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Oscillation theorems of quasilinear elliptic equations with arbitrary nonlinearities

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1 Introduction and Main Results

In asymptotic theory of differential equations it is an important probrem to determine whether solutions of equations under consideration are oscillatory or not. We will establish oscillation criteria for solutions of quasilinear elliptic equations with the leading term $\Delta_m u = \mathrm{div}(|Du|^{m-2}Du)$. To begin with we give the definition of oscillation precisely:

Definition. A continuous function defined in an exterior domain in $\mathbb{R}^N$, $N \geqq 2$, is said to be *oscillatory* if there is a sequence of its zeros diverging to ∞; otherwise *nonoscillatory.*

Let us consider the equation

$$\Delta_m u + a(x) f(u) = 0 \tag{1}$$

under the following conditions:

(i) $N \geqq 2, m > 1$ and $N > m$;

(ii) a is a nonnegative continuous function difined in an exterior domain in $\mathbb{R}^N$;

(iii) $f \in C(\mathbb{R}\backslash\{0\}; \mathbb{R}\backslash\{0\})$ is an odd function satisfying $f(u) > 0$ for $u > 0$.

Throughout the article by a solution of (1) is meant a function u which is defined near ∞ and satisfies (1) there.

Notation. Let $a_*(r)$ and $a^*(r)$ be continuous functions defined near $+\infty$ satisfying

$$0 \leqq a_*(|x|) \leqq a(x) \leqq a^*(|x|), \qquad |x| \geqq r_0,$$

where $r_0 > 0$ is a sufficiently large number.

When $f(u) = |u|^{\sigma-1}u$, $\sigma > 0$, oscillation criteria for (1), which can be regarded as generalizations of earilier results in [1, 5], have been obtained in [4]. The case of $m = 2$ has been treated in [3]. The arguments developed in these works are mainly based on asymptotic analysis of ordinary differential equations. We here intend to unify these results by proceeding further in this direction.

Our main results are as follows:

Theorem 1. *Let a_* be nondecreasing near $+\infty$. Then every solution of (1) is oscillatory if*

$$\int^{\infty} r^{N-1} a_*(r) f\left(cr^{-\frac{N-m}{m-1}}\right) dr = \infty \quad \textit{for all } c > 0. \tag{2}$$

To see the sharpness of Theorem 1 we give an existence theorem of nonoscillatory weak solutions:

Theorem 2. *Let $r^{\frac{m(N-1)}{m-1}} a_*(r)$ be nondecreasing near $+\infty$. Then (1) has a positive (weak) solution u satisfying*

$$c_1|x|^{-\frac{N-m}{m-1}} \leqq u(x) \leqq c_2|x|^{-\frac{N-m}{m-1}} \quad a.e.{-x} \tag{3}$$

near ∞ for some constants $c_1, c_2 > 0$ provided that

$$\int^{\infty} r^{N-1} a^*(r) f\left(cr^{-\frac{N-m}{m-1}}\right) dr < \infty \quad \textit{for some } c > 0. \tag{4}$$

Remark 1. When $a(x)$ has radial symmetry, we can construct the positive solution referred in Theorem 2 as a radial function.

For the autonomous equation

$$\Delta_m u + f(u) = 0 \tag{5}$$

we can completely characterize oscillatory behavior of every solution via Theorems 1 and 2 as shown below:

Corollary 1. *Every solution of (5) is oscillatory if and only if*

$$\int^{\infty} r^{N-1} f\left(r^{-\frac{N-m}{m-1}}\right) dr = \infty.$$

Theorem 1 is not applicable to (1) when a_* is not nondecreasing. However the following may be applicable in such a case:

Theorem 3. *Let $N > 2$ (and $m = 2$), and*

$$\liminf_{|x|\to\infty} |x|^l a(x) > 0 \quad \textit{for some } l \leqq 2.$$

Then every solution of

$$\Delta u + a(x) f(u) = 0$$

is oscillatory if

$$\int^\infty r^{N-1-l-\varepsilon} f\left(r^{2-N}\right) dr = \infty \quad \textit{for some } \varepsilon > 0. \tag{6}$$

Remark 2. (i) We conjecture that analogous results to Theorem 3 hold for (1) with $m \neq 2$.
(ii) The condition "$\varepsilon > 0$" in (6) can not be weakened to "$\varepsilon \geqq 0$".

Example. Let us consider the equation

$$\Delta u + \frac{\lambda}{|x|^2} u = 0, \quad N \geqq 3 \tag{7}$$

for $|x| \geqq 1$, where $\lambda > 0$ is a constant. It is known that:
(a) every solution of (7) is oscillatory if

$$\lambda > (N-2)^2/4;$$

(b) there is a positive solution of (7) of the form $|x|^\rho$, where ρ is a real root of the quadratic equation $\rho^2 + (N+2)\rho + \lambda = 0$ if

$$\lambda \leqq (N-2)^2/4.$$

These facts show that the monotonicity of a_* required in the assumption of Theorem 1 can not be dropped, and that (ii) of Remark 2 is true.

2 Sketch of proofs

We only give the main ideas of the proofs here. The detailed proofs will appear in forthcoming papers. To prove Theorem 1 we prepare an improtant proposition of comparison type.

Proposition 1. *If PDE (1) has a nonoscillatory solution u, then the ordinary differential equation*

$$r^{1-N}\left(r^{N-1}|v'|^{m-2}v'\right)' + a_*(r) f(v) = 0 \tag{8}$$

has a positive solution v satisfying

$$0 < v(r) \leqq \min_{|x|=r} |u(x)|$$

for sufficiently large r.

The following, which reduces oscillation criteria for PDE (1) to those for ODE (8), is an immediate consequence of Proposition 1:

Corollary 2. *If ODE (8) does not have eventually positive solutions near* $+\infty$*, then every solution of PDE (1) is oscillatory.*

To prove Theorem 1 it suffices to show that ODE (8) has no eventually positive solutions under condition (2). Theorem 3 can be proved similarly. The details are, however, omitted.

We turn to the proof of Theorem 2. Our proof is based on the supersolution-subsolution method which is described in [2], for example.

Let $v(r)$ be a positive solution of the ODE

$$r^{1-N}\left(r^{N-1}|v'|^{m-2}v'\right)' + a^*(r)f(v) = 0 \tag{9}$$

for $r \geqq r_0$, sufficiently large. Then, the function $\bar{u}(x) \equiv v(r)$, $r = |x|$, is a (weak) supersolution of PDE (1). In fact, we obtain

$$\begin{aligned}
&\Delta_m \bar{u}(x) + a(x)f(\bar{u}(x)) \\
&= r^{1-N}\left(r^{N-1}|v'(r)|^{m-2}v'(r)\right)' + a(x)f(v(r)) \\
&\leqq r^{1-N}\left(r^{N-1}|v'(r)|^{m-2}v'(r)\right)' + a^*(r)f(v(r)) = 0.
\end{aligned}$$

We seek a positve solution of (9) as a positive solution of the integral equation

$$v(r) = \frac{N-m}{m-1}\int_r^\infty s^{-\frac{N-1}{m-1}}\left\{b^{m-1} - \left(\frac{m-1}{N-m}\right)^{m-1}\int_s^\infty t^{N-1}a^*(t)f(v(t))dt\right\}^{\frac{1}{m-1}} ds,$$

where $b > 0$ is a suitable constant. Indeed, by employing the Schauder-Tychonoff fixed point theorem we can construct a positive solution $v_1(r)$ of this integral equation under condition (4) satisfying $v_1(r) \sim b_1 r^{-\frac{N-m}{m-1}}$ as $r \to \infty$ with some constant $b_1 > 0$. On the other hand, for a constant $b_2 \in (0, b_1)$ the function $v_2(r) = b_2 r^{-\frac{N-m}{m-1}}$, $r = |x|$, is obviously a subsolution of (1). Since $v_1(|x|) \geqq v_2(|x|)$ near ∞, we get a positive (weak) solution u of (1) satisfying $v_1(|x|) \leqq u(x) \leqq v_2(|x|)$ a.e.$-x$ near ∞ by the supersolution-subsolution method. This $u(x)$ clearly satisfies (3) for some c_1 and $c_2 > 0$.

References

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