha \notin F_0$. Hence $F_0 \cap F_1 = \emptyset$.

Let $G_{i,n}$ be an open set of X and $F_i \subseteq G_{i,n}$ for every $i = 0, 1$ and $n < \omega$. We want to see that $\bigcap_{i=0,1;n<\omega} G_{i,n} \neq \emptyset$. For $i = 0, 1$, let $\mathrm{pr}_{\sigma_i} : \omega_1^{m_i} \longrightarrow \omega_1^l$ denote the mapping such that $\mathrm{pr}_{\sigma_i}(y) = \langle y(\sigma_i(k)) \mid k < l\rangle$ for every $y \in \omega_1^{m_i}$. And let Y_i be the set of all $y \in \omega_1^{m_i}$ such that $\mathrm{pr}_{\sigma_i}(y) \in F_i$ and $y(j_0) < y(j_1)$ for every $j_0 < j_1 < m_i$. It is easy to see that Y_i is stationary in $\omega_1^{m_i}$, that is $Y_i \cap D^{m_i} \neq \emptyset$ for every club set D of ω_1.

For $n < \omega$, $G_{i,n}$ is open in X, so there is a function $f_{i,n} : Y_i \longrightarrow (\omega_1 \cup \{-1\})^{m_i}$ such that $X_{i,n}(y) \subseteq G_{i,n}$ and $f_{i,n}(y)(j) < y(j)$ for every $y \in Y_i$ and $j < m_i$, where

$$X_{i,n}(y) = X \cap \Pi_{k<l}(f_{i,n}(y)(\sigma_i(k)), y(\sigma_i(k))].$$

By using Fodor's Pressing Down Lemma generalized by Fleissner, Kemoto, and Terasawa (see [1]), we can pick a stationary tree $U_{i,n}$ in $\omega_1^{m_i}$ and a function $g_{i,n} : \bigcup_{j<m_i} \mathrm{Lv}_j(U_{i,n}) \longrightarrow \omega_1 \cup \{-1\}$ such that for every $u \in \mathrm{Lv}_{m_i}(U_{i,n})$, $u \in Y_i$ and $f_{i,n}(y) = \langle g_{i,n}(u \restriction j) \mid j < m_i\rangle$ hold. Here for $m < \omega$, we say that U is a *stationary tree* in ω_1^m if there is a family $\langle \mathrm{Lv}_j(U) \mid j \leq m \rangle$ such that:

- $\mathrm{Lv}_j(U) \subseteq \omega_1^j$ for every $j \leq m$,
- $U = \bigcup_{j \leq m} \mathrm{Lv}_j(U)$,
- $u \restriction j' \in \mathrm{Lv}_{j'}(U)$ for every $j \leq m$, $u \in \mathrm{Lv}_j(U)$, and $j' \leq j$,
- $\emptyset \in \mathrm{Lv}_0(U)$,
- $\mathrm{Move}_U(u) = \langle \alpha < \omega_1 \mid u^\frown\langle \alpha \rangle \in U \rangle$ is stationary in ω_1 for every $j < m$ and $u \in \mathrm{Lv}_j(U)$.

Inductively, we can pick $x \in \Pi_{k<l} A_k$ and $u_{i,n} \in \mathrm{Lv}_{m_i}(U_{i,n})$ such that $g_{i,n}(u_{i,n} \restriction j) < x(k) \leq u_{i,n}(j)$ for every $i = 0, 1$, $j < m_i$, and $k < l$ with $\sigma_i(k) = j$. For instance, we determine them in case $n = 5$ in the order $g_{i,n}(u_{i,n} \restriction \emptyset) = g_{i,n}(\emptyset), x(0), u_{1,n}(0), g_{1,n}(u_{1,n} \restriction 1), x(1), u_{0,n}(0)$, $g_{0,n}(u_{0,n} \restriction 1), x(2), u_{1,n}(1), g_{1,n}(u_{1,n} \restriction 2), x(3), u_{0,n}(1), g_{0,n}(u_{0,n} \restriction 2), x(4)$, $u_{i,n}(2)$.

$u_{i,n} \in Y_i$, $f_{i,n}(u_{i,n})(\sigma_i(k)) = g_{i,n}(u_{i,n} \restriction j)$, and $u_{i,n}(j) = u_{i,n}(\sigma_i(k))$, so we have $x \in X_{i,n}(u_{i,n})$ for every $i = 0, 1$ and $n < \omega$. Hence $x \in \bigcap_{i=0,1; n<\omega} G_{i,n}$. □

The problem below is still remained.

PROBLEM 6. *Let $\langle A_n \mid n < \omega \rangle$ be a pairwise disjoint family of stationary subspaces of ω_1. Is $\Pi_{n<\omega} A_n$ mildly normal?*

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