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# Identification problems for nonlinear perturbed sine-Gordon equations

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## 1. Introduction

In Ha and Nakagiri [9] we studied the identification problems of the damped sine-Gordon equation

$$\frac{\partial^2 y}{\partial t^2}+\alpha\frac{\partial y}{\partial t}-\beta\Delta y+\gamma\sin y=\delta f, \tag{1.1}$$

where $\alpha,\beta,\gamma,\delta$ are unknown constant parameters. In [9] the existence and the necessary conditions of optimality for the optimal parameter $q^*=(\alpha,\beta^*,\gamma^*,\delta^*)$ is established for the appropriate cost without including the cost of parameters $q=(\alpha,\beta,\gamma,\delta)$.

Several types of perturbed sine-Gordon equations differently from (1.1) are proposed to describe the dynamics of the phase difference in the Josephson junctions in various situations. We refer to, e.g. [1], [3]-[6], [11]. In Kivshar and Malomed [5] the perturbed equation

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+\sin y=\epsilon\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial t}\right) \tag{1.2}$$

is proposed by taking into account of losses or dissipation due to the current along a dielective barrier in Josephson junctions. The nonlinear perturbation

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+\sin y=\epsilon\sin 2y \tag{1.3}$$

is also proposed by Kivshar and Malomed [4] to determine the inelastic interaction of a fast kink and a weakly bounded breather. The additional nonlinear perturbations $\sum_{i=1}^{L}\epsilon_i\sin\kappa_i y$ are possible in (1.3).

Recently in Ramos [10] the numerical analysis of perturbed sine-Gordon equation of the generalized form

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+\sin y=\epsilon_1\frac{\partial y}{\partial t}+\epsilon_2 y+\epsilon_3\sin 2y+\epsilon_4\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial t}\right) \tag{1.4}$$

subject to homogeneous Neumann boundary conditions in the finite line is studied rather completely based on the implicit finite difference methods. There are various interesting observations of solutions in [10] according to the differences of perturbations for $\epsilon_i$ terms. It is an important physical problem to identify such constant parameters $\epsilon_i$.

In this paper we study the problems of identification of a general equation described by

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial \Delta y}{\partial t} - \beta \Delta y + \sum_{i=1}^{L} \gamma_i \sin \kappa_i y + \delta y = \nu f \tag{1.5}$$

in $R^n$, where $\alpha, \beta, \gamma_i, \delta, \kappa_i$ and $\nu$ are constants and $f$ is a prescribed source function. In our identification problems all parameters $\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu$ are assumed to be unknown but the number $L$ is prescribed. The objective of this paper is to extend the results in [9] to the the equations (1.5) under the homogeneous Neumann boundary conditions in $n$-dimensions.

## 2. Perturbed sine-Gordon equations

Let $\Omega$ be an open bounded set of $\mathbf{R}^n$ with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. We consider the Kivshar-Malomed type perturbed sine-Gordon equations described by

$$\frac{\partial^2 y}{\partial t^2} - \alpha \frac{\partial \Delta y}{\partial t} - \beta \Delta y + \sum_{i=1}^{L} \gamma_i \sin \kappa_i y + \delta y = f \quad \text{in} \quad Q, \tag{2.1}$$

where $\alpha, \beta > 0$, $\delta, \gamma_i, \kappa_i \in \mathbf{R}, i = 1, \cdots, L$,$\Delta$ is a Laplacian in $\mathbf{R}^n$ and $f$ is a given function. The boundary condition is the homogeneous Neumann condition

$$\frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \Sigma. \tag{2.2}$$

The initial values are given by

$$y(0,x) = y_0(x) \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \quad \Omega. \tag{2.3}$$

First we introduce two Hilbert spaces $H$ and $V$ by $H = L^2(\Omega)$ and $V = H^1(\Omega)$, respectively. We endow the space $H = L^2(\Omega)$ with the inner product and norm

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\psi| = (\psi,\psi)^{1/2}, \quad \forall \phi, \psi \in L^2(\Omega). \tag{2.4}$$

For $\phi, \psi \in V = H^1(\Omega)$ we define

$$(\!(\psi, \phi)\!) = \sum_{i=1}^{n} \int_\Omega \frac{\partial}{\partial x_i}\psi(x) \frac{\partial}{\partial x_i}\phi(x)dx. \tag{2.5}$$

The duality pairing between $V$ and $V'$ is denoted by $\langle \cdot, \cdot \rangle$. The inner product and norm of $V = H^1(\Omega)$ are defined by

$$(\!(\psi, \phi)\!)_1 = (\!(\psi, \phi)\!) + (\psi, \phi), \quad \|\psi\| = (\!(\psi, \psi)\!)_1^{1/2}, \quad \forall \phi, \psi \in H^1(\Omega). \tag{2.6}$$

Then the pair $(V, H)$ is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. The norm of the dual space $V'$ is denoted by $\| \cdot \|_*$.

Now we introduce the bilinear form

$$a(\phi,\varphi)=\int_\Omega \nabla\phi\cdot\nabla\varphi dx = ((\phi,\varphi)), \quad \forall \phi,\varphi\in H^1(\Omega). \tag{2.7}$$

Then we can define the bounded operator $A\in\mathcal{L}(V,V')$ through (2.7). The operator $A$ is an isomorphism from $V$ onto $V'$ and it is also considered as a self-adjoint operator in $H=L^2(\Omega)$ with dense domain $\mathcal{D}(A)$ in $V$ and in $H$,

$$\mathcal{D}(A)=\{\phi\in V:\ A\phi\in H\}=\{\phi\in H^2(\Omega):\ \frac{\partial\phi}{\partial n}=0 \ \text{ on } \Gamma\}.$$

Also we define the sine function for $z\in H=L^2(\Omega)$ by

$$(\sin z)(x)=\sin z(x) \quad \text{for a.e. } x\in\Omega.$$

Using the operator $A$ and the sine function $\sin y$, the problem (2.1), (2.2), (2.3), is converted to the following Cauchy problem in $H$:

$$\begin{cases} \dfrac{d^2y(t)}{dt^2}+\alpha A\dfrac{dy(t)}{dt}+\beta Ay(t)+\displaystyle\sum_{i=1}^{L}\gamma_i\sin\kappa_i y+\delta y=f(t), & t\in(0,T),\\ y(0)=y_0,\quad \dfrac{dy}{dt}(0)=y_1. \end{cases} \tag{2.8}$$

The solution space should be introduced in this perturbed case is defined by

$$W_V(0,T)=\{g|g\in L^2(0,T;V), g'\in L^2(0,T;V), g''\in L^2(0,T;V')\}$$

with inner product

$$(f,g)_{W_V(0,T)}=\int_0^T \left( ((f(t),g(t)))+((f'(t),g'(t)))+(f''(t),g''(t))_{V'}\right) dt,$$

where $(\cdot,\cdot)_{V'}$ is the inner product of $V'$. We denote by $\mathcal{D}'(0,T)$ the space of distributions on $(0,T)$. The definition of weak solutions of the problem (2.8) is as follows.

**Defintion 2.1.** *A function $y$ is said to be a weak solution of (2.8) if $y\in W_V(0,T)$ and $y$ satisfies*

$$\langle y''(\cdot),\phi\rangle+((\alpha y'(\cdot),\phi))+((\beta y(\cdot),\phi))+\sum_{i=1}^{L}(\gamma_i\sin\kappa_i y(\cdot),\phi)+(\delta y(\cdot),\phi)=\langle f(\cdot),\phi\rangle$$

$$\text{for all } \phi\in V \text{ in the sense of } \mathcal{D}'(0,T),$$

$$y(0)=y_0,\quad y'(0)=y_1.$$

For the existence, uniqueness and regularity of weak solutions for (2.8), we can prove the following theorem. For a proof, see Ha and Nakagiri [8].

**Theorem 2.1.** *Let $\alpha,\beta>0$, $\delta,\gamma_i,\kappa_i\in\mathbf{R}, i=1,\cdots,L$ and $f$, $y_0$, $y_1$ be given satisfying*

$$f\in L^2(0,\ T;V'),\quad y_0\in H^1(\Omega),\quad y_1\in L^2(\Omega). \tag{2.9}$$

*Then the problem* (2.8) *has a unique weak solution $y$ in $W_V(0,T)$. The solution $y$ has the regularity*

$$y\in C([0,T];H^1(\Omega))),\quad y'\in C([0,T];L^2(\Omega)). \tag{2.10}$$

## 3. Identification of constant parameters

In this section we study the identification problems for perturbed sine-Gordon equations described by

$$\begin{cases} y'' + (\alpha_0 + \alpha^2)Ay' + (\beta_0 + \beta^2)Ay + \sum_{i=1}^{L} \gamma_i \sin \kappa_i y + \delta y = \nu f \ \text{ in } \ (0,T), \\ y(0) = y_0, \ \ y'(0) = y_1, \end{cases} \tag{3.1}$$

where $\alpha_0 > 0$ and $\beta_0 > 0$ are fixed. In (3.1) we multiply the constant $\delta$ to the forcing term $f$ and replace the diffusion parameters $\alpha$ to $\alpha_0 + \alpha^2$ and $\beta$ to $\beta_0 + \beta^2$ to obtain the linear space of parameters $\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu$. Hence the diffusion terms in (3.1) never disappear and are uniformly coercive for all $\alpha, \beta \in \mathbf{R}$.

For the setting of the identification problems for (3.1), we assume that the parameters $\alpha, \beta, \gamma_i, \delta, \kappa_i$ and $\nu$ appeared in (3.1) are unknown and we take $\mathcal{P} = \mathbf{R}^{2L+4}$ as the set of parameters $q = (\alpha, \beta, \gamma_1, \cdots, \gamma_L, \delta, \kappa_1, \cdots, \kappa_L, \nu)$. The Euclidean norm and the inner productof $\mathcal{P}$ are denoted simply by $|\cdot|$ and $(\cdot,\cdot)$, respectively. For simplicity of notations we write $q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu) \in \mathcal{P}$.

By Theorem 2.2, for each $q \in \mathcal{P}$ there exists a unique weak solution $y = y(q) \in W_V(0,T)$ of (3.1). Then we can uniquely define the solution map $q \to y(q)$ of $\mathcal{P}$ into $W_V(0,T)$.

Let $K$ be a Hilbert space of observations and let $\|\cdot\|_K$ be its norm. The observation of $y(q)$ is assumed to be given by

$$z(q) = \mathcal{C}y(q) \in K, \tag{3.2}$$

where $\mathcal{C}$ is a bounded linear observation operator of $W_V(0,T)$ into $K$.

The cost functional attached to (3.1) with (3.2) is given by

$$J(q) = \|\mathcal{C}y(q) - z_d\|_K^2 + (Mq, q) \ \text{ for } \ q \in \mathcal{P}, \tag{3.3}$$

where $z_d \in K$ is a desired value of $y(q)$ and $M$ is a symmetric and non-negative $(2L+4) \times (2L+4)$ matrix on $\mathcal{P} = \mathbf{R}^{2L+4}$.

Assume that an admissible subset $\mathcal{P}_{ad}$ of $\mathcal{P}$ is convex and closed. As in [9] we study the existence and characterization problems for the perturbed sine-Gordon equations. That is, the following two problems:

(i) Find an element $q^* \in \mathcal{P}_{ad}$ such that

$$\inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*); \tag{3.4}$$

(ii) Give a characterization to such the $q^*$.

As usual we call $q^*$ the optimal parameter and $y(q^*)$ the optimal state. In order to solve (ii), we shall derive the necessary conditions on $q^*$. If $J(q)$ is Gâteaux differentiable at $q^*$ in the direction $q - q^*$, then $q^*$ has to satisfy

$$DJ(q^*)(q - q^*) \geq 0 \ \text{ for all } \ q \in \mathcal{P}_{ad}, \tag{3.5}$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$ in the direction $q - q^*$.

### 3.1. Existence of optimal parameters

The following theorem shows the continuity of solution map $q \to y(q)$, which is crucial to solve the problems (i) and (ii).

**Theorem 3.1.** *The map $q \to y(q) : \mathcal{P} \to W_V(0,T)$ is weakly continuous. That is, $y(q_n) \to y(q)$ weakly in $W_V(0,T)$ as $q_n \to q$ in $\mathbf{R}^{2L+4}$.*

The following theorem follows immediately from Theorem 3.1 and the lower semi-continuity of norms.

**Theorem 3.2.** *If $\mathcal{P}_{ad} \subset \mathcal{P} = \mathbf{R}^{2L+4}$ is compact or $M$ is a positive and symmetric on $\mathbf{R}^{2L+4}$, then there exists at least one optimal parameter $q^* \in \mathcal{P}_{ad}$ for the cost* (3.3).

### 3.2. Necessary conditions

For proving that $J(q)$ is Gâteaux differentiable at $q^*$ in the space of parameters, we have to estimate the quotients $z_\lambda = (y(q_\lambda) - y(q^*))/\lambda$ in the space $W_V(0,T)$, where $q_\lambda = q^* + \lambda(q - q^*)$, $\lambda \in (0,1]$ and $q, q^* \in \mathcal{P}$. We set $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$ for simplicity.

Let us begin to prove the weak Gâteaux differentiability of the solution map $q \to y(q)$ of $\mathcal{P}$ into $W_V(0,T)$.

**Theorem 3.3.** *The map $q \to y(q)$ of $\mathcal{P}$ into $W_V(0,T)$ is weakly Gâteaux differentiable. That is, for fixed $q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu)$ and $q^* = (\alpha^*, \beta, \gamma_i^*, \delta^*, \kappa_i^*, \nu^*)$ in $\mathcal{P}$ the weak Gâteaux derivative $z = Dy(q^*)(q - q^*)$ of $y(q)$ at $q = q^*$ in the direction $q - q^*$ exists in $W_V(0,T)$ and it is a unique weak solution of the evolution equation*

$$\left\{\begin{aligned} & z'' + (\alpha^{*2} + \alpha_0)Az' + (\beta^{*2} + \beta_0)Az + \sum_{i=1}^{L}(\gamma_i^*\kappa_i^* \cos\kappa_i^* y^*)z + \delta^* z \\ = \; & 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* + \textstyle\sum_{i=1}^{L}(\gamma_i^* - \gamma_i)\sin\kappa_i^* y^* \\ & + \sum_{i=1}^{L}(\gamma_i^* \cos\kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f \quad in \quad (0,T), \\ & z(0) = z'(0) = 0, \end{aligned}\right. \tag{3.6}$$

*where $y^* = y(q^*)$.*

Since the map $q \to y(q) : \mathcal{P} \to W_V(0,T)$ is Gâteaux differentiable at $q^*$ in the direction $q - q^*$, the inequality (3.5) is equivalent to

$$\langle Cy(q^*) - z_d, Cz\rangle_{K',K} \geq 0, \quad \forall q \in \mathcal{P}_{ad}, \tag{3.7}$$

where $z$ is the solution of (3.6). To avoid the identification problem to be complicated we study the problem according to two types of simple observations as follows:

1. Observe the distributed state $Cy(q) = y(q) \in L^2(0,T;H)$ and take $K = L^2(0,T;H)$;
2. Observe the time terminal state $Cy(q) = y(q;T) \in H$ and take $K = H$.

*1. Case of $Cy(q) = y(q) \in L^2(0,T;H)$*

In this case we give the cost functional by

$$J(q) \quad = \quad r\|y(q) - z_d\|^2_{L^2(0,T;H)} + (Mq, q), \tag{3.8}$$

where $z_d \in L^2(0,T;H)$ and $r > 0$. Then the necessary condition (3.7) with respect to (3.8) is written by

$$r(y(q^*) - z_d, z)_{L^2(0,T;H)} + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \tag{3.9}$$

Hence by standard arguments we have the following theorem.

**Theorem 3.4.** *The optimal parameter $q^*$ for the cost* (3.8) *is characterized by the two states $y = y(q^*), p = p(q^*)$ of equations*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^{L} \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f \quad \text{in} \quad (0,T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \tag{3.10}$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^{L}(\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)p + \delta^* p = r(y(q^*) - z_d) \quad \text{in} \quad (0,T), \\ p(T) = p'(T) = 0. \end{cases} \tag{3.11}$$

*and one inequality*

$$\begin{aligned} &\int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* \\ &+ \sum_{i=1}^{L}(\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^{L}(\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f \rangle \, dt \\ &+ (Mq^*, q^* - q) \geq 0 \quad \text{for all } q \in \mathcal{P}_{ad}. \end{aligned} \tag{3.12}$$

2. *Case of $Cy(q) = y(q;T) \in H$*

In this case the cost functional is given by

$$J(q) = r|y(q;T) - z_d|^2 + (Mq, q), \tag{3.13}$$

where $z_d \in H$ and $r > 0$. Then the necessary condition (3.7) with respect to (3.13) is written by

$$r(y(q^*;T) - z_d, z(T)) + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \tag{3.14}$$

Thus we have the following theorem.

**Theorem 3.5.** *The optimal parameter $q^*$ for the cost* (3.13) *is characterized by the two states $y = y(q^*), p = p(q^*)$ of equations*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^{L} \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f \quad \text{in} \quad (0,T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \tag{3.15}$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^{L}(\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)p + \delta^* p = 0 \quad in \quad (0,T), \\ p(T) = 0, \quad p'(T) = -r(y(q^*;T) - z_d). \end{cases} \tag{3.16}$$

*and one inequality*

$$\int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* \tag{3.17}$$
$$+ \sum_{i=1}^{L}(\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^{L}(\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f\rangle \, dt$$
$$+ (Mq^*, q - q^*) \geq 0, \qquad \forall q \in \mathcal{P}_{ad}.$$

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