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Massera Criterion for Abstract Functional Differential Equations with Advance and Delay

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abstract

The paper is concerned with the Massera criterion for periodic mild solutions of the equations of the form $\dot{u}(t) = Au(t) + \int_{-\infty}^{\infty} dB(\eta)u(t+\eta) + f(t)$, where A generates an analytic semigroup on a Banach space $\mathbb{X}$, B is an $L(\mathbb{X})$ valued function of bounded variation and f is a continuous 1-periodic function. The obtained results extend recent ones on the subject.

1. INTRODUCTION

This paper is concerned with the existence of 1-periodic mild solutions to the equation of the form

$$\frac{du(t)}{dt} = Au(t) + \int_{-\infty}^{\infty} dB(\eta)u(t+\eta) + f(t), \quad t \in \mathbb{R}, \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on a given Banach space $\mathbb{X}$, $B : \mathbb{R} \to L(\mathbb{X})$ is of bounded variation, and f is an $\mathbb{X}$-valued continuous 1-periodic function with Fourier coefficients

$$\tilde{f}_k = \int_0^1 e^{-2ik\pi t} f(t)dt, \quad k = 0, \pm 1, \pm 2, \cdots .$$

Moreover by setting T_B as

$$T_B(\theta) := \lim_{\sigma \to -\infty} Var(B, [\sigma, \eta]),$$

we impose the following conditions for B:

$$\exists \varepsilon_1 \in (0,1) : \int_0^\infty dT_B(\eta) e^{\varepsilon_1 \eta} < \infty \tag{2}$$

and

$$\exists \varepsilon_2 \in (0,1) : \int_{-\infty}^0 dT_B(\eta) e^{-\varepsilon_2 \eta} < \infty. \tag{3}$$

It has been known for decades (see [11]) that a linear ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + f(t),$$

where $A(t)$ and $f(t)$ are continuous and periodic with the same period τ, has a τ-periodic solution if and only if it has a solution bounded on the positive half line. As shown in [5], for a larger class of equations, the assumption on the existence of a solution bounded on the positive half line yields a solution bounded on the whole line.

Recently, there have been many works devoted on the extension of the above Massera criterion to various classes of differential equations. For example, in [2] the Massera criterion has been proved for functional differential equations. It has been shown that this result holds for equations with delay and advance (see [9, 10]), for abstract functional differential equations in Banach spaces [6, 12, 15] with infinite delay, and for almost periodic solutions [13].

In this paper [14] we first extend a result in [4, 7] to equations of the form (1) which characterizes the existence of a periodic mild solutions in terms of the solvability of a finitely many algebraic equations in the phase space $\mathbb{X}$ (see Theorem 3.1 below). Our method of study in this paper is to find such conditions that the characterization of the existence of periodic solutions in Theorem 3.1 holds. The main result of this paper is stated in Theorem 3.6. The novelty of our result is that by using a new method of study, we do not need to impose any conditions on the inter-relation between Fourier exponents of f and the spectrum of the corresponding homogeneous equation. And thus, our result extends and complements recent results on the subject (see e.g. [2, 6, 12, 13, 15]). Finally, we given an example showing that our result can apply to an abstract functional differential equation with advance which has not been covered by other works so far.

2. Preliminaries

2.1. **Notation and Definitions.** In this paper we use the following notations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ stand for the set of natural, integer, real, complex numbers, respectively; $\mathbb{X}$ will denote a given complex Banach space. If T is a linear operator on $\mathbb{X}$, then $D(T)$ stands for its domain. Given two Banach spaces $\mathbb{Y}, \mathbb{Z}$ by $L(\mathbb{Y}, \mathbb{Z})$ we will denote the space of all bounded linear operators from $\mathbb{Y}$ to $\mathbb{Z}$ and $L(\mathbb{X}, \mathbb{X}) := L(\mathbb{X})$. As usual, $\sigma(T), \rho(T), R(\lambda, T)$ are the notations of the spectrum, resolvent set and resolvent of the operator T.

2.2. **Functional differential equations.** For the sake of simplicity we denote

$$[\mathcal{B}u](t) := \int_{-\infty}^{\infty} dB(\eta)u(t+\eta), \quad t \in \mathbb{R}. \tag{4}$$

Definition 2.1. Let A be a closed operator on $\mathbb{X}$. An $\mathbb{X}$-valued continuous function u on $\mathbb{R}$ is said to be a *mild solution* of Eq.(1) on $\mathbb{R}$ if for every s, $\int_s^t u(\xi)d\xi \in D(A)$ and

$$u(t) = u(s) + A\int_s^t u(\xi)d\xi + \int_s^t [[\mathcal{B}u](\xi) + f(\xi)]d\xi, \quad \forall t \geq s.$$

If A is the generator of a C_0-semigroup, by [6, Lemma 2.11] this condition is equivalent to the condition that, for every $s \in \mathbb{R}$,

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\xi)[[\mathcal{B}u](\xi) + f(\xi)]d\xi \quad \forall t \geq s.$$

Consider the homogeneous equation of Eq.(1)

$$\frac{du(t)}{dt} = Au(t) + [\mathcal{B}u](t). \tag{5}$$

3. Massera Theorem for Equations with Advance and Delay

We begin this section by presenting a necessary and sufficient condition for the existence of 1-periodic solutions to the inhomogeneous equation (1) which was essentially established in our previous work [14].

Theorem 3.1. *Let A be the generator of an analytic semigroup. Then, Eq.(1) has a 1-periodic mild solution if and only if for every $k \in \mathbb{Z}$, the equation*

$$\Delta(2ik\pi)x = \tilde{f}_k \tag{6}$$

has solutions $x \in \mathbb{X}$, where $\Delta(\lambda)x := (\lambda I - A - \int_{-\infty}^{\infty} dB(\eta)e^{i\lambda\eta})x, \ x \in D(A)$.

If x_k is a solution of Eq.(6) for $k \in \mathbb{Z}$,then $\sum_{k=-\infty}^{\infty} x_k e^{2ik\pi t}$ is the Fourier series of a 1-periodic mild solution of Eq.(1).

We will use the above result to prove the Massera Theorem for Eq. (1). To this end, we first estimate the spectrum of a bounded mild solution of Eq. (1).

Lemma 3.2. *Assume that (2) and (3) hold. Then the set $\rho(\Delta)$, the set of λ such that $\Delta^{-1}(\lambda) \in L(\mathbb{X})$ exists, is open in $H := \{\lambda \in \mathbb{C} : -\varepsilon_2 < \Re\lambda < \varepsilon_1\}$ and $\Delta^{-1}(\lambda)$ is analytic in $\rho(\Delta) \cap H$.*

By Lemma3.2 and [6, Lemma 2.21], we have the following Lemma.

Lemma 3.3. *Let u be a bounded mild solution of Eq.(1). Then, the following estimate holds:*

$$sp(u) \subset \sigma_i(\Delta) \cup sp(f), \tag{7}$$

where, $sp(u) := \{\xi \in \mathbb{R} : \forall \epsilon > 0\ \exists f \in L^1(\mathbb{R}), supp\mathcal{F}f \subset (\xi - \epsilon, \xi + \epsilon), f * u \neq 0\}$ and $\sigma_i(\Delta) := \{\xi \in \mathbb{R} : i\xi \notin \rho(\Delta)\}$

Definition 3.4. A complex Banach space $\mathbb{X}$ is said to *contain* c_0 if there is a closed linear subspace of $\mathbb{X}$ which is isomorphic to c_0.

From [1, Theorem 4.8.7], if $sp(u)$ is discrete, then u is almost periodic; or from [8, Theorem 4, p.92], a uniformly continuous function u is almost periodic provided one of the following conditions holds:

i) $sp(u)$ is countable and $\mathbb{X}$ does not contain any subspaces isomorphic to c_0,
ii) $sp(u)$ is countable and the range of $u(t)$ is relatively weekly compact in $\mathbb{X}$.

Combining the above results with Lemma 3.3, we have the following Corollary.

Corollary 3.5. Let u be a uniformly continuous and bounded mild solution on $\mathbb{R}$ of Eq.(1). Then, u is almost periodic provided one of the following conditions holds:

i) $\sigma_i(\Delta)$ is discrete,
ii) $\sigma_i(\Delta)$ is countable and $\mathbb{X}$ does not contain any subspaces isomorphic to c_0,
iii) $\sigma_i(\Delta)$ is countable and the range of $u(t)$ is relatively weekly compact in $\mathbb{X}$.

Consequently, using Theorem3.1 and Corollary3.5, we get the Massera Theorem.

Theorem 3.6. *Let Eq.(1) have a bounded and uniformly continuous mild solution on $\mathbb{R}$ and let one of the conditions listed in Corollary 3.5 holds. Then there exists a 1-periodic mild solution to Eq.(1).*

4. Examples

4.1. **Ordinary Functional Differential Equations.** In this case we assume that $A = 0$ and $\mathbb{X} = \mathbb{C}^n$. Obviously, every bounded solution on $\mathbb{R}$ has relatively compact range and the spectrum of equation coincides with the set of zeros of function

$$\det \Delta(\lambda) = 0,$$

which is countable thanks to the analyticity of the function $\det \Delta(\lambda)$. Thus, Condition (iii) in Corollary 3.5 is satisfied. By Theorem 3.6, Eq. (8) has a 1-periodic mild solution if and only if it has a bounded, uniformly continuous mild solution on the real line. And we get the Massera Theorem for equations with advance and delay (see also [9]).

4.2. **Abstract Functional Differential Equations.**

Example 4.1. Let $\mathbb{X} = L^2[0, \pi]$. Consider the abstract functional differential equation with advance

$$\dot{x}(t) = Ax(t) + bx(t+1) + f(t), \quad t \in [0, \pi], \tag{8}$$

where $b \in \mathbb{R}$ and A is defined as $Ax = \ddot{x} + x$ for $x \in \mathbb{X}$ such that x is continuously differentiable, the derivative $\dot{x}$ is absolutely continuous, $\ddot{x} \in \mathbb{X}$, and that $x(0) = x(\pi) = 0$. Then

$$\sigma(A) = \{1 - n^2 : n = 1, 2, \cdots\},$$

and A generates a compact and analytic semigroup $T(t)$ on $\mathbb{X}$. The characteristic operator $\Delta(\lambda)$ becomes

$$\Delta(\lambda)x = (\lambda I - A - be^{\lambda})x, \quad x \in D(A).$$

Therefore the set $\sigma_i(\Delta)$ is determined from the set of imaginary solutions of the equations

$$\lambda + be^{\lambda} = 1 - n^2, \quad n = 1, 2, \cdots . \tag{9}$$

In (9) if we let $\lambda = i\tau, \tau \in \mathbb{R}$, then

$$1 - n^2 = i\tau + be^{i\tau} = i\tau + b(\cos\tau + i \sin\tau).$$

Now we consider the equation

$$b\cos\tau = 1 - n^2, \quad \tau + b\sin\tau = 0. \tag{10}$$

A simple computation shows that

$$\tau = \pm\sqrt{b^2 - (n^2 - 1)^2}. \tag{11}$$

Hence τ has at most two points. Therefore, $\sigma_i(\Delta)$ is a finite set. Thus, Condition (ii) in Corollary 3.5 is satisfied. By Theorem 3.6, Eq.(8) has a 1-periodic mild solution if and only if it has a bounded, uniformly continuous mild solution on the real line. We also note that in [3] abstract functional differential equations with delay are treated. So, our example complements the one considered in [3, Examples].

References

1. W. Arendt, C.J.K Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Birkhäser, Basel-Boston-Berlin, 2001.
2. S. N. Chow, J. K. Hale, Strongly limit-compact maps. *Funkc. Ekvac.* **17** (1974), 31-38.
3. T. Furumochi, T. Naito, Nguyen Van Minh, Boundedness and almost periodicity of solutions of partial functional differential equations. *J. Differential Equations* **180** (2002), 125-152.
4. L. Hatvani, T. Krisztin On the existence of periodic solutions for linear inhomogeneous and quasilinear functional differential equations. *J. Differential Equations* **97** (1992), 1–15.
5. Y. Hino, S. Murakami , Periodic solutions of a linear Volterra system. *Differential equations (Xanthi, 1987)*, 319–326, Lecture Notes in Pure and Appl. Math., **118**, Dekker, New York, 1987.
6. Y. Hino, T. Naito, Nguyen Van Minh, J. S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Taylor and Francis, London-New York, 2002.
7. C. Langenhop, Periodic and almost periodic solutions of Volterra integral differential equations with infinite memory. *J. Differential Equations* **58** (1985), 391–403.
8. B. M. Levitan, V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press 1982.
9. Y. Li, Z. Lin, Z. Li, A Massera type criterion for linear functional differential equations with advanced and delay. *J. Math. Anal. Appl.*, **200** (1996), 715-725.
10. Y. Li, F. Cong, Z. Lin, W. Liu, Periodic solutions for evolution equations. *Nonlinear Anal.* **36** (1999), 275–293.
11. J. L. Massera, The existence of periodic solutions of systems of differential equations. *Duke Math. J.* **17** (1952), 457–475.
12. T. Naito, Nguyen Van Minh, R. Miyazaki, J. S. Shin, A decomposition theorem for bounded solutions and the existence of periodic solutions to periodic differential equations. *J. Differential Equations* **160** (2000), 263-282.
13. T. Naito, Nguyen Van Minh, J. S. Shin, New spectral criteria for almost periodic solutions of evolution equations. *Studia Mathematica* **145** (2001), 97-111.
14. T. Nishikawa, Nguyen Van Minh, T. Naito, On the asymptotic periodic solutions of abstract functional differential equations. to appear in Funkcial. Ekvac. .
15. J. S. Shin, T. Naito, Semi-Fredholm operators and periodic solutions for linear functional differential equations. *J. Differential Equations* **153** (1999), 407-441.