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Questions on quotient compact images of metric spaces, and symmetric spaces

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In this paper, spaces are *regular* and T_1, and maps are continuous and onto.

We make a survey around quotient compact images of metric spaces and symmetric spaces, and we review related questions, by adding some comments. First, let us recall some definitions used in this paper.

For a map $f : X \longrightarrow Y$,

f is called a compact map (resp. s-map) if every $f^{-1}(y)$ is compact (resp. separable).

f is a compact-covering map [M1] if every compact subset of Y is the image of some compact subset of X.

f is a sequence-covering map [Si] if every convergent sequence in Y is the image of some convergent sequence in X.

f is a pseudo-sequence-covering map [ILiT] if every convergent sequence in Y is the image of some compact subset of X.

Every open map from a first countable space is sequence-covering. While, every open map from a locally compact space is compact-covering. Every open compact map from a metric space; or every quotient s-map from a locally compact paracompact space is compact-covering in view of [N1]; or [N2] respectively. But, sequence-covering maps and compact-covering maps are exclusive. Not every pseudo-sequence-covering quotient compact map from a separable metric space onto a compact metric space is compact-covering in view of [M2] (or [L4; Example 3.4.7]). Also, not every compact-covering quotient compact image of a locally compact, separable metric space is a sequence-covering quotient compact image of a metric space; see [TGe], for example.

For a covering $\mathcal{C}$ of a space X,

$\mathcal{C}$ is a k-network [O] if it satisfies (*): For every compact set K, and for every open set U containing K, $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{C}$.

$\mathcal{C}$ is a *cfp*-network [Y1] if $\mathcal{C}$ satisfies the above (*), but there exists a finite closed cover of the compact set K which refines the finite cover $\mathcal{F}$.

$\mathcal{C}$ is a cs^*-network (resp. cs-network) if, for every open set U and every sequence L converging to a point $x \in U$, L is frequently (resp. eventually) in some $C \in \mathcal{C}$ such that $x \in C \subset U$.

A space X is called an $\aleph$-space [O] (resp. $\aleph_0$-space [M1]) iff X has a σ-locally finite (resp. a countable) k-network.

For the theory around sequence-covering maps; k-networks, see [L4]; [T8] respectively, for example.

Let X be a space, and let $\{\mathcal{C}_n : n \in N\}$ be a sequence of covers of X such that each $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$. Then $\mathcal{C} = \bigcup\{\mathcal{C}_n : n \in N\}$ is called a σ-strong network [ILiT] for X if $\{st(x, \mathcal{C}_n) : n \in N\}$ is a local network at each point x in X. "σ-strong networks" is similar as "point-star networks" in the sense of [LY1]. For a σ-strong network $\mathcal{C} = \bigcup\{\mathcal{C}_n : n \in N\}$ for a space X, $\mathcal{C}$ is called a σ-point-finite strong cs^*-network if $\mathcal{C}$ is a cs^*-network for X such that each cover $\mathcal{C}_n$ is point-finite.

A space X is called symmetric (resp. semi-metric) if there exists a non-negative real valued function d defined on $X \times X$ such that $d(x, y) = 0$ iff $x = y$; $d(x, y) = d(y, x)$ for $x, y \in X$; and $G \subset X$ is open in X iff, for every $x \in G$, $S_{1/n}(x) \subset G$ (resp. $x \in int\ S_{1/n}(x) \subset G$) for some $n \in N$, where $S_{1/n}(x) = \{y \in X : d(x, y) < 1/n\}$. Every symmetric space is sequential. Every Fréchet symmetric space is precisely semi-metric. Every Lindelöf symmetric space is hereditarily Lindelöf (thus, any closed set is a G_δ-set) ([Ne]). For symmetric spaces, see [A], [G], [Ne], and [T6], etc.

Let (X, d) be a symmetric space. A sequence $\{x_n\}$ in X is called d-Cauchy if for every $\varepsilon > 0$, there exists $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > k$. Then (X, d) is called a symmetric space satisfying the weak condition of Cauchy (simply, weak Cauchy symmetric) if, $F \subset X$ is closed if some $S_{1/n}(x) \bigcap F = \{x\}$ for all $x \in F$; equivalently, every convergent sequence has a d-Cauchy subsequence. Every semi-metric space can be considered as a weak Cauchy semi-metric space ([B]). Around these matters, see [T6], for example.

We recall some results around quotient compact images of metric spaces. (1) is well-known. (2) is shown in [TGe], for example. (3) is known or routinely shown, but the parenthetic part holds by use of [B; Theorem 1] and (2). For (4), (a) $\Leftrightarrow$ (b) is due to [Co], [V], [J], or [L1]. (a) $\Leftrightarrow$ (c) is due to [ILiT]. For characterizations for various kinds of quotient images of metric spaces, see [L4] (or [T8]).

Results: (1) Every perfect image of a metric space is metric. While, every open compact image of a locally separable metric space is locally separable metric.

(2) Every Fréchet space Y which is a quotient compact image of a metric (resp. locally separable metric) space is metacompact, developable (resp. locally separable metric).

(3) Every quotient compact image of a metric space is weak Cauchy symmetric (but, the converse doesn't hold).

(4) For a space X, the following are equivalent.

(a) X is a quotient compact image of a metric space.

(b) X has a σ-point-finite strong network $\mathcal{C} = \bigcup\{\mathcal{C}_n : n \in N\}$ such that $G \subset X$ is open if for each $x \in G$, $St(x, \mathcal{C}_n) \subset G$ for some $n \in N$.

(c) X is a sequential space with a σ-point-finite strong cs^*-network.

Remark 1: (1) In (b) of the previous result (4), if we omit the "point-finite" of the covers $\mathcal{C}_n$ of X, then such a space X is precisely weak Cauchy symmetric, and it is characterized as a quotient π-image of a (locally compact) metric space; see [Co], [T6], etc. Here, a map f from a metric space (M, d) onto a space X is called a π-map (with respect to d) if, for any $x \in X$, and for any open nbd U of x, $d(f^{-1}(x), M - f^{-1}(U)) > 0$; see [A], for example. Every compact map from a metric space is a π-map.

(2) Every quotient s-image of a metric space is characterized as a sequential space with a point-countable cs^*-network ([T3], [L2]). If we replace "cs^*-network" by "cs-network (resp. cfp-network)", then we can add "sequence-covering (resp. compact-covering) before "quotient" ([LLi]) (resp. [YL]). For other topoics around these, see [L4], [LY2], [LY3], and [TGe], etc. We note that not every Lindelöf space with a σ-disjoint open base (hence, cs-network) is a quotient π-image of a metric space.

We recall two classical problems on symmetric spaces and quotient s-images of metric spaces. (P1) is stated in [BD], [DGNy], or [St], etc. (P2) is posed in [MN].

Problems: (P1) (Michael, Arhangelskii) Is every point of a symmetric space X a G_δ-set ?

(P2) (Michael and Nagami) Is every quotient s-image Y of a metric space X also a compact-covering quotient s-image of a metric space ?

Partial answers to Problems: (1) For (P1), if X is locally separable, under (CH), (P1) is affirmative by [St]. If X is locally Lindelöf or locally hereditarily separable, then (P1) is also affirmative, for every Lindelöf or hereditarily separable symmetric space is hereditarily Lindelöf ([Ne]).

(2) For (P2), Y is, at least, a pseudo-sequence-covering quotient s-image of a metric space ([GMT]). If X is separable, then (P2) is affirmative ([M1]). Also, every sequence-covering quotient s-image of a metric space is a compact-covering (and sequence-covering) quotient s-image of a metric space ([LLi]).

Related the above problems, we have the following questions on quotient compact images of metric spaces. (Q1) is posed in [T5], and (Q2) is given in [TGe]. (As far as the author knows, these questions has not yet been answered).

Questions: Let $f : X \longrightarrow Y$ be a quotient compact map such that X is metric.

(Q1) (a) Is every point of Y a G_δ-set ?

(b) Suppose X is moreover locally compact. Is Y a σ-space (or other nice space) ? Is every point of Y a G_δ-set ?

(Q2) Is Y a compact-covering quotient compact image of a metric space ?

Remark 2 (1) For (Q1), if the Y is Lindelöf or Fréchet, then (b) is affirmative under the quotient map f being an s-map with the metric domain X locally separable. In fact, Y is a cosmic space (resp. topological sum of $\aleph_0$-spaces [GMT]) if Y is Lindelöf (resp. Fréchet). Here, a space is called cosmic if it has a countable network.

(2): For (Q2), the same partial answers as in (P2) hold for the quotient compact images ([ILiT] and [TGe]). Also, if Y is an $\aleph$-space, then (Q2) is affirmative (in view of [TGe]).

Remark 3: (1) If the space Y in (Q1) and (Q2) is *Hausdorff*, (Q1) and (Q2) are negative, then so are (P1) and (P2). In fact, (Q1) is negative by refering to [DGNy; Example 3.2], and (Q2) is also negative by [C]. (Also, there exists a *Hausdorff* (resp. T_1-) symmetric space which has a closed subset (resp. point) that is not a G_δ-set ([Bo]))

(2) A classical Michael's question whether every "closed subset" of a symmetric space is a G_δ-set is negatively answered by [DGNy]; see later Example 6(4).

As an application of the first partial answer to (P1), let us consider the following remark. (1) holds by means of this partial answer. (2) is given in [T2; Corollary 4.15], which is shown by means of (1). Recall that the Arens' space S_2 is the quotient space obtained from the topological sum of countably many copies C_n $(n = 0, 1, \cdots)$ of $C_0 = \{0\} \cup \{1/n : n \in N\}$, by identifying $1/n \in C_0$ with $0 \in C_n$ for each $n \in N$.. The space S_2 is symmetric, but not Fréchet, hence not semi-metric.

Remark 4: (1) (CH). A symmetric space X is semi-metric iff X contains no (closed) copy of the Arens' space S_2.

(2) (CH): If X^ω is symmetric, then X^ω is semi-metric.

(3) The following questions are posed in [T4].

(a) For a separable symmetric space X, is every point of X a G_δ-set ?

(b) Is it possible to omit (CH) in (1) (or (2)) ?

Question (a) is positive under (CH), and is also positive (actually, X is hereditarily Lindelöf) if X is nornal under $2^{\omega_1} > c = 2^\omega$ (see [St]); collectionwise normal; or meta-Lindelöf. While, question (b) is positive if question (a) is positive. Besides, (b) is also positive if X is hereditarily normal, or each point of X is a G_δ-set; or X is a quotient s-image of a locally separable metric space. Here, if (1) holds, then so does (2) without (CH). For case where X is a quotient compact image of a metric space, the author has a *question* whether (1) or (2) holds without (CH).

Related to (a) in (Q1), for certain quotient spaces (or symmetric spaces) Y, let us consider the pseudo-character $\psi(Y)$ or character $\chi(Y)$ of Y, and pose some related questions in terms of products of k-spaces. Here, $\psi(Y) = sup\{\psi(y, Y) : y \in Y\}$, $\psi(y, Y) = min\{|\mathcal{U}| : \mathcal{U}$ is a family of open sets of Y with $\bigcap \mathcal{U} = \{y\}\}$. $\chi(Y)$ is similarly defined, but use

$\chi(y, Y) = min\{|\mathcal{U}| : \mathcal{U}$ is a nbd base at $y\}$ instead of $\psi(y, Y)$.

Remark 5: (1) Let $f : X \longrightarrow Y$ be a quotient s-map such that X is metric. If (a) X is locally separable; or (b) Y^2 is a k-space, then $\psi(Y) \leq c$, and $\chi(Y) \leq 2^c$. Also, if Y is Fréchet with X metric, and f is compact (resp. s-map), then $\chi(Y) \leq \omega$ (resp. $\chi(Y) \leq c$). These are shown in view of [TZ; Theorem 2.1], etc., but the result for (b) can be shown by use of [T7; Lemma 1], refering to [TZ], [GMT], etc. (We note that $\chi(Y) \leq \omega$ if Y^ω is a k-space by use of [L3; Corollary 3.9]).

(2) Related to case (b) in (1), the author has a *question* whether $\chi(Y) \leq c$ (or $\psi(Y) = \omega$) for a quotient compact image Y of a metric space such that Y^2 is a symmetric space (equivalently, k-space). We can't replace "$\chi(Y) = c$" by "$\chi(Y) \leq \omega$", by putting $Y = S_2$ in *Proposition* below. The author also has a similar *question* whether $\chi(Y) \leq c$ (or $\psi(Y) = \omega$, or $\chi(Y) \leq 2^c$) for a symmetric space Y such that Y^2 is a symmetric space (equivalently, k-space).

Proposition: For a quotient compact image Y of a locally compact, separable metric space, each product Y^n $(n \in N)$ is symmetric.

(3) For each infinite cardinal α, there exists a paracompact σ-metric space which is a quotient finite-to-one image Y of a metric space, but $\chi(Y) > \alpha$ by use of [TZ; Example 2.3]. However, the author doesn't know whether there exists some cardinal α such that $\psi(Y) \leq \alpha$ for any symmetric space (resp. any quotient compact images of metric space) Y. ((P1) (resp. (Q1)(a)) is the question for case $\alpha = \omega$).

Finally, let us recall some examples on quotient finite-to-one images of metric spaces.

Example 6: (1) An open finite-to-one map $f : X \longrightarrow Y$ such that X is metric, Y is σ-metric, but Y is not normal ([T1; Example 3.2]).

(2) A quotient finite-to-one map $f : X \longrightarrow Y$ such that X is locally compact metric and Y is paracompact σ-metric, but Y has no point-countable cs-networks ([LT; Remark 14]).

(3) A quotient finite-to-one map $f : X \longrightarrow Y$ such that X is locally compact metric and Y is separable σ-metric, but Y is not meta-Lindelöf, and not an $\aleph$-space ([GMT; Example 9.3]).

(4) A quotient finite-to-one map $f : X \longrightarrow Y$ such that X is metric,

but Y contains a closed set which is not a G_δ-set by refering to [DGNy; Example 3.1].

(5) (CH) A quotient finite-to-one map $f : X \longrightarrow Y$ such that X is locally separable metric and Y is σ-metric and cosmic, but Y is not an $\aleph$-space. (We note that (CH) is used to see the Y is *regular*. Without (CH), there exists a quotient finite-to-one map $f : X \longrightarrow Y$ such that X is locally compact metric, and Y is a *Hausdorff*, σ-compact and cosmic space, but Y is not an $\aleph$-space). For these, see [S] and [T7; Remark 2].

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