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The behaviour of dimension functions on unions of closed subsets I

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1 Introduction

All spaces we shall consider here are separable metrizable spaces.

It is well known that there exist (transfinite) dimension functions d such that $d(X_1 \cup X_2) > \max\{dX_1, dX_2\}$ even if the subspaces X_1 and X_2 are closed in the union $X_1 \cup X_2$.

Let $\mathcal{K}$ be a class of spaces, β, α be ordinals such that $\beta < \alpha$, and X be a space from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. Define $m(X, d, \beta, \alpha) = \min\{k : X = \bigcup_{i=1}^{k} X_i$, where X_i is closed in X and $dX_i \leq \beta\}$, $m_{\mathcal{K}}(d, \beta, \alpha) = \min\{m(X, d, \beta, \alpha) : X \in \mathcal{K}$ and $m(X, d, \beta, \alpha)$ exists$\}$ and $M_{\mathcal{K}}(d, \beta, \alpha) = \sup\{m(X, d, \beta, \alpha) : X \in \mathcal{K}$ and $m(X, d, \beta, \alpha)$ exists$\}$.

We will say that $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ do not exist if there is no space X from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. It is evident that either $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ satisfy $2 \leq m_{\mathcal{K}}(d, \beta, \alpha) \leq M_{\mathcal{K}}(d, \beta, \alpha) \leq \infty$ or they do not exist.

Two natural questions arise.

Question 1.1 *Determine the values of $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ for given $\mathcal{K}, d, \beta, \alpha$.*

Question 1.2 *Find a (transfinite) dimension function d having for given pair $2 \leq k \leq l \leq \infty$, $m_{\mathcal{K}}(d, \beta, \alpha) = k$ and $M_{\mathcal{K}}(d, \beta, \alpha) = l$.*

Let $\mathcal{C}$ be the class of metrizable compact spaces and $\mathcal{P}$ be the class of separable completely metrizable spaces. By trind(trInd) we denote Hurewicz's (Smirnov's) transfinite extension of ind (Ind) and Cmp is the large inductive compactness degree introduced by de Groot. We shall recall their definitions in the next section. Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural decomposition of the ordinal $\alpha \geq 0$ into the sum of a limit number $\lambda(\alpha)$(observe that λ(an integer $\geq 0) = 0)$ and a nonnegative integer $n(\alpha)$. Let $\beta < \alpha$ be ordinals, put $p(\beta,\alpha) = \frac{n(\alpha)+1}{n(\beta)+1}$ and $q(\beta,\alpha)$ = the smallest integer $\geq p(\beta,\alpha)$. We have the following theorems. The outline of the proof will be presented in section 2.

Theorem 1.1 *1. Let $0 \leq \beta < \alpha$ be finite ordinals. Then we have $m_{\mathcal{P}}(Cmp\ ,\beta,\alpha) = q(\beta,\alpha)$ and $M_{\mathcal{P}}(Cmp\ ,\beta,\alpha) = \infty$.*

2. Let $\beta < \alpha$ be infinite ordinals. Then we have

$$m_{\mathcal{C}}(trInd,\beta,\alpha) = \begin{cases} q(\beta,\alpha), & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise} \end{cases}$$

$$M_{\mathcal{C}}(trInd,\beta,\alpha) = \begin{cases} \infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise} \end{cases}$$

Theorem 1.2 *1. For every finite $\alpha \geq 1$ there exists a space $X_\alpha \in \mathcal{P}$ such that*

(a) $CmpX_\alpha = \alpha$;

(b) $X_\alpha = \bigcup_{i=1}^{\infty} Y_i$, where each Y_i is closed in X_α and $CmpY_i \leq 0$;

(c) $X_\alpha \neq \bigcup_{i=1}^{m} Z_i$, where each Z_i is closedin X_α and $CmpZ_i \leq \alpha-1$ and m is any integer ≥ 1.

2. For every infinite α with $n(\alpha) \geq 1$ there exists a space $X_\alpha \in \mathcal{C}$ such that

(a) $trIndX_\alpha = \alpha$;

(b) $X_\alpha = \bigcup_{i=1}^{\infty} Y_i$, where each Y_i is closed in X_α and finite-dimensional;

(c) $X_\alpha \neq \bigcup_{i=1}^{m} Z_i$, where each Z_i is closed in X_α and $trIndZ_i \leq \alpha - 1$ and m is any integer ≥ 1.

2 Evaluations of $m_{\mathcal{K}}(d,\beta,\alpha)$ and $M_{\mathcal{K}}(d,\beta,\alpha)$

The notation $X \sim Y$ means that the spaces X and Y are homeomorphic. At first we consider the following construction.

Step 1. Let X be a space without isolated points and P a countable dense subset of X. Consider Alexandroff's dublicate $D = X \cup X^1$ of X, where each point of X^1 is clopen in D. Remove from D those points of X^1 which do not correspond to any point from P. Denote the obtained space by $L(X,P)$. Observe that $L(X,P)$ is the disjoint union of X with the countable dense subset P^1 of $L(X,P)$ consisting of points from X^1 corresponding to the points from P. The space $L(X,P)$ is separable and metrizable. It will be compact if X is compact. Put $L_1(X,P) = L(X,P)$. Assume that X is a completely metrizable space (recall that the increment $bX \setminus X$ in any compactification bX of X is an F_σ-set in bX). Observe that $L(bX,P)$ is a compactification of $L(X,P)$ and the increment

$L(bX, P) \setminus L(X, P)$ ($\sim bX \setminus X$) is an F_σ-set in $L(bX, P)$. Hence $L(X, P)$ is also completely metrizable.

Step 2. Let X be a space with a countable subset R consisting of isolated points. Let Y be a space. Substitute each point of R in X by a copy of Y. The obtained set W has the natural projection $pr : W \to X$. Define the topology on W as the smallest topology such that the projection pr is continuous and each copy of Y has its original topology as a subspace of this new space. The obtained space is denoted by $L(X, R, Y)$. It is separable and metrizable and it will be compact (completely metrizable) if X and Y are the same. Moreover $L(X, R, Y)$ is the disjoint union of the closed subspace $X \setminus R$ of X (which we will call *basic* for the space $L(X, R, Y)$) and countably many clopen copies of Y.

Step 3. Let X be a space without isolated points and P be a countable dense subset of X. Define $L_n(X, P) = L(L_1(X, P), P^1, L_{n-1}(X, P)), n \geq 2$. Observe that for any open subset O of $L_n(X, P)$ meeting the basic subset X of $L_n(X, P)$ there is a copy of $L_{n-1}(X, P)$ contained in O. Put $L_*(X, P) = \{*\} \cup \oplus_{n=1}^{\infty} L_n(X, P)$. (Here by $\{*\} \cup \oplus_{i=1}^{\infty} X_i$ we mean the one-point extension of the free union $\oplus_{i=1}^{\infty} X_i$ such that a neighborhood base at the point $*$ consists of the sets $\{*\} \cup \oplus_{i=k}^{\infty} X_i, k = 1, 2, ...$). Observe that $L_*(X, P)$ is separable and metrizable, and it contains a copy of $L_q(X, P)$ for each q. $L_*(X, P)$ will be compact (completely metrizable) if X is the same.

All our dimension functions d are assumed to be monotone with respect to closed subsets and $d($ a point $) \leq 0$.

Lemma 2.1 *Let d be a dimension function and X be a space without isolated points which cannot be written as the union of $k \geq 1$ closed subsets with $d \leq \alpha$, where α is an ordinal. Let also P be a countable dense subset of X. Then*

(a) for every q we have $L_q(X, P) \neq \bigcup_{i=1}^{qk} X_i$, where each X_i is closed in $L_q(X, P)$ and $dX_i \leq \alpha$;

(b) $L_*(X, P) \neq \bigcup_{i=1}^{m} X_i$, where each X_i is closed in $L_*(X, P)$ and $dX_i \leq \alpha$, and m is any integer ≥ 1.

All our classes $\mathcal{K}$ of topological spaces are assumed to be monotone with respect to closed subsets and closed under operations $L(,)$ and $L(,,)$.

Lemma 2.2 *Let $\mathcal{K}$ be a class of topological spaces, α be an ordinal ≥ 0 and d be a dimension function such that $dL(L(S, P), P^1, T) \leq \alpha$ for any S, T from $\mathcal{K}$ with $dS \leq \alpha$, $dT \leq \alpha$ and any P. Let $X \in \mathcal{K}$ such that $X = \bigcup_{i=1}^{k} X_i$, where each X_i is closed in X, without isolated points and $dX_i \leq \alpha$. Let also P_i be a countable dense subset of X_i for each i. Then for each q the space $L_q(X, \bigcup_{i=1}^{k} P_i)$ exists and is the union of k^q closed subsets with $d \leq \alpha$.*

We will say that a dimension function d satisfies *the sum theorem of type A* if for any X being the union of two closed subspaces X_1 and X_2 with $dX_i \leq \alpha_i$, where each α_i is finite and ≥ 0, we have $dX \leq \alpha_1 + \alpha_2 + 1$. A space X is *completely decomposable in the sense of the dimension function d* if $dX = \alpha$, where α is an integer ≥ 1, and $X = \bigcup_{i=1}^{\alpha+1} X_i$, where each X_i is closed in X and $dX_i = 0$. Observe that if this space X belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq \alpha + 1$ for each β with $0 \leq \beta < \alpha$.

We will say that a transfinite dimension function d satisfies *the sum theorem of type A_{tr}* if for any X being the union of two closed subspaces X_1 and X_2 with $dX_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $dX \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $dX \leq \alpha_2 + n(\alpha_1) + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$. A space X is *completely decomposable in the sense of the transfinite dimension function d* if $dX = \alpha$, where α is an infinite ordinal with $n(\alpha) \geq 1$, and $X = \bigcup_{i=1}^{n(\alpha)+1} X_i$, where each X_i is closed in X and $dX_i = \lambda(\alpha)$. Observe that if this space X belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq n(\alpha) + 1$ for each β with $\lambda(\alpha) \leq \beta < \alpha$.

To every space X one assigns *the large inductive compactness degree* Cmp as follows.

(i) Cmp $X = -1$ iff X is compact;

(ii) Cmp $X = 0$ iff there is a base $\mathcal{B}$ for the open sets of X such that the boundary Bd U is compact for each U in $\mathcal{B}$;

(iii) Cmp $X \leq \alpha$, where α is an integer ≥ 1, if for each pair of disjoint closed subsets A and B of X there exists a partition C between A and B in X such that Cmp $C \leq \alpha - 1$;

(iv) Cmp $X = \alpha$ if Cmp $X \leq \alpha$ and Cmp $X > \alpha - 1$;

(v) Cmp $X = \infty$ if Cmp $X > \alpha$ for every positive integer α.

Recall also the definitions of *the transfinite inductive dimenions* trind and trInd.

(i) trInd$X = -1$ iff $X = \emptyset$;

(ii) trInd$X \leq \alpha$, where α is an ordinal ≥ 0, if for each pair of disjoint closed subsets A and B of X there exists a partition C between A and B in X such that trInd$C < \alpha$;

(iii) trInd$X = \alpha$ if trInd$X \leq \alpha$ and trInd$X \leq \beta$ holds for no $\beta < \alpha$;

(iv) trInd$X = \infty$ if trInd$X \leq \alpha$ holds for no ordinal α.

The definition of trind is obtained by replacing the set A in (ii) with a point of X.

Remark 2.1 *(i) Note that Cmp satisfies the sum theorem of type A ([ChH, Theorem 2.2]) and for each integer $\alpha \geq 1$ there exists a separable completely metrizable space C_α with Cmp $C_\alpha = \alpha$ which is completely decomposable in the sense of Cmp ([ChH, Theorem 3.1]). For the convenience of the reader, we recall that $C_\alpha = \{0\} \times ([0,1]^\alpha \setminus (0,1)^\alpha) \cup \bigcup_{i=1}^{\infty} \{x_i\} \times [0,1]^\alpha \subset I^{\alpha+1}$, where $\{x_i\}_{i=1}^{\infty}$ is a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all i and $\lim_{i \to \infty} x_i = 0$. Note that the closed subsets in the decomposition of C_α can be assumed without isolated points.*

(ii) Note also that trInd satisfies the sum theorem of type A_{tr} ([E, Theorem 7.2.7]) and for each infinite ordinal α with $n(\alpha) \geq 1$ there exists a metrizable compact space S^α (Smirnov's

compactum) with trInd$S^\alpha = \alpha$ which is completely decomposable in the sense of trInd ([Ch, Lemma 3.5]). Recall that Smirnov's compacta $S^0, S^1, ..., S^\alpha, ..., \alpha < \omega_1$, are defined by transfinite induction: S^0 is the one-point space, $S^\alpha = S^\beta \times [0,1]$ for $\alpha = \beta + 1$, and if α is a limit ordinal, then $S^\alpha = \{*_\alpha\} \cup \bigcup_{\beta<\alpha} S^\beta$ is the one-point compactification of the free union of all the previously defined S^β's, where $*_\alpha$ is the compactifying point. Note that the closed subsets in the decomposition of S^α can be assumed without isolated points.

(iii) Observe that trind satisfies another sum theorem. Namely, for any X being the union of two closed subspaces X_1 and X_2 with trind$X_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have trind$X \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and trind$X \leq \alpha_2 + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$ [Ch, Theorem 3.9].

Proposition 2.1 *(i) Let $\mathcal{K}$ be a class of topological spaces, d be a dimension function satisfying the sum theorem of type A, α be an integer ≥ 1 and X be a space from $\mathcal{K}$ with $dX = \alpha$ which is completely decomposable in the sense of d. Then for any integer $0 \leq \beta < \alpha$ we have $m_\mathcal{K}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$.*

(ii) Let $\mathcal{K}$ be a class of topological spaces, d be a transfinite dimension function satisfying the sum theorem of type A_{tr}, α be an infinite ordinal with $n(\alpha) \geq 1$ and X be a space from $\mathcal{K}$ with $dX = \alpha$ which is completely decomposable in the sense of d. Then for any infinite ordinal $\beta < \alpha$ we have $m_\mathcal{K}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$ if $\lambda(\beta) = \lambda(\alpha)$ and $m_\mathcal{K}(d, \beta, \alpha)$ does not exist otherwise.

The *deficiency* def is defined in the following way: For a space X,

$$\text{def } X = \min\{\dim(Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$

Recall that Cmp $X \leq$ def X and def $X = 0$ iff Cmp $X = 0$.

Lemma 2.3 *(i) def $L(L(X, P), P^1, Y) = \max\{$def X, def $Y\}$ for any X, P, Y. In particular, we have Cmp $L(L(X, P), P^1, Y) \leq 0$ if Cmp $X \leq 0$ and Cmp $Y \leq 0$.*

(ii) trInd$L(L(X, P), P^1, Y) = \max\{$trInd$X$, trInd$Y\}$ for any compacta X, Y and any P.

Proof. (i) Let bX and bY be metrizable compactifications of X and Y respectively such that $\dim(bX \setminus X) =$ def X and $\dim(bY \setminus Y) =$ def Y. Observe that the space $L(L(bX, P), P^1, bY)$ is a compactification of $L(L(X, P), P^1, Y)$ and the increment $Z = L(L(bX, P), P^1, bY) \setminus L(L(X, P), P^1, Y)$ is the union of countably many closed subsets, one of which is homeomorphic to $bX \setminus X$ and the others are homeomorphic to $bY \setminus Y$. So by the countable sum theorem for dim we get that $\dim Z = \max\{\dim(bX \setminus X), \dim(bY \setminus Y)\} = \max\{\text{def } X, \text{def } Y\}$. Hence def $L(L(X, P), P^1, Y) \leq \max\{\text{def } X, \text{def } Y\}$, thereby def $L(L(X, P), P^1, Y) = \max\{\text{def } X, \text{def } Y\}$.

(ii) At first let us prove the statement when Y is a singleton. Observe that in this case $L(L(X, P), P^1, Y) = L(X, P)$. Consider two disjoint closed subsets A and B of $L(X, P)$.

Recall that $L(X,P)$ contains a copy of X. Choose a partition C between $A\cap X$ and $B\cap X$ in X. Extend the partition to a partition C_1 between A and B in $L(X,P)$. Consider another partition C_2 between A and B in $L(X,P)$ which is "thin" (i.e. $\mathrm{Int}_{L(X,P)}C_2=\emptyset$) and is in C_1. Observe that $C_2\subset C$. Hence $\mathrm{trInd}L(X,P)=\mathrm{trInd}X$.

Now let us consider the general case. Assume that A and B are disjoint closed subsets in $L(L(X,P),P^1,Y)$. Recall that there is the natural continuous projection pr : $L(L(X,P),P^1,Y)\to L(X,P)$. Consider the closed subsets prA and prB of $L(X,P)$. If they are disjoint, choose a partition C_2 between prA and prB in $L(X,P)$ like in the previous part. Observe that $pr^{-1}C_2$ is a partition between A and B in $L(L(X,P),P^1,Y)$ such that $pr^{-1}C_2$ is homeomorphic to a closed subset of C. Assume now that $prA\cap prB\neq\emptyset$. Note that $Q^1=prA\cap prB$ is finite and $L(L(X,P),P^1,Y)$ is the free union of $L(L(X,(P\setminus Q)),P^1\setminus Q^1,Y)$, where Q is the finite subset of P corresponding to Q^1 and finitely many copies of Y. Choose a partition between A and B in X and a partition between A and B in each of the copies of Y corresponding to points of Q. It follows from the foregoing discussion that the free union of these partitions constitutes a partition in $L(L(X,P),P^1,Y)$ between A and B. We conclude that $\mathrm{trInd}L(L(X,P),P^1,Y)=\max\{\mathrm{trInd}X,\mathrm{trInd}Y\}$. □

Proof of Theorem 1.1.

(i) Because of Remark 2.1 and Proposition 2.1, we need only establish that $M_{\mathcal{P}}(\mathrm{Cmp},\beta,\alpha)=\infty$. Consider the space $C_\alpha=\bigcup_{i=1}^{\alpha+1}X_i$, where each X_i is closed in X, without isolated points and $\mathrm{Cmp}\,X_i=0$, from Remark 2.1. Let P_i be a countable dense subset of X_i. Put $P=\bigcup_{i=1}^{\alpha+1}P_i$. Recall that def $C_\alpha=\alpha$ ([ChH, Theorem 3.1]). So by Lemma 2.3 for any integer q we have def $L_q(C_\alpha,P)=\alpha$ and hence Cmp $L_q(C_\alpha,P)=\alpha$. Observe that by Lemmas 2.2 and 2.3, we get that the completely metrizable space $L_q(C_\alpha,P)$ is the union of $(\alpha+1)^q$ many closed subspaces with Cmp ≤ 0. Hence $m(L_q(C_\alpha,P),\mathrm{Cmp},\beta,\alpha)\leq(\alpha+1)^q$. Since Cmp satisfies the sum theorem of type A, C_α cannot be represented as α-many closed subsets with Cmp ≤ 0. By Lemma 2.1, we have $m(L_q(C_\alpha,P),\mathrm{Cmp},\beta,\alpha)\geq q\alpha\geq q$. Since $\lim_{q\to\infty}q=\infty$ we get $M_{\mathcal{P}}(\mathrm{Cmp},\beta,\alpha)=\infty$.

(ii) By similar arguments as in the proof of (i) one can prove $M_{\mathcal{C}}(\mathrm{trInd},\beta,\alpha)=\infty$, if $\lambda(\beta)=\lambda(\alpha)$; and does not exist otherwise. □

Proof of Theorem 1.2.

(i) Put $X_\alpha=\{*\}\cup\oplus_{i=1}^\infty L_i(C_\alpha,P)$. Observe that X_α is completely metrizable and is the union of countably many closed subspaces with Cmp ≤ 0. Since def $X_\alpha=\alpha$, we have Cmp $X_\alpha=\alpha$. Now observe that $\lim_{i\to\infty}m(L_i(C_\alpha,P),\mathrm{Cmp}\,,\alpha-1,\alpha)=\infty$. Hence X_α cannot be written as the finite union of closed subsets with Cmp $\leq\alpha-1$.

(ii) Put $X_\alpha=\{*\}\cup\oplus_{i=1}^\infty L_i(S^\alpha,P)$. Observe that X_α is compact and is the union of countably many finite-dimensional closed subspaces (recall that S^α and therefore $L_i(S^\alpha,P)$ have the same property). Since for each i, $\mathrm{trInd}L_i(S^\alpha,P)=\alpha$, we have $\mathrm{trInd}X_\alpha=\alpha$. Now observe that $\lim_{i\to\infty} m(L_i(S^\alpha, P), \text{trInd}, \alpha-1, \alpha) = \infty$. Hence X_α cannot be written as the finite union of closed subsets with $\text{trInd} \le \alpha - 1$. □

Remark 2.2 *Let Q be the set of rational numbers of the closed interval $[0,1]$. Recall that for the spaces $X = Q \times [0,1]^n$ and $Y = ([0,1] \setminus Q) \times I^n$ we have $Cmp\, X = def\, X = Cmp\, Y = def\, Y = n$ ([AN, p. 18 and 56]). It is easy to observe that X satisfies points (a)-(c) of Theorem 1.2 (i). However, X is not completely metrizable. Note that Y is completely metrizable and satisfies points (a) and (c) of Theorem 1.2 (i) but not (b). Observe that Smirnov's compactum S^α with $n(\alpha) \ge 1$ satisfies points (a) and (b) of Theorem 1.2 (ii) but not (c). Note also that any Cantor manifold Z with $trIndZ = \alpha$, where α is infinite ordinal with $n(\alpha) \ge 1$, (see for such spaces for example in [O]) satisfies points (a) and (c) of Theorem 1.2 (ii) but not (b).*

Let d be a (transfinite) dimension function. A space X with $dX \neq \infty$ is said *to have property* $(*)_d$ if for every open nonempty subset O of the space X there exists a closed in X subset $F \subset O$ with $dF = dX$.

Observe that the spaces X, Y from Remark 2.2 have property $(*)_{Cmp}$ and Z has property $(*)_{trInd}$.

Proposition 2.2 *Let X be a completely metrizable space with $dX \neq \infty$. Then $X \neq \bigcup_{i=1}^\infty X_i$, where each X_i is closed in X and $dX_i < dX$ iff there exists a closed subspace Y of X such that*

(i) $dY = dX$ and

(ii) Y has the property $(*)_d$.

Remark 2.3 *This remark concerns non-metrizable compact spaces. Using the construction of Lokucievskij's example ([E, p. 140]), Chatyrko, Kozlov and Pasynkov [ChKP, Remark 3.15 (b)] presented for each $n = 3, 4, \ldots$ a compact Hausdorff space X_n such that $ind\, X_n = 2$ and $m(X_n, ind, 1, 2) = n$. Hence it is clear that $m_{\mathcal{N}}(ind, 1, 2) = 2$ and $M_{\mathcal{N}}(ind, 1, 2) = \infty$, where $\mathcal{N}$ is the class of compact Hausdorff spaces. In [K] Kotkin constructed a compact Hausdorff space X with $ind\, X = 3$ which is the union of three one-dimensional in the sense of ind closed subspaces. Hence, $m_{\mathcal{N}}(ind, 1, 3) = 3$ and $m_{\mathcal{N}}(ind, 2, 3) = 2$. Filippov in [F] presented for every n a compact Hausdorff space F_n with $ind\, F_n = n$, which is the union of finitely many one-dimensional in the sense of ind closed subspaces, thereby $m_{\mathcal{N}}(ind, k, n) < \infty$ for each $1 \le k < n$. By the sum theorem from Remark 2.1 (iii) for ind which is valid in fact for all regular spaces, one can get that $m_{\mathcal{N}}(ind, 1, n) \ge 2^{n-2} + 1$ for each n.*

参考文献

[AN] J. M. Aarts and T. Nishiura, Dimension and Extensions, North-Holland, Amsterdam, 1993

[Ch] V. A. Chatyrko, On finite sum theorems for transfinite inductive dimensions, Fund. Math. 162 (1999) 91-98

[ChH] V. A. Chatyrko and Y. Hattori, On a question of de Groot and Nishiura, Fund. Math. 172 (2002) 107-115

[ChKP] V. A. Chatyrko, K. L. Kozlov and B. A. Pasynkov, On an approach to constructing compacta with different dimensions dim and ind, Topology Appl. 107 (2000) 39-55

[E] R. Engelking, Theory of dimensions, finite and infinite, Heldermann Verlag, Lemgo, 1995.

[F] V. V. Filippov, On compacta with unequal dimensions ind and dim, Soviet Math. Dokl. vol. 11 (1970) N 3, 687-691

[K] S. V. Kotkin, Summation theorem for inductive dimensions, Math. Notes vol. 52 (1992) N 3, 938-942

[O] W. Olshewski, Cantor manifolds in the theory of transfinite dimension, Fund. Math. 145 (1994) 39-64