数理解析研究所講究録 1370 巻 2004 年 22-35

FIBREWISE HOMOTOPY IN THE FIBREWISE CATEGORY MAP

TAKUO MIWA

1. Introduction

For a base space B, the category $\mathbf{TOP}_B$ is the fibrewise topology over B. For Fibrewise Homotopy in $\mathbf{TOP}_B$, see [6],[2]. For General Topology of Continuous Maps or Fibrewise General Topology, see B.A.Pasynkov[9],[10]. In [1], D. Buhagiar studied fibrewise topology in the category of all continuous maps, called **MAP** by him (as a way of thinking of a category, **MAP** can be seen in earlier works, see for example [12],[5]). The study of fibrewise topology in **MAP** is a generalization of it in the category $\mathbf{TOP}_B$. For the merit of treating in **MAP**, we clarify that in treating fibrewise homotopy and fibrewise pointed homotopy, we can freely consider $I \times B$ as base spaces, and therefore need not consider some complicated procedures:$(I \times B) \times_B X = I \times X$ and the reduced fibrewise cylinder $I \tilde{\times} X$ for constructing sections, and also we need not to consider any generalized concept of fibrewise non-degenerate spaces [6; section 22], and we can give a simple proof of the generalized formula for the suspension of fibrewise product spaces.

Main purpose of this paper is that we give some definitions, propositions and theorems in [4] and [7].

The objects of **MAP** are continuous maps from any topological space into any topological space. For two objects $p : X \to B$ and $p' : X' \to B'$, a morphism from p into p' is a pair (ϕ, α) of continuous maps $\phi : X \to X', \alpha : B \to B'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\alpha} & B' \end{array}$$

is commutative. We note that this situation is a generalization of the category $\mathbf{TOP}_B$ since the category $\mathbf{TOP}_B$ is isomorphic to the particular case of $\mathbf{MAP}$ in which the spaces $B' = B$ and $\alpha = id_B$. We call an object $p : X \to B$ an $\mathbf{M}$-*fibrewise space* and denote (X, p, B). Also, for two $\mathbf{M}$-fibrewise spaces $(X, p, B), (X', p', B')$, we call the morphism (ϕ, α) from p into p' an $\mathbf{M}$-*fibrewise map*, and denote $(\phi, \alpha) : (X, p, B) \to (X', p', B')$.

Furthermore, in this paper we often consider the case that an $\mathbf{M}$-fibrewise space (X, p, B) has a section $s : B \to X$, we call it an $\mathbf{M}$-*fibrewise pointed space* and denote (X, p, B, s). For two $\mathbf{M}$-fibrewise pointed spaces (X, p, B, s), (X', p', B', s'), if an $\mathbf{M}$-fibrewise map $(\phi, \alpha) : (X, p, B) \to (X', p', B')$ satisfies $\phi s = s'\alpha$, we call it an $\mathbf{M}$-*fibrewise pointed map* and denote $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$.

In this paper, we assume that all spaces are topological spaces, all maps are continuous and id is the identity map of $I = [0, 1]$ into itself. Moreover, we use the following notation : For any $t \in I$, the maps $\sigma_t : X \to I \times X$ and $\delta_t : B \to I \times B$ are defined by

$$\sigma_t(x) = (t, x), \ \delta_t(b) = (t, b) \ (x \in X, \ b \in B).$$

For other undefined terminology, see [3], [4] and [6].

2. M-FIBREWISE POINTED HOMOTOPY

In this section, first we shall define an $\mathbf{M}$-fibrewise pointed homotopy, $\mathbf{M}$-fibrewise pointed cofibration and $\mathbf{M}$-fibrewise pointed cofibred pair which are introduced in [4]. Next, we shall introduce some concepts, for example, $\mathbf{M}$-fibrewise pointed mapping cylinder, $\mathbf{M}$-fibrewise pointed collapse, $\mathbf{M}$-fibrewise pointed cone and $\mathbf{M}$-fibrewise pointed nulhomotopic. Last, we shall give some propositions without proofs.

Definition 2.1. (1)([4; Definition 5.1]) Let $(\phi, \alpha), (\theta, \beta) : (X, p, B, s) \to (X', p', B', s')$ be $\mathbf{M}$-fibrewise pointed maps. If there exists an $\mathbf{M}$-fibrewise pointed map $(H, h) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B', s')$ such that (H, h) is an $\mathbf{M}$-fibrewise homotopy of (ϕ, α) into (θ, β) (that is; $H\sigma_0 = \phi, \ H\sigma_1 = \theta, \ h\delta_0 = \alpha, \ h\delta_1 = \beta$), we call it an $\mathbf{M}$-*fibrewise pointed homotopy* of (ϕ, α) into (θ, β). If there exists an $\mathbf{M}$-fibrewise pointed homotopy of

(ϕ,α) into (θ,β), we say (ϕ,α) is **M**-*fibrewise pointed homotopic* to (θ,β) and write $(\phi,\alpha)\simeq^{\mathbf{M}}_{(\mathbf{P})}(\theta,\beta)$.

(2)([4;Definition 5.2]) An **M**-fibrewise pointed map $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ is called an **M**-*fibrewise pointed homotopy equivalence* if there exists an **M**-fibrewise pointed map $(\theta,\beta):(X',p',B',s')\to(X,p,B,s)$ such that $(\theta\phi,\beta\alpha)\simeq^{\mathbf{M}}_{(\mathbf{P})}(id_X,id_B)$, $(\phi\theta,\alpha\beta)\simeq^{\mathbf{M}}_{(\mathbf{P})}(id_{X'},id_{B'})$. Then we denote $(X,p,B,s)\cong^{\mathbf{M}}_{(\mathbf{P})}(X',p',B',s')$.

It is obvious that the relations $\simeq^{\mathbf{M}}_{(\mathbf{P})}$ and $\cong^{\mathbf{M}}_{(\mathbf{P})}$ are equivalence relations.

Definition 2.2. ([4;Definition 5.6]) An **M**-fibrewise pointed map $(u,\gamma):(X_0,p_0,B_0,s_0)\to(X,p,B,s)$ is an **M**-*fibrewise pointed cofibration* if (u,γ) has the following **M**-fibrewise homotopy extension property : Let $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ be an **M**-fibrewise pointed map and $(H,h):(I\times X_0,id\times\gamma,I\times B_0,id\times s_0)\to(X',p',X',s')$ an **M**-fibrewise pointed homotopy such that the following two diagrams

$$\begin{array}{ccc} X_0 & \xrightarrow{\sigma_0} & I\times X_0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle H} \\ X & \xrightarrow{\phi} & X' \end{array}\qquad\qquad \begin{array}{ccc} B_0 & \xrightarrow{\delta_0} & I\times B_0 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{\alpha} & B' \end{array}$$

are commutative. Then there exists an **M**-fibrewise pointed homotopy $(K,k):(I\times X,id\times p,I\times B,I\times s)\to(X',p',B',s')$ such that $K\kappa_0=\phi, K(id\times u)=H, k\rho_0=\alpha, k(id\times\gamma)=h$, where $\kappa_0:X\to I\times X$ and $\rho_0:B\to I\times B$ are defined by $\kappa_0(x)=(0,x)$ and $\rho_0(b)=(0,b)$ for $x\in X,\ b\in B$.

Definition 2.3. ((([4;Definition 2.3 and p.210])

(1) Let (X,p,B) be an **M**-fibrewise space. If $X_0\subset X, B_0\subset B$ and $p(X_0)\subset B_0$, we call $(X_0,p|X_0,B_0)$ an **M**-*fibrewise subspace of* (X,p,B). We sometimes use the notation (X_0,p_0,B_0) instead of $(X_0,p|X_0,B_0)$. By the same way, we define an **M**-*fibrewise pointed subspace.*

(2) For an **M**-fibrewise pointed subspace (X_0,p_0,B_0,s_0) of (X,p,B,s), the pair $((X,p,B,s),(X_0,p_0,B_0,s_0))$ is called by an **M**-*fibrewise pointed pair.* If X_0 is closed in X and B_0 is closed in B, it is called a *closed* **M**-*fibrewise pointed pair.* For an **M**-fibrewise pointed pair $((X,p,B,s),(X_0,p_0,B_0,s_0))$,

if the inclusion map $(u,\gamma):(X_0,p_0,B_0,s_0)\to(X,p,B,s)$ is an **M**-fibrewise pointed cofibration, we call the pair $((X,p,B,s),(X_0,p_0,B_0,s_0))$ an **M**-*fibrewise pointed cofibred pair.*

Proposition 2.4. *Let* $((X,p,B,s),(X_0,p_0,B_0,s_0))$ *and* $((X',p',B',s'),(X_0',p_0',B_0',s_0'))$ *be two closed* **M**-*fibrewise pointed cofibred pairs. Then*

$$((X\times X',p\times p',B\times B',s\times s'),(X_0\times X'\cup X\times X_0',\overline{p},B_0\times B'\cup B\times B_0',\overline{s})$$

is also an **M**-*fibrewise pointed cofibred pair, where*

$$\overline{p}=p\times p'|X_0\times X'\cup X\times X_0',\overline{s}=s\times s'|B_0\times B'\cup B\times B_0'.$$

For cotriad, see [6]. We can also define the **M**-fibrewise push-out of a cotriad as same as the fibrewise push-out in [6] as follows:

Definition 2.5. (cf. [4;p.208~ 9]) For an **M**-fibrewise pointed map $(u,\gamma):(X_0,p_0,B_0,s_0)\to(X_1,p_1,B_1,s_1)$, we can construct the **M**-fibrewise pointed push-out (M,p,B,s) of the cotraids

$$(I\times X_0,\mathrm{id}\times p_0,I\times B_0,\mathrm{id}\times s_0)\xleftarrow{(\sigma_0,\delta_0)}(X_0,p_0,B_0,s_0)\xrightarrow{(u,\gamma)}(X_1,p_1,B_1,s_1)$$

where (σ_0,δ_0) is an **M**-fibrewise embedding to 0-level, as follows : $M=(I\times X_0+X_1)/\sim)$ and $B=(I\times B_0+B_1)/\approx$, where $(0,a)\sim u(a)$ for $a\in X_0$ and $(0,b)\approx\gamma(b)$ for $b\in B_0$, and $p:M\to B$ and $s:B\to M$ are defined, respectively, by

$$p(x)=\begin{cases}[\gamma p_0(a)] & \text{if } x=[u(a)],a\in X_0\\ [t,p_0(a)] & \text{if } x=[t,a],t\neq 0\\ [p_1(x)] & \text{if } x\in X_1-u(X_0),\end{cases}$$

$$s(b)=\begin{cases}[s_1\alpha(d)] & \text{if } b=[\alpha(d)],d\in B_0\\ [t,s_0(d)] & \text{if } x=[t,d],t\neq 0\\ [s_1(b)] & \text{if } b\in B_1-\alpha(B_0).\end{cases}$$

where [*] is an equivalence class. Then it is easily verified that p and s are well-defined and continuous. We call the **M**-fibrewise pointed push-out of the cotriad the **M**-*fibrewise pointed mapping cylinder* of (u,γ), and denote by $M(u,\gamma)$.

Now we shall consider the case in which (X_0, p_0, B_0, s_0) is an $\mathbf{M}$-fibrewise pointed subspace of (X_1, p_1, B_1, s_1) and $(u, \gamma) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$ is the inclusion. We can define an $\mathbf{M}$-fibrewise pointed map (e, ϵ) :
$(M, p, B, s) \to (0 \times X_1 \cup I \times X_0, id \times p_1, 0 \times B_1 \cup I \times B_0, id \times s_1)$ by

$$e(x) = \begin{cases} (0, a) & \text{if } x = [u(a)],\ a \in X_0 \\ (t, a) & \text{if } x = [t, a],\ t \neq 0 \\ (0, x) & \text{if } x \in X_1 - u(X_0), \end{cases}$$

$\epsilon(b)$ is defined by a similar way. Moreover if X_0 is closed in X_1 and B_0 is closed in B_1, the maps e and ϵ are homeomorphisms and we may identify (M, p, B, s) with $(0 \times X_1 \cup I \times X_0, id \times p_1, 0 \times B_1 \cup I \times B_0, id \times s)$.

For each $\mathbf{M}$-fibrewise pointed map $(u, \gamma) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$, we can define an $\mathbf{M}$-fibrewise pointed map $(k, \xi) : (M, p, B, s) \to (I \times X_1, id \times p_1, I \times B_1, id \times s_1)$ by

$$k(x) = \begin{cases} (0, u(a)) & \text{if } x = [u(a)],\ a \in X_0 \\ (t, u(a)) & \text{if } x = [t, a],\ t \neq 0 \\ (0, x) & \text{if } x \in X_1 - u(X_0), \end{cases}$$

$$\xi(b) = \begin{cases} (0, \gamma(d)) & \text{if } b = [\gamma(d)],\ d \in B_0 \\ (t, \gamma(d)) & \text{if } x = [t, d],\ t \neq 0 \\ (0, b) & \text{if } b \in B_1 - \gamma(B_0). \end{cases}$$

Then we can obtain the following proposition by the same method of the proof in [4; Theorem 3.1].

Proposition 2.6. *The $\mathbf{M}$-fibrewise pointed map $(u, \gamma) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$ is an $\mathbf{M}$-fibrewise pointed cofibration if and only if there is an $\mathbf{M}$-fibrewise pointed map $(L, l) : (I \times X_1, id \times p_1, I \times B_1, id \times s_1) \to (M, p, B, s)$ such that $Lk = id_M, l\xi = id_B$, where (M, p, B, s) is the same one in Definition 2.5 and $(k, \xi) : (M, p, B, s) \to (I \times X_1, id \times p_1, I \times B_1, id \times s_1)$ is the same one in the above.*

We now define $\mathbf{M}$-fibrewise pointed collapse, $\mathbf{M}$-fibrewise pointed cone and $\mathbf{M}$-fibrewise pointed nulhomotopic.

Definition 2.7. (1) Let (X, p, B, s)be an $\mathbf{M}$-fibrewise pointed space and (X_0, p_0, B_0, s_0) a closed $\mathbf{M}$-fibrewise pointed subspace. Let $\tilde{X}$ be a set $\cup_{b\in B} X_b/X_{0b}$, where $X_b = p^{-1}(b)$ and $X_{0b} = p_0^{-1}(b)$ for $b \in B$ (or $b \in B_0$). We introduce the set $\tilde{X}$ the quotient topology of X and put $\tilde{B} = B$. If we define maps $\tilde{p} : \tilde{X} \to \tilde{B}$ and $\tilde{s} : \tilde{B} \to \tilde{X}$ indeced by p and s respectively, then $(\tilde{X}, \tilde{p}, \tilde{B}, \tilde{s})$ is an $\mathbf{M}$-fibrewise pointed space. We call $(\tilde{X}, \tilde{p}, \tilde{B}, \tilde{s})$ an $\mathbf{M}$-*fibrewise pointed collapse* of (X, p, B, s) with respect to (X_0, p_0, B_0, s_0) and denoted by

$$(X, p, B, s)/_{\mathbf{M}}(X_0, p_0, B_0, s_0).$$

(For fibrewise collapse, see [6; section 5].)

(2) For an $\mathbf{M}$-fibrewise pointed space (X, p, B, s), we call the $\mathbf{M}$-fibrewise pointed collapse

$$(I \times X, id \times p, I \times B, id \times s)/_{\mathbf{M}}(1 \times X, id \times p|1 \times X, 1 \times B, id \times s|1 \times B)$$

the $\mathbf{M}$-*fibrewise pointed cone* of (X, p, B, s) and denote by $\Gamma(X, p, B, s)$.

(We denote the total space of $\Gamma(X, p, B, s)$ by CX.)

Definition 2.8. Let $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ be an $\mathbf{M}$-fibrewise pointed map. Then we call (ϕ, α) to be $\mathbf{M}$-*fibrewise pointed nulhomotopic* if there is an $\mathbf{M}$-fibrewise pointed map $(c, \alpha_c) : (X, p, B, s) \to (X', p', B', s')$ such that $c = s'\alpha_c p$ and $(\phi, \alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (c, \alpha_c)$.

We now prove the following proposition which is used in next section.

Proposition 2.9. *Let $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ be an $\mathbf{M}$-fibrewise pointed map. Then (ϕ, α) is $\mathbf{M}$-fibrewise pointed nulhomotopic if and only if $(\phi, \alpha) \circ (i, \epsilon)^{-1}$ can be extended to an $\mathbf{M}$-fibrewise pointed map of $\Gamma(X, p, B, s)$ to (X', p', B', s'), where $(i, \epsilon) : (X, p, B, s) \to \Gamma(X, p, B, s)$ is the natural embedding to 0-level of $\Gamma(X, p, B, s)$.*

3. Puppe exact sequence and application

In this section, the main purpose is that we give the theorems of Puppe Exact Sequence (Theorem 3.12) and Hilton's Formula (Theorem 3.21). For this purpose we need some definitions and propositions, which we give in this section without proofs.

Definition 3.1. For an **M**-fibrewise pointed map $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$, we call $(CX \cup_\phi X', \tilde{p}, (I \times B) \cup_\alpha B', \tilde{s})$ the **M**-*fibrewise pointed mapping cone* of (ϕ, α), and denote $\Gamma(\phi, \alpha)$, where CX is the space in Definition 2.7.

In this definition, the maps

$$\tilde{p} : CX \cup_\phi X' \to (I \times B) \cup_\alpha B', \quad \tilde{s} : (I \times B) \cup_\alpha B' \to CX \cup_\phi X'$$

are defined as the **M**-fibrewise pointed maps induced by the following maps, respectively:

$$\overline{p} : CX + X' \to (I \times B) + B', \quad \overline{s} : (I \times B) + B' \to CX + X'$$

$$\begin{array}{ll} \overline{p}([t, x]) = (t, p(x)) & ([t, x] \in CX) \\ \overline{p}(x') = p'(x') & (x' \in X') \\ \overline{s}(t, b) = [t, s(b)] & ((t, b) \in I \times B) \\ \overline{s}(b') = s'(b') & (b' \in B'), \end{array}$$

where $CX \cup_\phi X'$, $(I \times B) \cup_\alpha B'$ are adjunction spaces, respectively, determined by

$$\phi : X = 0 \times X \to X', \quad \alpha : B = 0 \times B \to B'.$$

For fibrewise adjunction space, see [8].

For an **M**-fibrewise pointed map $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ and the **M**-fibrewise mapping cone $\Gamma(\phi, \alpha)$, it is easy to see that there is the natural embedding

$$(\phi', \alpha') : (X', p', B', s') \to \Gamma(\phi, \alpha).$$

For two **M**-fibrewise pointed spaces (X, p, B, s) and (X', p', B', s'), we consider the set of all **M**-fibrewise pointed homotopy classes of **M**-fibrewise pointed maps from (X, p, B, s) to (X', p', B', s'), and denote it by $\pi((X, p, B, s), (X', p', B', s'))$. Then for **M**-fibrewise pointed map $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ and any **M**-fibrewise pointed space (X'', p'', B'', s''), we can define an induced map

$$(\phi, \alpha)^* : \pi((X', p', B', s'), (X'', p'', B'', s'')) \to \pi((X, p, B, s), (X'', p'', B'', s''))$$

of (ϕ,α) by $(\phi,\alpha)^*([\phi',\alpha']) = [\phi'\phi,\alpha'\alpha]$. It is easy to see that this map is well-defined. Now we shall define exactness of a sequence of **M**-fibrewise pointed maps.

Definition 3.2. A sequence of **M**-fibrewise pointed maps

$$(X_1,p_1,B_1,s_1) \xrightarrow{(\phi_1,\alpha_1)} (X_2,p_2,B_2,s_2) \xrightarrow{(\phi_2,\alpha_2)} (X_3,p_3,B_3,s_3) \xrightarrow{(\phi_3,\alpha_3)} \cdots$$

is *exact* if for any **M**-fibrewise pointed space (X',p',B',s') the induced sequence from one in the above

$$\begin{aligned}\pi((X_1,p_1,B_1,s_1),(X',p',B',s')) &\xleftarrow{(\phi_1,\alpha_1)^*} \pi((X_2,p_2,B_2,s_2),(X',p',B',s'))\\ &\xleftarrow{(\phi_2,\alpha_2)^*} \pi((X_3,p_3,B_3,s_3),(X',p',B',s'))\\ &\xleftarrow{(\phi_3,\alpha_3)^*} \cdots\end{aligned}$$

is exact.

Proposition 3.3. *For* **M**-*fibrewise pointed map* $(\phi,\alpha) : (X,p,B,s) \to (X',p',B',s')$ *and* **M**-*fibrewise pointed space* (X'',p'',B'',s''), *the sequence*

$$\begin{aligned}\pi((X,p,B,s),(X'',p'',B'',s'')) &\xleftarrow{(\phi,\alpha)^*} \pi((X',p',B',s'),(X'',p'',B'',s''))\\ &\xleftarrow{(\phi',\alpha')^*} \pi(\Gamma(\phi,\alpha),(X'',p'',B'',s''))\end{aligned}$$

is exact.

Definition 3.4. An **M**-fibrewise pointed space (X,p,B,s) is **M**-*fibrewise pointed contractible* if there is an **M**-fibrewise pointed space (B',p',B',p'^{-1}) (where p' is a homeomorphism) such that

$$(X,p,B,s) \cong^{\mathbf{M}}_{(\mathbf{P})} (B',p',B',p'^{-1}).$$

Proposition 3.5. *An* **M**-*fibrewise pointed space* (X,p,B,s) *is* **M**-*fibrewise pointed contractible if and only if* (id_X, id_B) *is* **M**-*fibrewise pointed nulhomotopic.*

Proposition 3.6. *For an* **M**-*fibrewise pointed space* (X,p,B,s), *the* **M**-*fibrewise pointed cone* $\Gamma(X,p,B,s)$ *is* **M**-*fibrewise pointed contractible.*

From now on, to prove Puppe exact sequence in **MAP**, we shall give some propositions.

Proposition 3.7. *Let $((X,p,B,s),(X_0,p_0,B_0,s_0))$ be a closed* **M**-*fibrewise pointed cofibred pair. Assume that (X_0,p_0,B_0,s_0) is* **M**-*fibrewise pointed contractible. Then the natural projection*

$$(\pi, id_B) : (X,p,B,s) \to (X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0)$$

is an **M**-*fibrewise pointed homotopy equivalence.*

Proposition 3.8. *Let $(\phi,\alpha) : (X,p,B,s) \to (X',p',B',s')$ be an* **M**-*fibrewise pointed cofibration. Then the natural projection*

$$(\pi, id) : \Gamma(\phi,\alpha) \to \Gamma(\phi,\alpha)/_{\mathbf{M}}\Gamma(X,p,B,s) \cong^{\mathbf{M}}_{(\mathbf{P})} (X',p',B',s')/_{\mathbf{M}}(X,p,B,s)$$

is an **M**-*fibrewise pointed homotopy equivalence.*

Proposition 3.9. *For an* **M**-*fibrewise pointed map $(\phi,\alpha) : (X,p,B,s) \to (X',p',B',s')$, the inclusion $(\phi',\alpha') : (X',p',B',s') \to \Gamma(\phi,\alpha)$ is an* **M**-*fibrewise pointed cofibration.*

Definition 3.10. For an **M**-fibrewise pointed space (X,p,B,s), let $\overline{X} = \{0,1\} \times X$, $\overline{p} = id \times p|\{0,1\} \times X$, $\overline{B} = \{0,1\} \times B$ and $\overline{s} = id \times s|\{0,1\} \times B$. Then the **M**-fibrewise pointed collapse

$$(I \times X, id \times p, I \times B, id \times s)/_{\mathbf{M}}(\overline{X},\overline{p},\overline{B},\overline{s})$$

is called to be **M**-*fibrewise pointed suspension*, and denoted by $\Sigma(X,p,B,s)$. (We denote the total space of $\Sigma(X,p,B,s)$ by ΣX, and the projection of $\Sigma(X,p,B,s)$ by Σp.)

Proposition 3.11. *Let $(\phi,\alpha) : (X,p,B,s) \to (X',p',B',s')$ be an* **M**-*fibrewise pointed map, where α is a bijection. Then for the inclusion $(\phi',\alpha') : (X',p',B',s') \to \Gamma(\phi,\alpha)$ $\Gamma(\phi',\alpha')$ is* **M**-*fibrewise pointed equivalent to the* **M**-*fibrewise pointed suspension $\Sigma(X,p,B,s)$.*

In the process in the above, $((\phi')',(\alpha')')$ is transformed into an **M**-fibrewise pointed map

$$(\phi'',\alpha'') : \Gamma(\phi,\alpha) \to \Sigma(X,p,B,s).$$

Repeating this process we find that $\Gamma((\phi')', (\alpha')')$ is **M**-fibrewise pointed equivalent to **M**-fibrewise pointed suspension $\Sigma(X', p', B', s')$, and in the process $(((\phi')')', ((\alpha')')')$ is transformed into the **M**-fibrewise pointed suspension

$$(\Sigma\phi, id \times \alpha) : \Sigma(X, p, B, s) \to \Sigma(X', p', B', s'),$$

where $\Sigma\phi$ is the map from ΣX to $\Sigma X'$. Thus we obtain the main theorem of Puppe exact sequence in **MAP**.

Theorem 3.12. *For an* **M***-fibrewise pointed map* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *where* α *is a bijection, the following sequence is exact.*

$$\begin{aligned} (X, p, B, s) &\xrightarrow{(\phi,\alpha)} (X', p', B', s') \\ &\xrightarrow{(\phi',\alpha')} \Gamma(\phi, \alpha) \\ &\xrightarrow{(\phi'',\alpha'')} \Sigma(X, p, B, s) \\ &\xrightarrow{(\phi''',\alpha''')} \Sigma(X', p', B', s') \\ &\longrightarrow \cdots \end{aligned}$$

From now on, to prove Hilton's formula in **MAP**, we shall give some definitions and propositions.

Definition 3.13. For two **M**-fibrewise pointed spaces (X, p, B, s) and (X', p', B', s'), let

$$X \vee^{\mathbf{M}} X' = \cup_{(b,b')\in B\times B'}(X_b \times s'(b') \cup s(b) \times X'_{b'}),$$

$$p \vee^{\mathbf{M}} p' = p \times p'|X \vee^{\mathbf{M}} X'.$$

The **M**-fibrewise pointed space $(X \vee^{\mathbf{M}} X', p \vee^{\mathbf{M}} p', B \times B', s \times s')$ is called the **M**-*fibrewise pointed coproduct* of (X, p, B, s) and (X', p', B', s'), and denoted by $(X, p, B, s) \vee^{\mathbf{M}} (X', p', B', s')$. The **M**-fibrewise pointed collapse

$$(X, p, B, s) \times (X', p', B', s')/_{\mathbf{M}}(X, p, B, s) \vee^{\mathbf{M}} (X', p', B', s')$$

is called the **M**-*fibrewise smash product* of (X, p, B, s) and (X', p', B', s'), and denoted by $(X, p, B, s) \wedge^{\mathbf{M}} (X', p', B', s')$. Note that $(X, p, B, s) \times (X', p', B', s') = (X \times X', p \times p', B \times B', s \times s')$.

In this paper, we set up the following hypothesis.

HYPOTHESIS: By Definitions 3.10 and 3.13, the base space of $\Sigma\{(X,p,B,s)\vee^{\mathbf{M}}(X',p',B',s')\}$ and $\Sigma\{(X,p,B,s)\wedge^{\mathbf{M}}(X',p',B',s')\}$ is $I\times(B\times B')$, but the base space of $\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s')$ and $\Sigma(X,p,B,s)\wedge^{\mathbf{M}}\Sigma(X',p',B',s')$ is $(I\times B)\times(I\times B')$. So, since $\Sigma\{(X,p,B,s)\vee^{\mathbf{M}}(X',p',B',s')\}$ and $\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s')$, or $\Sigma\{(X,p,B,s)\wedge^{\mathbf{M}}(X',p',B',s')\}$ and $\Sigma(X,p,B,s)\wedge^{\mathbf{M}}\Sigma(X',p',B',s')$ have different base spaces, those **M**-fibrewise pointed spaces are different, respectively. We want to identify those spaces as **M**-fibrewise pointed space, we set up the following hypothesis: In $\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s')$ and $\Sigma(X,p,B,s)\wedge^{\mathbf{M}}\Sigma(X',p',B',s')$, we always restrict the base space $(I\times B)\times(I\times B')$ to $\cup\{(t\times B)\times(t\times B')|t\in I\}$. By this hypothesis, the following equalities always hold.

(3.1)
$$\Sigma\{(X,p,B,s)\vee^{\mathbf{M}}(X',p',B',s')\}=\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s'),$$
(3.2)
$$\Sigma\{(X,p,B,s)\wedge^{\mathbf{M}}(X',p',B',s')\}=\Sigma(X,p,B,s)\wedge^{\mathbf{M}}\Sigma(X',p',B',s').$$

In the rest of this paper, consider automatically these identifications if necessary.

Definition 3.14. For two sequences of **M**-fibrewise pointed maps

$$\mathcal{F}:(X_1,p_1,B_1,s_1)\to(X_2,p_2,B_2,s_2)\to\cdots\to(X_n,p_n,B_n,s_n)\to\cdots$$
$$\mathcal{F}':(X_1',p_1',B_1',s_1')\to(X_2',p_2',B_2',s_2')\to\cdots\to(X_n',p_n',B_n',s_n')\to\cdots$$

if there are **M**-fibrewise pointed homotopy equivalences

$$(\phi_n,\alpha_n):(X_n,p_n,B_n,s_n)\to(X_n',p_n',B_n',s_n')$$

such that all diagrams induced by these maps are commutative, $\mathcal{F}$ and $\mathcal{F}'$ *have the same* **M**-*fibrewise pointed homotopy type.*

The following proposition is obvious from the construction of Puppe exact sequence.

Proposition 3.15. *Let $\alpha : B \to B'$ be a bijection. The* $\mathbf{M}$*-fibrewise pointed homotopy type of Puppe exact sequence (in the sense of* $\mathbf{M}$*-fibrewise pointed) induced from an* $\mathbf{M}$*-fibrewise pointed map* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *is only depend on the* $\mathbf{M}$*-fibrewise pointed homotopy class of* (ϕ, α).

In particular, if (ϕ, α) *is* $\mathbf{M}$*-fibrewise pointed nulhomotopic, the* $\mathbf{M}$*-fibrewise pointed homotopy type of the sequence induced from* (ϕ, α) *has the same* $\mathbf{M}$*-fibrewise pointed homotopy type of the sequence induced from an* $\mathbf{M}$*-fibrewise constant map* (c, α_c)

$$\begin{aligned}(X,p,B,s) &\xrightarrow{(c,\alpha_c)} (X',p',B',s') \\ &\longrightarrow (X',p',B',s') \vee^{\mathbf{M}} \Sigma(X,p,B,s) \\ &\longrightarrow \Sigma(X,p,B,s) \\ &\longrightarrow \cdots\end{aligned}$$

As an application of this proposition, we prove the generalized formula for the M-fibrewise pointed suspension of M-fibrewise pointed product spaces, Let

$$(u, id) : (X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s') \to (X,p,B,s) \times (X',p',B',s')$$

be an M-fibrewise pointed embedding. We denote the M-fibrewise pointed mapping cone $\Gamma(u, id)$ of (u, id) by $(X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s')$. The Puppe sequence (in the sense of M-fibrewise pointed) of (u, id) is as follows:

$$\begin{aligned}(X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s') &\xrightarrow{(u,id)} (X,p,B,s) \times (X',p',B',s') \\ &\xrightarrow{(v,\alpha_v)} (X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s') \\ &\xrightarrow{(w,id)} \Sigma(X,p,B,s) \vee^{\mathbf{M}} \Sigma(X',p',B',s') \\ &\longrightarrow \cdots\end{aligned}$$

Then we obtain the following.

Proposition 3.16. (w, id) *in the above is* $\mathbf{M}$*-fibrewise pointed nulhomotopic.*

From Propositions 3.15 and 3.16, we find that the **M**-fibrewise pointed mapping cone $\Gamma(w, id)$ of (w, id) has the same **M**-fibrewise pointed homotopy type of

$$\Sigma(X,p,B,s) \vee^{\mathbf{M}} \Sigma(X',p',B',s') \vee^{\mathbf{M}} \Sigma\{(X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s')\}.$$

Since the **M**-fibrewise pointed mapping cone $\Gamma(w,\mathrm{id})$ has the same **M**-fibrewise pointed homotopy type of $\Sigma\{(X,p,B,s) \times (X',p',B',s')\}$, we have the following.

Proposition 3.17.

$$\Sigma\{(X,p,B,s) \times (X',p',B',s')\} \cong^{\mathbf{M}}_{(\mathbf{P})}$$
$$\Sigma(X,p,B,s) \vee^{\mathbf{M}} \Sigma(X',p',B',s') \vee^{\mathbf{M}} \Sigma\{(X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s')\}.$$

Definition 3.18. An **M**-fibrewise pointed space (X,p,B,s) is called **M**-*fibrewise well-pointed* if $(s, id_B) : (B, id_B, B, id_B) \to (X,p,B,s)$ is an **M**-fibrewise pointed cofibration and $s(B)$ is closed in X.

Proposition 3.19. *Assume that two* **M**-*fibrewise pointed spaces* (X,p,B,s) *and* (X',p',B',s') *are* **M**-*fibrewise well-pointed. Then the natural projection*

$$\begin{aligned}
&(X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s') \\
&\to (X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s')/_{\mathbf{M}}\Gamma\{(X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s')\} \\
&= (X,p,B,s) \times (X',p',B',s')/_{\mathbf{M}}(X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s') \\
&= (X,p,B,s) \wedge^{\mathbf{M}} (X',p',B',s')
\end{aligned}$$

is an **M**-*fibrewise pointed homotopy equivalence.*

We have the following from Propositions 3.17 and 3.19.

Corollary 3.20. *Assume that two* **M**-*fibrewise pointed spaces* (X,p,B,s) *and* (X',p',B',s') *are* **M**-*fibrewise well-pointed. Then the next formula holds.*

$$\Sigma\{(X,p,B,s) \times (X',p',B',s')\} \cong^{\mathbf{M}}_{(\mathbf{P})}$$
$$\Sigma(X,p,B,s) \vee^{\mathbf{M}} \Sigma(X',p',B',s') \vee^{\mathbf{M}} \Sigma\{(X,p,B,s) \wedge^{\mathbf{M}} (X',p',B',s')\}.$$

By repeatedly using this formula and (3.1),(3.2), we can obtain the following formula.

Theorem 3.21. *Assume that* $\mathbf{M}$*-fibrewise pointed spaces* (X_i, p_i, B_i, s_i) $(i = 1, \cdots n)$ *are* $\mathbf{M}$*-fibrewise well-pointed. Then the next formula holds.*

$$\sum \left\{ \prod_{i=1}^{n} (X_i, p_i, B_i, s_i) \right\} \cong^{\mathbf{M}}_{(\mathbf{P})} \bigvee_{N}{}^{\mathbf{M}} \sum \Big(\bigwedge_{i \in N}{}^{\mathbf{M}} (X_i, p_i, B_i, s_i) \Big)$$

where N *runs through all nonempty subsets of* $\{1, \cdots, n\}$.

References

[1] D.Buhagiar : The category $\mathcal{MAP}$, Mem. Fac. Sci.& Eng., Shimane Univ., 34(2001), 1-19.

[2] M.Crabb and I.M.James : Fibrewise Homotopy Theory, Springer Monographs in Mathematics, 1998

[3] R.Engelking : General Topology, Heldermann, Berlin, rev. ed., 1989

[4] T.Hotta and T.Miwa : A new approach to fibrewise fibrations and cofibrations, Topology and its Appl., 122(2002), 205-222.

[5] I.M.James: General Topology and Homotopy Theory, Springer-Verlag, 1984

[6] I.M.James : Fibrewise Topology, Cambridge Univ. Press, Cambridge, 1989

[7] Y.Konami and T.Miwa: Puppe exact sequence and its application in the fibrewise category **MAP**, submitted for publication.

[8] T.Miwa : On fibrewise retraction and extension, Houston J. Math., 26(2000), 811-831.

[9] B.A.Pasynkov : On extensions to mappings of certain notions and assertions concerning spaces, Mappings and Functions (Moscow), Izdat. MGU, Moscow, 1984, (in Russian), pp.72-102.

[10] B.A.Pasynkov : Elements of the general topology of continuous maps, On Compactness and Completeness Properties of Topological Spaces, " FAN" Acad. of Science of the Uzbek. Rep., Tashkent, 1994, (in Russian), pp.50-120.

[11] D.Puppe : Homotopiemengen und ihre induzierten Abbildungen. I, Math. Z., 69(1958), 299-344.

[12] E.H.Spanier : Algebraic Topology, McGraw-Hill, 1966

Takuo Miwa, Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: `miwa@math.shimane-u.ac.jp`