On dense subsets of the boundary of a Coxeter system

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The purpose of this note is to introduce main results of my recent paper [10] about dense subsets of the boundary of a Coxeter system.

A Coxeter group is a group $W$ having a presentation

$$(S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S),$$

where $S$ is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

1. $m(s,t) = m(t,s)$ for each $s, t \in S$,
2. $m(s,s) = 1$ for each $s \in S$, and
3. $m(s,t) \geq 2$ for each $s, t \in S$ such that $s \neq t$.

The pair $(W, S)$ is called a Coxeter system. Let $(W, S)$ be a Coxeter system. For a subset $T \subset S$, $W_T$ is defined as the subgroup of $W$ generated by $T$, and called a parabolic subgroup. If $T$ is the empty set, then $W_T$ is the trivial group. A subset $T \subset S$ is called a spherical subset of $S$, if the parabolic subgroup $W_T$ is finite. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$, where $\ell(w)$ is the minimum length of word in $S$ which represents $w$. For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

Let $(W, S)$ be a Coxeter system and let $S'$ be the family of spherical subsets of $S$. We denote $WS'$ as the set of all cosets of the form $wW_T$, with $w \in W$ and $T \in S'$. The sets $S'$ and $WS'$ are partially ordered by inclusion. Contractible simplicial complexes $K(W, S)$ and $\Sigma(W, S)$ are
defined as the geometric realizations of the partially ordered sets $S^f$ and $WS^f$, respectively ([7, §3], [5]). The natural embedding $S^f \to WS^f$ defined by $T \mapsto WT$ induces an embedding $K(W, S) \to \Sigma(W, S)$ which we regard as an inclusion. The group $W$ acts on $\Sigma(W, S)$ via simplicial automorphism. Then $\Sigma(W, S) = WK(W, S)$ ([5], [7]). For each $w \in W$, $wK(W, S)$ is called a chamber of $\Sigma(W, S)$. If $W$ is infinite, then $\Sigma(W, S)$ is noncompact. In [12], G. Moussong proved that a natural metric on $\Sigma(W, S)$ satisfies the CAT(0) condition. Hence, if $W$ is infinite, $\Sigma(W, S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W, S)$ ([6, §4], [8]). This boundary $\partial\Sigma(W, S)$ is called the boundary of $(W, S)$. We note that the natural action of $W$ on $\Sigma(W, S)$ is properly discontinuous and cocompact ([5], [6]), and this action induces an action of $W$ on $\partial\Sigma(W, S)$.

A subset $A$ of a space $X$ is said to be dense in $X$, if $\overline{A} = X$. A subset $A$ of a metric space $X$ is said to be quasi-dense, if there exists $N > 0$ such that each point of $X$ is $N$-close to some point of $A$.

Let $(W, S)$ be a Coxeter system. Then $W$ has the word metric $d_w$ defined by $d_w(w, w') = \ell(w^{-1}w')$ for each $w, w' \in W$.

Here we obtained the following theorems in [10].

**Theorem 1.** Let $(W, S)$ be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in $W$ with respect to the word metric and $m(s_0, t_0) = \infty$ for some $s_0, t_0 \in S$. Then there exists $\alpha \in \partial\Sigma(W, S)$ such that the orbit $W\alpha$ is dense in $\partial\Sigma(W, S)$.

Suppose that a group $\Gamma$ acts properly and cocompactly by isometries on a CAT(0) space $X$. Every element $\gamma \in \Gamma$ such that the order $o(\gamma) = \infty$ is a hyperbolic transformation of $X$, i.e., there exists a geodesic axis $c : \mathbb{R} \to X$ and a real number $a > 0$ such that $\gamma \cdot c(t) = c(t + a)$ for each $t \in \mathbb{R}$ ([3]). Then, for all $x \in X$, the sequence $\{\gamma^i x\}$ converges to $c(\infty)$ in $X \cup \partial X$. We denote $\gamma^\infty = c(\infty)$.

**Theorem 2.** Let $(W, S)$ be a Coxeter system. If the set

$$\bigcup \{W^{\{s\}} | s \in S \text{ such that } m(s, t) = \infty \text{ for some } t \in S\}$$
is quasi-dense in \( W \), then \( \{ w^\infty \mid w \in W \text{ such that } o(w) = \infty \} \) is dense in \( \partial \Sigma(W, S) \).

**Remark.** For a negatively curved group \( G \) and the boundary \( \partial G \) of \( G \),

1. we can show that \( G\alpha \) is dense in \( \partial G \) for each \( \alpha \in \partial G \) by an easy argument, and
2. it is known that \( \{ g^\infty \mid g \in G \text{ such that } o(g) = \infty \} \) is dense in \( \partial G \) ([2]).

**Example.** Let \( S = \{ s, t, u \} \) and let
\[
W = \langle S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.
\]
Then \( (W, S) \) is a Coxeter system and \( W^{\{s\}} \) is quasi-dense in \( W \). On the other hand, for any \( \alpha \in \partial \Sigma(W, S) \), \( W\alpha \) is a finite-points set and not dense in \( \partial \Sigma(W, S) \) which is a circle. Thus we can not omit the assumption "\( m(s_0, t_0) = \infty \)" in Theorem 1.

We showed the following lemma in [10].

**Lemma 3.** Let \( (W, S) \) be a Coxeter system. Suppose that there exist a maximal spherical subset \( T \) of \( S \) and \( s_0 \in S \) such that \( m(s_0, t) \geq 3 \) for each \( t \in T \) and \( m(s_0, t_0) = \infty \) for some \( t_0 \in T \). Then \( W^{\{s_0\}} \) is quasi-dense in \( W \).

As an application of Theorems 1 and 2, we can obtain the following corollary from Lemma 3.

**Corollary 4.** Let \( (W, S) \) be a Coxeter system. Suppose that there exist a maximal spherical subset \( T \) of \( S \) and an element \( s_0 \in S \) such that \( m(s_0, t) \geq 3 \) for each \( t \in T \) and \( m(s_0, t_0) = \infty \) for some \( t_0 \in T \). Then

1. \( W\alpha \) is dense in \( \partial \Sigma(W, S) \) for some \( \alpha \in \partial \Sigma(W, S) \), and
2. \( \{ w^\infty \mid w \in W \text{ such that } o(w) = \infty \} \) is dense in \( \partial \Sigma(W, S) \).

**Example.** The Coxeter system defined by the diagram in Figure 1 is not hyperbolic in Gromov sense, since it contains a copy of \( \mathbb{Z}^2 \), and it satisfies the condition of Corollary 4.
REFERENCES


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