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Bifurcation structure of stationary solutions of a Lotka-Volterra competition model with diffusion

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In order to understand the mechanism of phenomena in various fields, we often discuss the existence and stability of stationary solutions for the system of reaction-diffusion equations

$$\begin{cases} \mathbf{w}_t = \varepsilon^2 D\, \mathbf{w}_{xx} + \mathbf{f}(\mathbf{w}), & x \in (0,1), \quad t > 0, \\ \mathbf{w}_x = \mathbf{0}, \quad x = 0,1, & t > 0 \end{cases} \tag{1}$$

with suitable initial condition, where $\mathbf{w} \in \mathbf{R}^N$, $\varepsilon > 0$, D is a diagonal matrix whose elements are positive, and $\mathbf{f} : \mathbf{R}^N \to \mathbf{R}^N$ is smooth function.

When $N = 1$ is satisfied, we can comparatively easily study the existence of stationary solutions for (1) and their spatial profile by the analysis of motions in the phase plane, because the so-called *comparison principle* holds. Furthermore it is well-known that for suitable $f(w)$, the global attractor $\mathcal{A}$ of (1) is represented as $\mathcal{A} = \bigcup_{e \in E} W^u(e)$, where E is the set of stationary solutions for (1), and $W^u(e)$ is an unstable manifold of (1) at $w = e$ (for example, see Hale [2, Chapter 4]). This fact means that one of important problems is to seek all stationary solutions of (1).

In general, the comparison principle does not always hold for the case $N \geq 2$, so we have the considerable complexity for studying the existence and stability of stationary solutions for (1). In this report, as a first step to approach the problem for $N \geq 2$, we treat the stationary problem

$$\begin{cases} 0 = \varepsilon^2 d_w\, w'' + (1 - w^n - c\, z^n)\, w, \\ 0 = \varepsilon^2 d_z\, z'' + (1 - b\, w^n - z^n)\, z, \quad x \in (0,1), \\ w' = 0, \quad z' = 0, \quad x = 0,1 \end{cases} \tag{2}$$

of a Lotka-Volterra competition model which is most simple within the framework of reaction-diffusion equations, where $' = \frac{d}{dx}$, and every parameter is a positive constant. As $\mathbf{w}$ means the population density for two competing species, we restrict our discussion to *positive* solutions which satisfy $w(x) > 0$ and $z(x) > 0$ for any $x \in [0,1]$. It is obvious that (2) has constant solutions $(0,0)$, $(0,1)$, $(1,0)$, and $\hat{\mathbf{w}} = (\hat{w}, \hat{z})$ with

$$\hat{w} = \sqrt[n]{\frac{1-c}{1-b\,c}}, \quad \hat{z} = \sqrt[n]{\frac{1-b}{1-b\,c}}$$

which is positive for $\max(b,c) < 1$ or $\min(b,c) > 1$. Furthermore the maximum principle leads to the fact that every solution of (2) with $w(x) \geq 0$ and $z(x) \geq 0$ for any $x \in [0,1]$ must be a constant function in x, when $\min(b,c) < 1$ is satisfied.

Let us consider the case

$$\mu = (b,c) \in \mathcal{M} \equiv \{\, (b,c) \mid \min(b,c) > 1 \,\}.$$

After simple calculations, we see that for any $\mathbf{d} \in \mathcal{D}_0(\mu)$, the linearized operator of (2) around $\mathbf{w} = \hat{\mathbf{w}}$ has only one eigenvalue and at least two eigenvalues in the right half-plane for any ε with $\varepsilon > 1$ and $0 < \varepsilon < 1$, respectively, where $R_+ = (0, +\infty)$, $\mathbf{d} = (d_w, d_z)$, and

$$\mathcal{D}_0(\mu) = \{\, \mathbf{d} \in R_+^2 \mid \det(-\pi^2 D + \mathbf{f_u}(\hat{\mathbf{w}})) = 0 \,\}.$$

The bifurcation theory says that nonconstant positive solutions of (2) which look like $\pm \mathbf{v} \cos(\pi x)$ perturbations from $\mathbf{w} = \hat{\mathbf{w}}$ bifurcate at $\varepsilon = 1$ for any $\mathbf{d} \in \mathcal{D}_0(\mu)$, where $\mathbf{v}$ is an eigenvector of the linearized operator corresponding to the eigenvalue 0. As the multi-existence of nonconstant positive solutions for (2) is suggested, we shall in this report establish the bifurcation structure of positive solutions for (2) with respect to ε for arbitrarily fixed $\mu \in \mathcal{M}$ and $\mathbf{d} \in \mathcal{D}_0(\mu)$.

Let us prepare definitions and notations to state the main result of this report. We define the order relation $\preceq$ by

$$(w_1, z_1) \preceq (w_2, z_2) \Longleftrightarrow w_1 \leq w_2,\ z_1 \geq z_2,$$

and denote by $\prec$ the relation obtained from the above definition by replacing $\leq$ with $<$. We set

$$\rho = (\mu, \mathbf{d}), \quad \mathcal{N} = \bigcup_{\mu \in \mathcal{M}} \{\, \mu \,\} \times \mathcal{D}_0(\mu), \quad E_0(\rho) = R_+ \times \{\, \hat{\mathbf{w}} \,\},$$
$$X = \{\, \mathbf{w}(.) \in C^2([0,1]) \mid \mathbf{w}'(0) = \mathbf{0} = \mathbf{w}'(1) \,\}.$$

For each $\rho \in \mathcal{N}$, we denote by $E(\rho)$ the set of $(\varepsilon, \mathbf{w}(.)) \in R_+ \times X$ such that $\mathbf{w}(x)$ is a positive solution of (2) for ε, and by $E_k(\rho)$ ($k \in \mathbf{N}$) the set of $(\varepsilon, \mathbf{w}(.)) \in E(\rho)$ such that there exists $\ell \in \{0,1\}$ such that $(-1)^{j+\ell}\mathbf{w}'(x) \succ \mathbf{0}$ is satisfied for any $j \in \mathbf{Z}$ and $x \in (j/k, (j+1)/k)$. By definition, we see that $\bigcup_{k \geq 0} E_k(\rho) \subset E(\rho)$ holds for any $\rho \in \mathcal{N}$, and that for any $\rho \in \mathcal{N}$ and $k \in \mathbf{N}$, $(\varepsilon, \mathbf{w}(.)) \in E_k(\rho)$ is equivalent to $(k\,\varepsilon, \mathbf{w}(./k)) \in E_1(\rho)$.

Theorem 1. *$E(\rho) = \bigcup_{k \geq 0} E_k(\rho)$ is satisfied for any $n \geq 1$ and $\rho \in \mathcal{N}$.*

The above theorem says that for each $n \geq 1$ and $\rho \in \mathcal{N}$, we can understand the complete structure of $E(\rho)$ by using the information on the structure of $E_1(\rho)$. While the structure of $E_1(\rho)$ was completely established for the case $n = 1$ in the previous paper [3], the following is for the case $n \geq 2$:

Theorem 2. *For each $n \geq 2$ and $\rho \in \mathcal{N}$, there exist continuous functions $\mathbf{w}_-(.,\varepsilon)$ and $\mathbf{w}_+(.,\varepsilon)$ defined on $(0,1)$ such that*
(i) $E_1(\rho) = \{(\varepsilon, \mathbf{w}_\pm(.,\varepsilon)) \mid \varepsilon \in (0,1)\}$,
(ii) $\pm\mathbf{w}'_\pm(x,\varepsilon) \prec \mathbf{0}$ *for any* $x \in (0,1)$ *and* $\varepsilon \in (0,1)$, *and*
(iii) $\lim_{\varepsilon\to 1} \mathbf{w}_\pm(.,\varepsilon) = \hat{\mathbf{w}}$.
are satisfied (see Figure 1).

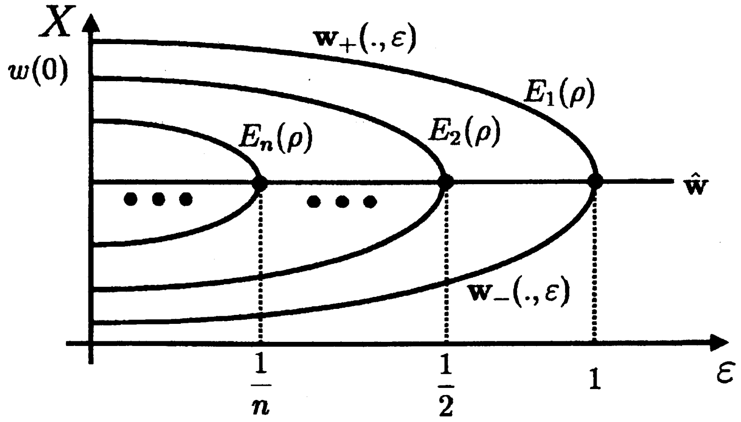


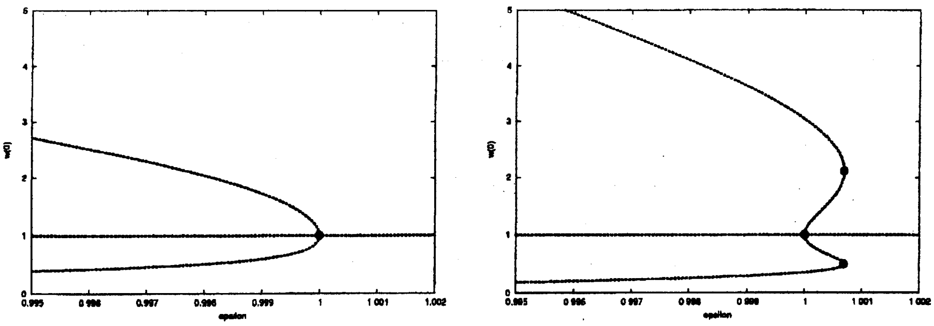
Figure 1. Global bifurcation structure.

In consideration of Chafee and Infante [1], we see that the bifurcation structure of positive solutions for (2) with respect to ε for arbitrarily fixed $n \geq 2$ and $\rho \in \mathcal{N}$ is similar to that for

$$\begin{cases} 0 = \varepsilon^2 u'' + u(1-u)(u-a), & x \in (0,1), \\ u'(0) = 0, \quad u'(1) = 0, \end{cases}$$

where $0 < a < 1$. Furthermore it follows from Kishimoto and Weinberger [4] that $\mathbf{w}_-(.,\varepsilon)$ and $\mathbf{w}_+(.,\varepsilon)$ are unstable stationary solutions for (1).

Figure 2 (a) and (b) are numerical bifurcation diagrams of $E_1(\rho)$ for the case where the assumption of Theorem 2 is violated, and show that the structure of $E_1(\rho)$ depends on the interspecific competition rates b and c in case of $1 < n < 2$.



(a) $b = c = 200.0$ (b) $b = c = 2000.0$

Figure 2. Bifurcation structure, where $n = 1.1$ and $d_w = d_z$.

In the proof of Theorem 2, one of important parts is to determine the geometrical position of the curve of positive solutions for (2) bifurcating from $\mathbf{w} = \hat{\mathbf{w}}$ at $\varepsilon = 1$. In general, as the equation which describes the geometrical position is very complex even if we can explicitly write down it, we have difficulty in analyzing the geometrical position theoretically. In the proof, to determine the geometrical position, we employ the numerical verification by the help of the interval arithmetic built into *Mathematica.*

References

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