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Analytic Solutions of Nonlinear Difference Equation

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1 Introduction

We consider the following second order nonlinear difference equation,

$$u(t+2)=f(u(t),u(t+1)), \tag{1.1}$$

where f is a holomorphic function for $u(t)$, $u(t+1)$. Put u^* as a equilibrium point of (1.1). And we suppose that (1.1) has a equilibrium point $u^*=0$ and $f(x,y)=-\beta x-\alpha y+g(x,y)$, ($\alpha,\beta$ are constants, $\beta\neq 0$), where g is higher order terms for x, y such that $g(x,y)=\sum_{i,j\geqq 0,i+j\geqq 2}b_{i,j}x^iy^j$. Here we consider analytic solutions such that $u(t)\to 0$ when $t\to+\infty$ or $t\to-\infty$.

The Characteristic equation of (1.1) is

$$D(\lambda)=\lambda^2+\alpha\lambda+\beta=0. \tag{1.2}$$

Let λ_1, λ_2 be roots of the characteristic equation and $|\lambda_1|\leqq|\lambda_2|$. Then we consider following two case i) $|\lambda_1|<1$, and ii) $|\lambda_2|>1$. Of course, some characteristic equations have properties both i) and ii).

Here we consider solutions such that i) $u(t)\to 0$, as $\mathbb{R}[t]\to+\infty$, and ii) $u(t)\to 0$, as $\mathbb{R}[t]\to-\infty$.

2 Existence of an analytic solution

If (1.1) is a real Model, then the "t" of equation (1.1) represent "time" and t is of course a real variable. But in this section we consider t to be a complex variable, and we will prove existence of an analytic solution of (1.1) which converge to 0 with methods of complex analysis.

When we consider a real Model, after we have solutions of (1.1), we take t such as $t\in\mathbb{R}$. Then we can have solutions which are real values.

2.1 A formal solution

In case i) w e put $\lambda = \lambda_1$, in case ii) w e put $\lambda = \lambda_2$. Then we can define a firmal solution such as

$$u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}, \tag{2.1}$$

in both cases. Where α_1 : arbirary, $\alpha_k \cdot D(\lambda^k) = C_k(\alpha_1, \cdots, \alpha_{k-1})$, $(k = 2, \cdots)$, and $C_k(\alpha_1, \cdots, \alpha_{k-1})$ are polynomials for $\alpha_1, \cdots, \alpha_{k-1}$ with coefficients $b_{i,j}\lambda^l$, $0 \leqq i \leqq k$, $0 \leqq j \leqq k$, $0 \leqq l \leqq k$, $2 \leqq i+j \leqq k$. Here we suppose that $\alpha_1 \neq 0$.

2.2 Map T and its Fixed Point

Here we put $u(t) = s, u(t+1) = w, u(t+2) = z$, and $H(s, w, z) = -z + f(s, w)$. Then the equation (1.1) can be written such as $H(u(t), u(t+1), u(t+2)) = 0$.

$H(s, w, z)$ is holomorphic in a neighborhood of $(0, 0, 0)$, and we have $H(0, 0, 0) = 0$, easily. Furthermore we have

$$\frac{\partial H}{\partial s}(0,0,0) = \left.\frac{\partial f}{\partial s}\right|_{s=w=0} = -\beta \neq 0.$$

So we have a holomorphic function ϕ such that $s = \phi(w, z)$ for $|w|, |z| \leqq \rho$, (for $\exists \rho > 0$). Furthermore we have a constant K such that $|s| = |\phi(w, z)| \leqq K(|w|+|z|)$ for $|w|, |z| \leqq \rho$.

Let N be a positive integer. Put the partial sum of formal solution as $P_N(t) = \sum_{n=1}^{N} \alpha_n \lambda^{nt}$, and put $p_N(t) = u(t) - P_N(t)$. Here we rewrite $p(t) = p_N(t)$.

Moreover we define following sets,

$$S(\eta) = \{t \in \mathbb{C} : |\lambda^t| \leqq \eta\}$$
$$J(A, \eta) = \{p : p(t) \text{ is holomorphic and } |p(t)| \leqq A|\lambda^t|^{N+1} \text{for } t \in S(\eta)\}.$$

in which $A > 0$ and η, $0 < \eta < 1$, are constants to be determined later.

2.2.1 The case i) $|\lambda| < 1$

In this case, our aim is to prove the existence of $u(t)$ when $\mathbb{R}[t] \to \infty$, such that

$$u(t) = \phi(u(t+1), u(t+2)).$$

If we have the analytic solution $u(t)$, then it is the solution of (1.1), and have a solution p of following equation,

$$p(t) = \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t).$$

Conversely if $p(t)$ which satisfies above equation would exist, then we have a solution $u(t)$ of (1.1) which has the expansion (2.1) by $u(t) = p(t) + P_N(t)$.

For $p(t) \in J(A, \eta)$, put

$$T_1[p](t) = \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t).$$

Lemma 1. *We have a fixed point $p(t) = p_N(t) \in J(A, \eta)$ of T_1, which depends on N.*

Proof. Since ϕ is holomorphic on $|w| \leqq \rho$, $|z| \leqq \rho$ we have

$$\left|\frac{\partial \phi}{\partial w}\right|, \left|\frac{\partial \phi}{\partial z}\right| \leqq \frac{8K}{\rho} \qquad \text{for} \quad |w|, |z| \leqq \frac{\rho}{2}.$$

Next we take A, and take η sufficiently small such that $A\eta^{N+1} < \frac{\rho}{4}$. Then for sufficiently large t, we have $|p(t)| \leqq A|\lambda^t|^{N+1} < \frac{\rho}{4}$, $|p(t+1)| \leqq A|\lambda|^{N+1}|\lambda^t|^{N+1} < \frac{\rho}{4}$, $|p(t+2)| \leqq A|\lambda|^{2(N+1)}|\lambda^t|^{N+1} < \frac{\rho}{4}$. Furthermore we can obtain $|w|$, $|z| \leqq \frac{\rho}{2}$. So we have

$$|T_1[p](t)| \leqq \left(\frac{16K}{\rho} A|\lambda|^{N+1} + K_2\right)|\lambda^t|^{N+1}. \tag{2.2}$$

where K_2 is constant, depends on N. Hence we have If we suppose N is so large that $\frac{16K}{\rho}|\lambda|^{N+1} < \frac{1}{4}$, furthermore we take A so large that $A > \frac{4}{3}K_2$, then

$$|T_1[p](t)| < A|\lambda^t|^{N+1}.$$

So we obtain that T_1 maps $J(A, \eta)$ into itself, The map T_1 is continuous if $J(A, \eta)$ is endowed with topology of uniform convergence on compact set in $S(\eta)$, and $J(A, \eta)$ is convex, and is relatively compact set.

Thus by Schauder's fixed point theorem in [2], we obtain the existence of a fixed point $p(t) = p_N(t) \in J(A, \eta)$ of T_1.□

2.2.2 The case ii) $|\lambda| > 1$

In this case, our aim is to prove the existence of $u(t)$ when $\mathbb{R}[t] \to \infty$, such that $u(t) = f(u(t-2), u(t-1))$.

If we have an the analytic solution $u(t)$, then it is the solution of (1.1). And we have a solution p of following equation,

$$p(t) = f(p(t-2) + P_N(t-2), p(t-1) + P_N(t-1)) - P_N(t).$$

Conversely if $p(t)$ which satisfies above equation would exist, then we have a solution $u(t)$ of (1.1) which has the expansion (2.1) by $u(t) = p(t) + P_N(t)$.

For $p(t) \in J(A, \eta)$, put

$$T_2[p](t) = f(p(t-2) + P_N(t-2), p(t-1) + P_N(t-1)) - P_N(t).$$

Lemma 2. *We have a fixed point $p(t) = p_N(t) \in J(A,\eta)$ of T_2, which depends on N.*

Proof. Here we put $s = u(t-2)$, $w = u(t-1)$, $z = u(t)$. Since f is holomorphic on $|s| \leqq \rho$, $|w| \leqq \rho$ we have Hence we have

$$\left|\frac{\partial f}{\partial s}\right|, \left|\frac{\partial f}{\partial w}\right| \leqq \frac{8K_1}{\rho} \quad \text{for} \quad |s|, |w| \leqq \frac{\rho}{2},$$

where K_1 is a constant. Next we take A, and take η sufficiently small such that $A\eta^{N+1} < \frac{\rho}{4}$.

Then for sufficiently large $-t$, we have

$$|T_2[p](t)| \leqq \left(\frac{16K_1}{\rho} A|\lambda|^{-(N+1)} + K_3\right)|\lambda^t|^{N+1}.$$

with a constant K_3 which depends on N.

If we suppose N is so large that $\frac{16K_1}{\rho}|\lambda|^{N+1} < \frac{1}{4}$, and we take A so large that $A > \frac{4}{3}K_3$, then

$$|T_2[p](t)| < A|\lambda^t|^{N+1}.$$

So we obtain that T_2 maps $J(A,\eta)$ into itself, T_2 maps $J(A,\eta)$ into itself, The map T_2 is continuous if $J(A,\eta)$ is endowed with topology of uniform convergence on compact set in $S(\eta)$, and $J(A,\eta)$ is convex, and is relatively compact set.

Thus by Schauder's fixed point theorem in [3], we We obtain the existence of a fixed point $p(t) = p_N(t) \in J(A,\eta)$ of T_2.□

2.3 Uniqueness of the Fixed Point

We can have following two lemmas.

Lemma 3. *The fixed point $p_N(t) \in J(A,\eta)$ of T_1 is unique for each N.*

Lemma 4. *The fixed point $p_N(t) \in J(A,\eta)$ of T_2 is unique for each N.*

2.4 Proof that the solution $u(t) = p_N(t) + P_N(t)$ is independent of N

Finally we will show that the solution $u(t)$, given by $u(t) = p_N(t) + P_N(t)$ does not depend on N. Then we obtain that (2.1) gives an exact solution of (1.1).

Lemma 5. *The solution $u_N(t) = p_N(t) + P_N(t)$ of (1.1) is independent of N.*

2.5 the analytic solution $u(t)$ of (1.1)

From **lemma 1-lemma 5**, we have proved that a solution $u(t)$ is defined and holormorphic in $S(\eta)$ for a $\eta > 0$, which has the expansion $u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}$. Hence we have the following Theorem 6.

Theorem 6. *Let λ_1, λ_2 be roots of the characteristic equation of (1.1) and $|\lambda_1| \leqq |\lambda_2|$. If $|\lambda_1| < 1$ or $|\lambda_2| > 1$, then we have the holomorphic solution $u(t)$ of (1.1) in $S(\eta)$ for a $\eta(>0)$, which has the expansion $u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}$.*

However, we cannot assume the condition $\frac{\partial H}{\partial s}(s, w, z) \neq 0$, for all So in case i), if $\frac{\partial H}{\partial s}(s, w, z) = 0$, for some, w, z, then the (w, z) are branch points. The solution $u(t)$ can be continued analytically by making use of the relation

$$u(t-2) = \phi(u(t-1), u(t)),$$

keeping out of branch points, up to $\mathbb{R}[t] \geqq 0$. The solution obtained may be multivalued.

3 Analytic General Solutions

Theorem 7. *Suppose that $u(\tau)$ is the solution of (1.1) which we have in Theorem 6, and has the expansion $u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}$. Further suppose that $\chi(t)$ is an analytic solution of (1.1) such that $\chi(t+n) \to 0$ as when $\lambda < 1$, $n \to +\infty$, and as when $\lambda > 1$, $n \to -\infty$ uniformly on any compact set.*

Then there is a periodic entire function $\pi(t), (\pi(t+1) = \pi(t))$, such that

$$\chi(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda} + t)} = \sum_{n=1}^{\infty} \alpha_n \pi(t)^n \lambda^{nt},$$

where $\pi(t)$ is an arbitrarily periodic function whose period is one.

Conversely, if we put

$$\chi(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda} + t)} = \sum_{n=1}^{\infty} \alpha_n \pi(t)^n \lambda^{nt},$$

where π is a periodic function whose period is one, then $\chi(t)$ is a solution of (1.1).

Proof. Here we prove in the case $\lambda < 1$.

Let $u(\tau)$ be the solution of (1.1) in above argument. And suppose $\chi(t)$ be a solution of (1.1) such that $\chi(t+n) \to 0$ as $n \to +\infty$ uniformly on any compact set.

We put

$$u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt} = U(\lambda^t), \quad \alpha_1 \neq 0,$$

then U, χ are open maps, and $U(0) = 0$. So we have $\chi(t) = U(\tau) = U(\lambda^\sigma)$ (for $\exists \tau = \lambda^\sigma$). Since $\alpha_1 \neq 0$, we have $\sigma = \log_\lambda U^{-1}(\chi(t)) := l(t)$.

Here according to [3], ([5]), we can prove existence of Ψ such that

$$\Psi(F(\chi, \Psi(\chi))) = G(\chi, \Psi(\chi)),$$

where $F(s,w) = w$, $G(s,w) = f(s,w)$. Then we obtain the following first order difference equation from (1.1)

$$\chi(t+1) = \Psi(\chi(t)).$$

And we obtain

$$l(t) = t + \pi(t) \quad (\pi : \text{ arbitrarily period one}).$$

Now we put $\lambda^{\pi(t)}$ into $\pi(t)$. Then $\chi(t)$ can be written as

$$\chi(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda}+t)} = \sum_{n=1}^{\infty} \alpha_n \pi(t)^n \lambda^{nt},$$

where π is an arbitrarily periodic function whose period is one. □

References

[1] L.V. Ahfors," Complex Analysis", New York : McGraw-Hill, 1966.

[2] D.R. Smart," Fixed point theorems", Cambridge Univ. Press, 1974 .

[3] M.Suzuki, "Holomorphic solutions of some functional equations", Nihonkai Mathematical Journal, **5** ,1994,109-114.

[4] M.Suzuki, "On some Difference equations in economic model", Mathematica Japonica, **43**, 1996, 129-134.

[5] M. Suzuki, "Holomorphic solutions of some system of n functional equations with n variables related to difference systems", Aequationes Mathematicae, **57**, 1999, 21-36.

[6] M. Suzuki, "Difference Equation for A Population Model", Discrete Dynamics in Nature and Society, **5**, 2000, 9-18.

[7] N. Yanagihara, "Meromorphic solutions of some difference equations", Funkcial. Ekvac., **23**, 1980, ,309-326.