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Oscillation Theorems for Nonlinear Differential Equations with p-Laplacian and Its Application to Elliptic Equation

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1. Introduction. We are concerned with the oscillation problem for the nonlinear differential equation

$$(\phi_p(x'))' + \frac{1}{t^p} g(x) = 0, \qquad t > 0, \tag{1}$$

where $\phi_p(y)$ is a real-valued function defined by $\phi_p(y) = |y|^{p-2}y$ with $p > 1$ a fixed real number, and $g(x)$ is a continuous function on $\mathbb{R}$ satisfying the signum condition

$$xg(x) > 0 \quad \text{if } x \neq 0 \tag{2}$$

and a suitable smoothness condition for the uniqueness of solutions of the initial value problem. By virtue of a continuation result in [3, 7], we can prove that all solutions of (1) are continuable in the future. Hence, it is worth while to discuss whether solutions of (1) are oscillatory or not.

By an *oscillatory solution* we mean one having an infinite number of zeros on $0 < t < \infty$. Otherwise, the solution is said to be *nonoscillatory*. Hence, a nonoscillatory solution eventually keeps either positive or negative. It is called a *positive* (or *negative*) *solution*.

To begin with, we consider a very simple case. When $p = 2$ and $g(x) = \lambda x$ with $\lambda > 0$ a parameter, equation (1) reduces to the Euler differential equation

$$x'' + \frac{\lambda}{t^2} x = 0, \qquad t > 0. \tag{3}$$

It is well-known that all nontrivial solutions of (3) are oscillatory if $\lambda > 1/4$ and are nonoscillatory if $\lambda \leq 1/4$. In other words, $1/4$ is the lower bound for all nontrivial solutions of (3) to be oscillatory. Such a number is generally called the *oscillation constant* (for example, see [4, 9]).

Two natural questions now arise: (i) what is the oscillation constant for the linear equation

$$x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \lambda\delta(t)\right\} x = 0, \qquad t > 0, \tag{4}$$

where λ is a positive parameter, and $\delta(t)$ is a positive and continuous function? (ii) what is the oscillation constant for the nonlinear equation

$$x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \lambda h(x)\right\} x = 0, \qquad t > 0, \tag{5}$$

where λ is a positive parameter, and $h(x)$ is a positive and continuous function?

As to the first question, by means of Sturm's comparison theorem, we see that if $\lim_{t\to\infty}\delta(t) > 0$, then all nontrivial solutions of (4) are oscillatory for any $\lambda > 0$. More delicate and interesting case is that

$$\delta(t) \searrow 0 \quad \text{as } t \to \infty.$$

When $\delta(t) = 1/(\log t)^2$, equation (4) is called the Riemann-Weber version of the Euler differential equation. It is famous that the oscillation constant is also $1/4$ for equation (4) with $\delta(t) = 1/(\log t)^2$.

Recently, Sugie and Kita [8] have studied the nonlinear differential equation

$$x'' + \frac{1}{t^2}g(x) = 0, \qquad t > 0. \tag{6}$$

Using the following their results, we can give an answer to the second question.

Theorem A. *Assume* (2) *and suppose that there exists a* λ *with* $\lambda > 1/16$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of* (6) *are oscillatory.*

Theorem B. *Assume* (2) *and suppose that*

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{1}{16(\log|x|)^2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then all nontrivial solutions of* (6) *are nonoscillatory.*

From Theorems A and B, we see that the oscillation constant for equation (5) is $1/16$ provided $h(x) = 1/(\log|x|)^2$ for $|x|$ sufficiently large. Of course, Theorems A and B cover almost the delicate case that

$$\frac{g(x)}{x} \searrow \frac{1}{4} \quad \text{as } |x| \to \infty.$$

Next, consider the case that $g(x) = \lambda\phi_p(x)$. Then equation (1) becomes the half-linear differential equation

$$(\phi_p(x'))' + \frac{\lambda}{t^p}\phi_p(x) = 0, \qquad t > 0. \tag{7}$$

Since equation (7) coincides with equation (3) when $p = 2$, we may regard (7) as a generalization of (3). As a matter of fact, the oscillation constant is $((p-1)/p)^p$ for equation (7) (see [1, 2]). This drives us to the further question what are the oscillation constants for the equations

$$(\phi_p(x'))' + \frac{1}{t^p}\left\{\left(\frac{p-1}{p}\right)^p + \lambda\delta(t)\right\}\phi_p(x) = 0, \qquad t > 0 \tag{8}$$

and

$$(\phi_p(x'))' + \frac{1}{t^p}\left\{\left(\frac{p-1}{p}\right)^p + \lambda h(x)\right\}\phi_p(x) = 0, \qquad t > 0, \tag{9}$$

respectively.

Elbert and Schneider [2] have already discussed the oscillation problem for equation (8) and gave the following result.

Theorem C. *Consider equation* (8) *with* $\delta(t) = 1/(\log t)^2$. *Then the oscillation constant is* $\gamma_p/2$, *where*

$$\gamma_p = \left(\frac{p-1}{p}\right)^{p-1}.$$

Theorem C is an improvement of the above result concerning the Riemann-Weber version of the Euler differential equation. Hence, it is safe to say that the question for equation (8) is solved. However, that for equation (9) remains unsettled. The purpose of this paper is to give an oscillation theorem which can be applied even to the case that

$$\frac{g(x)}{\phi_p(x)} \searrow \left(\frac{p-1}{p}\right)^p \quad \text{as } |x| \to \infty.$$

Our main result is stated as follows:

Theorem 1. *Assume* (2) *and suppose that*

$$\frac{g(x)}{\phi_p(x)} \geq \left(\frac{p-1}{p}\right)^p + \frac{\lambda}{(\log|x|)^2} \tag{10}$$

for $|x|$ *sufficiently large, where*

$$\lambda > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p+1}.$$

Then all nontrivial solutions of (1) *are oscillatory.*

Remark 1. Since Theorem 1 coincides with Theorem A when $p = 2$, Theorem 1 is a complete generalization of Theorem A.

2. Preliminary. To prove Theorem 1, we prepare some lemmas below. As space is limited, we have to omit the proofs.

Lemma 2. *Let* T *be a positive number. Suppose that a positive function* $f \in C^2[T,\infty)$ *satisfies*

$$(\phi_p(f'(t)))' < 0 \quad \textit{for } t \geq T.$$

Then $f'(t)$ *is also positive for* $t \geq T$.

From Lemma 2, we see that each positive solution of (1) has the following property.

Lemma 3. *Assume* (2) *and suppose that equation* (1) *has a positive solution. Then the solution tends to* ∞ *as* $t \to \infty$.

Using the so-called "Riccati technique" and a straightforward calculation, we have the following two lemmas on some differential inequalities of the first order.

Lemma 4. *Let* $s = \log t$. *Suppose that the differential inequality*

$$\dot{\xi} + (p-1)\left\{\xi^{\frac{p}{p-1}} - \xi + \frac{(p-1)^{p-1}}{p^p}\right\} \leq 0, \qquad \dot{} = \frac{d}{ds}$$

has a positive solution on $[s_0,\infty)$ *with* $s_0 > 0$. *Then the solution is decreasing and tends to* γ_p *as* $s \to \infty$.

Lemma 5. *Let $s = \log t$. Suppose that the differential inequality*

$$\dot{\xi} + (p-1)\left\{\xi^{\frac{p}{p-1}} - \xi + \frac{(p-1)^{p-1}}{p^p}\right\} + \lambda\delta(e^s) \le 0$$

has a positive solution on $[s_0, \infty)$ with $s_0 > 0$, where λ is a positive parameter and $\delta(e^s)$ is a positive and continuous function for $s \ge s_0$. Then all nontrivial solutions of (8) are nonoscillatory.

3. Proof of our main theorem. The proof is by contradiction. Suppose that equation (1) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. Let L be a large number satisfying the assumption (10) for $|x| > L$. By Lemma 3, there exists a $T > 0$ such that

$$x(t) > L \quad \text{for } t \ge T.$$

As in the proof of Lemma 2, we see that $x'(t) > 0$ for $t \ge T$.

Let $s = \log t$ and put $u(s) = x(t)$. Then equation (1) is transformed into the equation

$$(\phi_p(\dot{u}))^{\cdot} - (p-1)\phi_p(\dot{u}) + g(u) = 0 \tag{11}$$

and $u(s)$ is a positive solution of (11). Note that

$$u(s) > L \quad \text{and} \quad \dot{u}(s) = tx'(t) > 0$$

for $s \ge \log T$. Hence, $\phi_p(u(s)) = u(s)^{p-1}$ and $\phi_p(\dot{u}(s)) = \dot{u}(s)^{p-1}$ for $s \ge \log T$.

Define

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))}.$$

Then $\xi(s) > 0$ for $s \ge \log T$. Differentiating $\xi(s)$ and using (10) and (11), we have

$$\begin{aligned}
\dot{\xi}(s) &= \frac{(\phi_p(\dot{u}(s)))^{\cdot}\phi_p(u(s)) - (p-1)\phi_p(\dot{u}(s))u(s)^{p-2}\dot{u}(s)}{\phi_p(u(s))^2} \\
&= \frac{(\phi_p(\dot{u}(s)))^{\cdot}}{\phi_p(u(s))} - (p-1)\left(\frac{\dot{u}(s)}{u(s)}\right)^p \\
&= (p-1)\frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} - \frac{g(u(s))}{\phi_p(u(s))} - (p-1)\left(\frac{\dot{u}(s)}{u(s)}\right)^p \\
&\le (p-1)\xi(s) - \left(\frac{p-1}{p}\right)^p - \frac{\lambda}{\{\log u(s)\}^2} - (p-1)\xi(s)^{\frac{p}{p-1}} \\
&= -(p-1)\left\{\xi(s)^{\frac{p}{p-1}} - \xi(s) + \frac{(p-1)^{p-1}}{p^p}\right\} - \frac{\lambda}{\{\log u(s)\}^2}
\end{aligned} \tag{12}$$

for $s \ge \log T$. Hence, from Lemma 4 we see that

$$\xi(s) \searrow \gamma_p \quad \text{as } s \to \infty. \tag{13}$$

Since $\lambda > ((p-1)/p)^{p+1}/2$, we can choose an $\varepsilon_0 > 0$ so small that

$$\frac{1}{2}\left(\frac{p-1}{p}\right)^{p+1} < \frac{\gamma_p}{2}\left(\frac{p-1}{p} + 2\varepsilon_0\right)^2 < \lambda. \tag{14}$$

By (13) we can find an $s_1 > \log T$ such that

$$\xi(s)^{\frac{1}{p-1}} = \frac{\dot{u}(s)}{u(s)} \le \frac{p-1}{p} + \varepsilon_0$$

for $s \ge s_1$. Integrating the both sides of this inequality, we obtain

$$u(s) \le u(s_1) e^{\left(\frac{p-1}{p}+\varepsilon_0\right)(s-s_1)} \quad \text{for } s \ge s_1,$$

and therefore, there exists an $s_2 > s_1$ such that

$$L < u(s) \le e^{\left(\frac{p-1}{p}+2\varepsilon_0\right)s} \quad \text{for } s \ge s_2.$$

From this estimation and (12), we see that

$$\dot{\xi}(s) \le -(p-1)\left\{\xi(s)^{\frac{p}{p-1}} - \xi(s) + \frac{(p-1)^{p-1}}{p^p}\right\} - \frac{\lambda}{\left(\frac{p-1}{p}+2\varepsilon_0\right)^2 s^2}$$

for $s \ge s_2$. Hence, by Lemma 5 we conclude that all nontrivial solutions of the equation

$$(\phi_p(x'))' + \frac{1}{t^p}\left\{\left(\frac{p-1}{p}\right)^p + \frac{\lambda}{\left(\frac{p-1}{p}+2\varepsilon_0\right)^2 (\log t)^2}\right\}\phi_p(x) = 0, \qquad t > 0 \tag{15}$$

are nonoscillatory. However, by (14) we have

$$\frac{\lambda}{\left(\frac{p-1}{p}+2\varepsilon_0\right)^2} > \frac{\gamma_p}{2}.$$

Hence, from Theorem C, we see that all nontrivial solutions of (15) are oscillatory. This is a contradiction. We have thus proved the theorem.

4. Application to an elliptic equation. To apply our results, we consider an elliptic equation of the form

$$\Delta_p u + F(x,u) = 0, \qquad x \in \Omega, \tag{16}$$

where Ω is an exterior domain of $\mathbb{R}^N$ with $N \ge 2$, that is, it contains $G_a \overset{\text{def}}{=} \{x \in \mathbb{R}^N\colon |x| > a\}$ for some $a > 0$, Δ_p is a operator given by

$$\Delta_p u = \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right), \qquad \nabla = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_N),$$

and $F(x,u)$ is a continuous function on $\Omega \times \mathbb{R}$ satisfying the assumption

$$\begin{cases} \text{there is a continuous function } f\colon [a,\infty) \times \mathbb{R} \to \mathbb{R} \text{ such that} \\ uF(x,u) \ge uf(|x|,u) \ge 0 \text{ for } |x| \ge a \text{ and } u \in \mathbb{R}, \text{ and} \\ f(t,u) \text{ is nondecreasing with respect to } u \in \mathbb{R} \text{ for each fixed } t \ge a. \end{cases} \tag{17}$$

We call a function $u \in C^1(G_b)$ with $|\nabla u|^{p-2}\nabla u \in C^1(G_b)$ for some $b \ge a$ a *solution* of (16) in G_b if it satisfies equation (16) at every point $x \in G_b$. We say that a solution of (16) is *oscillation* if it keeps neither positive nor negative in any exterior domain.

A typical case of (16) is the half-linear partial differential equation

$$\Delta_p u + c(|x|)\phi_p(u) = 0, \qquad x \in \Omega. \tag{18}$$

It is clear that assumption (17) is satisfied with $f(t,u) = c(t)\phi_p(u)$ in this case. Kusano *et al.* [5, Theorem 2.1] have presented a comparison theorem of Sturm type for equation (18) and more general half-linear elliptic equations. By virtue of their work, we have the following result.

Proposition 6. *Let D be a bounded domain in Ω with piecewise smooth boundary ∂D. If there exists a nontrivial solution u of* (18) *such that $u = 0$ on ∂D, then every solution except a constant multiple of u vanishes at some point of D.*

From this and Theorem C, we can get a sufficient condition for all nontrivial solutions of the equation

$$\Delta_p u + \frac{1}{|x|^p}\left\{\left(\frac{p-N}{p}\right)^p + \frac{\mu}{(\log|x|)^2}\right\}\phi_p(u) = 0, \qquad x \in \Omega \tag{19}$$

to be oscillatory.

Theorem 7. *Let $N < p$. If*

$$\mu > \frac{p-1}{2p}\left(\frac{p-N}{p}\right)^{p-2}, \tag{20}$$

then all nontrivial solutions of (19) *are oscillatory.*

Proof. Let $u(x)$ be a radial solution of (19) and let $v(t)$ be the function defined by

$$v(t) = u(x), \qquad t = |x|.$$

Then we have $\nabla u(x) = v'(t)x/t$, and therefore,

$$\begin{aligned}\Delta_p(u(x)) &= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{x_i}{t}\phi_p(v'(t))\right)\\ &= (\phi_p(v'(t)))' + \frac{N-1}{t}\phi_p(v'(t)) = \frac{1}{t^{N-1}}\left(t^{N-1}\phi_p(v'(t))\right)'.\end{aligned}$$

Hence, we see that the function $v(t)$ is a solution of the equation

$$\left(t^{N-1}\phi_p(v')\right)' + t^{N-1-p}\left\{\left(\frac{p-N}{p}\right)^p + \frac{\mu}{(\log t)^2}\right\}\phi_p(v) = 0. \tag{21}$$

We next define

$$w(s) = v(t), \qquad s = t^{\frac{p-N}{p-1}}.$$

Then we obtain

$$\begin{aligned} v'(t) &= \frac{p-N}{p-1}t^{\frac{1-N}{p-1}}\dot{w}(s),\\ \phi_p(v'(t)) &= \left(\frac{p-N}{p-1}\right)^{p-1} t^{1-N}\phi_p(\dot{w}(s))\end{aligned}$$

and

$$\left(t^{N-1}\phi_p(v'(t))\right)' = \left(\frac{p-N}{p-1}\right)^p s^{\frac{1-N}{p-N}}(\phi_p(\dot{w}(s)))^{\cdot}.$$

From the last equality and (21) it turns out that $w(s)$ satisfies the equation

$$(\phi_p(\dot{w}))^{\cdot} + \frac{1}{s^p}\left\{\left(\frac{p-1}{p}\right)^p + \left(\frac{p-1}{p-N}\right)^{p-2}\frac{\mu}{(\log s)^2}\right\}\phi_p(w) = 0. \tag{22}$$

By (20) we have

$$\left(\frac{p-1}{p-N}\right)^{p-2}\mu > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} = \frac{\gamma_p}{2}.$$

Hence, using Theorem C, we can conclude that all nontrivial solutions of (22) are oscillatory. Of course, $w(s)$ is oscillatory, so that it has an infinite number of zeros clustering at $s = \infty$. Since $p > N \geq 2$, the variable t tends to ∞ as s increases, and therefore, $v(t)$ is also oscillatory. This means that all radial solutions of (19) are oscillatory. Thus, we can choose a sequence $\{t_m\}$ tending to ∞ such that $u(x) = 0$ for $|x| = t_m$. Denote

$$G(t_m, t_{m+1}) = \left\{x \in \mathbb{R}^N : t_m < |x| < t_{m+1}\right\}$$

for $m \in \mathbb{N}$. Then $u(x) = 0$ for $x \in \partial G(t_m, t_{m+1})$, which is smooth. Hence, from Proposition 6, we see that every nontrivial solution of (19) has at least one zero on the closure of $G(t_m, t_{m+1})$. Since this fact is true for arbitrary $m \in \mathbb{N}$, all (non-radial) solutions of (19) are oscillatory. The proof is complete.

Let us now return to equation (16). We will give an oscillation theorem for equation (16) without using Sturm's comparison method, such as Proposition 6.

Naito and Usami [6] have studied more general quasilinear elliptic equations than equation (16) and clarified the relation with associated quasilinear ordinary differential equations. The following is an immediate consequence of their result.

Proposition 8. *Assume that* (17) *holds. If every solution of*

$$\left(t^{N-1}\phi_p(u')\right)' + t^{N-1}f(t,u) = 0, \qquad t \geq a$$

is oscillatory, then every solution of (16) *is also oscillatory.*

Consider the case that $f(t,u) = g(u)/t^p$, namely, the equation

$$\left(t^{N-1}\phi_p(u')\right)' + t^{N-1-p}g(u) = 0, \tag{23}$$

where $g(u)$ is a continuous and nondecreasing function on $\mathbb{R}$ satisfying the signum condition (2). By putting $v(s) = u(t)$ and $s = t^{\frac{p-N}{p-1}}$, equation (23) becomes

$$(\phi_p(\dot{v}))^{\cdot} + \left(\frac{p-1}{p-N}\right)^p \frac{1}{s^p} g(v) = 0.$$

Let us assume that $g(u)$ satisfies

$$\frac{g(u)}{\phi_p(u)} \geq \left(\frac{p-N}{p}\right)^p + \frac{\mu}{(\log|u|)^2} \tag{24}$$

for $|u|$ sufficiently large. Then we have

$$\left(\frac{p-1}{p-N}\right)^p \frac{g(u)}{\phi_p(u)} \geq \left(\frac{p-1}{p}\right)^p + \left(\frac{p-1}{p-N}\right)^p \frac{\mu}{(\log|u|)^2}.$$

Let $\lambda = ((p-1)/(p-N))^p\mu$. If

$$\mu > \frac{p-1}{2p}\left(\frac{p-N}{p}\right)^p, \tag{25}$$

then $\lambda > ((p-1)/p)^{p+1}/2$. Hence, from Theorem 1, we conclude that all nontrivial solutions of (23) are oscillatory. Moreover, if $F(u,x)$ satisfies

$$uF(u,x) \geq \frac{ug(u)}{|x|^p} \tag{26}$$

for $|x| \geq a$ and $u \in \mathbb{R}$, then assumption (17) holds. Hence, by Proposition 8, we see that all nontrivial solutions of (16) are also oscillatory. Thus, combining Theorem 1 with Proposition 8, we can obtain the following result.

Theorem 9. *Let $N < p$. Suppose that there exist a nondecreasing function $g(u)$ and a positive number μ satisfying* (2) *and* (24)–(26). *Then all nontrivial solutions of* (16) *are oscillatory.*

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