数理解析研究所講究録 1373 巻 2004 年 71-79

Exact Formulation of Stochastic EMQ Model for an Unreliable Production System

B.C. Giri and T. Dohi
Department of Information Engineering, Hiroshima University,
1-4-1 Kagamiyama, Higashi-Hiroshi-ma 739-8527, Japan.

Abstract

The paper presents an exact formulation of stochastic EMQ model for an unreliable production system under a general framework in which the time to machine failure, corrective and preventive repair times are taken as random variables. The criteria for the existence and uniqueness of the optimal production time are derived under arbitrary as well as specific failure and repair time distributions. For exact financial implications of the lot sizing decisions, the model is further analyzed based on the net present value (NPV) approach. Numerical examples are devoted to find the optimal production policies of the developed models and examine the sensitivity of some model-parameters. Computational results show that the decision based on the NPV approach is superior to that based on the long-run average cost approach, though the performance level strongly depends on the pertinent failure and repair distributions.

1. Introduction

The Economic Manufacturing Quantity (EMQ) problem is one of the oldest inventory/production control problem being investigated from time to time under various realistic situations. Most of the EMQ models developed in the literature assume that the production process is perfectly reliable *i.e.*, the facility never fails and hence its maintenance is ignored. The issue of interdependence between production and maintenance policies was raised first by McCall [1] who suggested that simulation perhaps is the most appropriate tool for analyzing a production/inventory problem in an unreliable environment. Bielecki and Kumar [2] showed that there exists a range of parameter values describing an unreliable manufacturing system for which zero inventory policy is exactly optimal even when the production capacity is uncertain. The steady state distribution of the inventory level and some important system characteristics related to both machine utilization and service level to customers in an unreliable production environment were obtained by Posner and Berg [3]. Groenevelt *et al.* [4] analyzed the effects of machine breakdowns and corrective maintenance on economic lot sizing decisions. Assuming exponential inter-failure time and instantaneous repair, they showed that the optimal lot size is greater than that of the corresponding classical EMQ model. In the subsequent article [5], they investigated the issue of safety stocks required to meet a managerially prescribed service level under a simplified assumption of exponential failure time and randomly distributed repair time. After the seminal works by Groenevelt *et al.* [4,5], a number of EMQ models with stochastic machine breakdowns and repairs have been reported in the literature, *e.g.*, see Kim and Hong [6], Abboud [7], Dohi *et al.* [8,9], Makis and Fung [10] and their references.

The studies on EMQ problems with stochastic machine breakdown and repairs have been performed assuming negligible corrective repair time and without PM (Groenevelt *et al.* [4], Kim and Hong [6]), constant corrective repair time and without PM (Kim *et al.* [11]), arbitrarily distributed corrective repair time and without PM (Abboud [7]). Preventive maintenance (PM) with negligible time is defined as a part of machine set up in Dohi *et al.* [8]. EMQ models without PM or with negligible PM time can not provide appropriate production-maintenance strategies especially when the production environment is unreliable. On the other hand, optimal lot sizing policies of the EMQ models have been derived by minimizing the long-run average cost in the steady state, assuming that the system will continue to operate over an infinite planning horizon. The average cost approach does not reflect the time value of money. In economy, money is endowed with time and its value ought to reduce as time passes if a greater economic change or revolution does not take place. Since the cost of capital tied up in inventory is included as part of the inventory carrying cost, so in theory, a more correct approach would be to determine the control variables by minimizing the net present value (NPV) of the expected total cost over all future time.

The purpose of this paper is to present an exact formulation of stochastic EMQ model for an unreliable production system under a general framework in which the time to machine failure, corrective and preventive repair times are assumed to be random variables. Moreover, it is aimed to study the proposed model under the NPV or discounted cash flow (DCF) approach, for exact financial implications of the lot sizing decisions.

2. The general EMQ model

Notations:

X : non-negative i.i.d. random variable denoting time to machine failure

$F_X(t)$: failure time distribution with p.d.f. $f_X(t) = dF_X(t)/dt$

$G_1(l_1)$: corrective repair time distribution with p.d.f. $g_1(l_1)$ and mean m_1^{-1} (> 0)

$G_2(l_2)$: preventive repair time distribution with p.d.f. $g_2(l_2)$ and mean m_2^{-1} (> 0)

p (> 0) : production rate

d $(< p)$: demand rate

$C_0(> 0)$: set up cost

$C_1(> 0)$: corrective repair cost per unit time

$C_2(< C_1)$: preventive repair cost per unit time

$C_i(> 0)$: inventory holding cost per unit product per unit time

$C_s(> 0)$: shortage cost per unit product

Q (> 0) : order quantity

β (> 0) : discount factor

Model description

Consider a single-unit single-item production system in which the facility may fail at most once during a production phase. The production process starts at time $t = 0$ with the aim of producing a lot of size Q. If the machine failure does not occur until time $t = Q/p$ then the production process is stopped and preventive repair is carried out to return back the machine to the same initial working condition before the start of the next production cycle. If, however, the machine fails before producing Q units then the corrective repair action is started immediately. During preventive or corrective repair, the demand is met first from the accumulated inventory. If there is sufficient stock to meet the demand during machine repair then the next production starts when the on-hand inventory is exhausted. Since the repair time is assumed to be a random variable, the on-hand inventory may be depleted before the repair is completed. The shortages, if occured, are not delivered after machine repair. To avoid an unrealistic decision making, we assume $\underline{Q} \leq Q \leq \overline{Q}$, where $\underline{Q}$ and $\overline{Q}$, the lower and upper limits of the production lot size, respectively are prescribed in advance by the decision maker.

We define the time interval between two successive production start points as one (repeating) cycle. Then, by conditioning on the time to machine failure, the mean time length of one cycle can be obtained as

$$\begin{aligned} T_0(Q) &= \int_0^\infty \text{E[duration of a cycle} \mid X = t] \; f_X(t)dt \\ &= \int_0^{Q/p} \left[\int_0^{(p-d)t/d} \frac{pt}{d} \, dG_1(l_1) + \int_{(p-d)t/d}^\infty (t + l_1) \, dG_1(l_1) \right] dF_X(t) \\ &+ \int_{Q/p}^\infty \left[\int_0^{(p-d)Q/(pd)} \frac{Q}{d} \, dG_2(l_2) + \int_{(p-d)Q/(pd)}^\infty (Q/p + l_2) \, dG_2(l_2) \right] dF_X(t), \quad (1) \end{aligned}$$

where each term in the right hand side of (1) corresponds to an event represented by the bound of integration. Similarly, by conditioning on the time to machine failure, the expected total cost for one cycle can be obtained as

$$\begin{aligned} S_0(Q) &= C_0 + C_1 \int_0^{Q/p} \int_0^\infty l_1 \, dG_1(l_1) \, dF_X(t) + C_2 \int_{Q/p}^\infty \int_0^\infty l_2 \, dG_2(l_2) \, dF_X(t) \\ &+ C_i \left[\int_0^{Q/p} \frac{(p-d)pt^2}{2d} \, dF_X(t) + \int_{Q/p}^\infty \frac{(p-d)Q^2}{2pd} \, dF_X(t) \right] \\ &+ C_s d \int_0^{Q/p} \int_{(p-d)t/d}^\infty \left\{ l_1 - \frac{(p-d)t}{d} \right\} dG_1(l_1) \, dF_X(t) \\ &+ C_s d \int_{Q/p}^\infty \int_{(p-d)Q/(pd)}^\infty \left\{ l_2 - \frac{(p-d)Q}{pd} \right\} dG_2(l_2) \, dF_X(t). \quad (2) \end{aligned}$$

By the well-known renewal reward theorem (Ross [12]), the expected cost per unit time in the steady state is given by

$$C_0(Q) = \lim_{t \to \infty} \frac{\text{E[total cost on}(0, t)]}{t} = S_0(Q)/T_0(Q). \quad (3)$$

The problem is to seek the optimal production lot size Q^* which minimizes $C(Q)$, subject to $\underline{Q} \leq Q^* \leq \overline{Q}$. For convenience, let $t_0 = Q/p$. Then $\underline{t_0} = \underline{Q}/p$ and $\overline{t_0} = \overline{Q}/p$ become the lower and upper limits of t_0, respectively. Define the numerator of the derivative of $C(t_0) =$

$S(t_0)/T(t_0)$ where $S(t_0) = S_0(pt_0)$ and $T(t_0) = T_0(pt_0)$, with respect to t_0 divided by $1-F_X(t_0)$ as $q(t_0)$, *i.e.*,

$$\begin{aligned} q(t_0) &= \Bigg[r_X(t_0)\Big\{C_1 m_1^{-1} - C_2 m_2^{-1} + C_s d\left(m_1^{-1} - m_2^{-1}\right) + C_s d\int_0^{(p-d)t_0/d} G_1(l_1)dl_1 \\ &\quad - C_s d\int_0^{(p-d)t_0/d} G_2(l_2)dl_2\Big\} + (p-d)\left\{\frac{C_i p t_0}{d} - C_s\overline{G_2}\left(\frac{(p-d)t_0}{d}\right)\right\}\Bigg] T(t_0) \\ &\quad - \Bigg[r_X(t_0)\Big\{\left(m_1^{-1} - m_2^{-1}\right) + \int_0^{(p-d)t_0/d} G_1(l_1)dl_1 - \int_0^{(p-d)t_0/d} G_2(l_2)dl_2\Big\} \\ &\quad + \frac{(p-d)}{d} G_2\left(\frac{(p-d)t_0}{d}\right) + 1\Bigg] S(t_0), \end{aligned}$$

where $r_X(t) = f_X(t)/\left(1-F_X(t)\right)$ is the failure (hazard) rate which is assumed to be a continuous and differentiable function of time t. Differentiating $q(t_0)$ with respect t_0, we get

$$\frac{dq(t_0)}{dt_0} = \phi(t_0)T(t_0) + \psi(t_0)[C_s d\, T(t_0) - S(t_0)],$$

where $\phi(t_0) = \frac{dr_X(t_0)}{dt_0}\left(C_1 m_1^{-1} - C_2 m_2^{-1}\right) + \frac{C_i p(p-d)}{d}$,

$$\psi(t_0) = \frac{dr_X(t_0)}{dt_0}\left\{\left(m_1^{-1} - m_2^{-1}\right) + \int_0^{(p-d)t_0/d} G_1(l_1)dl_1 - \int_0^{(p-d)t_0/d} G_2(l_2)dl_2\right\} + \frac{(p-d)^2}{d^2}$$
$$\times\ g_2\left(\tfrac{(p-d)t_0}{d}\right) + \tfrac{(p-d)}{d}\, r_X(t_0)\left\{G_1\left(\tfrac{(p-d)t_0}{d}\right) - G_2\left(\tfrac{(p-d)t_0}{d}\right)\right\}.$$

The problem is now reduced to finding the optimal production time t_0^* ($\underline{t_0} \le t_0^* \le \overline{t_0}$) which minimizes $C(t_0)$. To derive the criteria for the existence and uniqueness of t_0^*, we make the following assumptions:

(A-1) $\phi(\overline{t_0}) > 0$.

(A-2) $G_1(\cdot)$ and $G_2(\cdot)$ $[G_1 > G_2]$ are both continuous and increasing functions in the interval $[0, (p-d)\underline{t_0}/d]$ such that $\psi(\underline{t_0}) \ge 0$.

(A-3) The opportunity loss per unit demand $C_s d$ and the long-run average cost $C(t_0)$ are such that $C_s d < C(t_0) < C_s d + \phi(\overline{t_0})/\psi(\underline{t_0})\ \forall\ t_0 \in [\underline{t_0}, \overline{t_0}]$.

Theorem 1: Suppose that the failure time distribution $F_X(t)$ is IFR (Increasing Failure Rate). Under assumptions (A-1)-(A-3), (i) if $q(\underline{t_0}) < 0$ and $q(\overline{t_0}) > 0$ then there exists a finite and unique optimal production time t_0^* ($0 < \underline{t_0} < t_0^* < \overline{t_0} < \infty$) satisfying the non-linear equation $q(t_0^*) = 0$.

(ii) If $q(\overline{t_0}) \le 0$ then $C(t_0)$ is a decreasing function of $t_0 \in [\underline{t_0}, \overline{t_0}]$ and therefore, the optimal production time is $t_0^* = \overline{t_0}$. If $q(\underline{t_0}) \ge 0$ then $C(t_0)$ is an increasing function of $t_0 \in [\underline{t_0}, \overline{t_0}]$ and hence the optimal production time is $t_0^* = \underline{t_0}$.

Proof: The proof of the first part of the theorem is straightforward as $q(t_0)$ is an increasing function in the interval $[\underline{t_0}, \overline{t_0}]$ by assumptions (A-1)-(A-3). The second part of the theorem follows directly.

Remarks: If $m_1 \le m_2$, *i.e.*, the preventive repair rate is not less than that of the corrective repair then the assumptions (A-1) and (A-2) are clearly validated. Again, these assumptions are trivially true when $m_2 \to \infty$ *i.e.* the regular repair is instantaneous.

The case of exponential failure and exponential repair:

Let $F_X(t) = 1 - \exp\{-\lambda t\}$, $\lambda\ (> 0)$; $G_1(l_1) = 1 - \exp\{-\mu_1 l_1\}$, $\mu_1 > 0$ and $G_2(l_2) = 1 - \exp\{-\mu_2 l_2\}$, $\mu_2 > 0$. Then, the expected cost per unit time in the steady state is $C_1(t_0) = S_1(t_0)\ /\ T_1(t_0)$, where

$$\begin{aligned} S_1(t_0) &= C_0 + \frac{C_1}{\mu_1}\left(1 - e^{-\lambda t_0}\right) + \frac{C_2}{\mu_2} e^{-\lambda t_0} + \frac{C_i(p-d)p}{2d}\left\{\frac{2}{\lambda^2}\left(1 - e^{-\lambda t_0}\right)\right. \\ &\quad \left. - \frac{2t_0}{\lambda} e^{-\lambda t_0}\right\} + \frac{C_s d\lambda}{\mu_1}\left\{\frac{1 - e^{-\{\lambda+\mu_1(\frac{p-d}{d})\}t_0}}{\lambda + \mu_1\left(\frac{p-d}{d}\right)}\right\} + \frac{C_s d}{\mu_2} e^{-\{\lambda+\mu_2(\frac{p-d}{d})\}t_0}, \\ T_1(t_0) &= \frac{p}{d\lambda}\left(1 - e^{-\lambda t_0}\right) + \frac{1}{\mu_2} e^{-\{\lambda+\mu_2(\frac{p-d}{d})\}t_0} + \frac{\lambda}{\mu_1}\left\{\frac{1 - e^{-\{\lambda+\mu_1(\frac{p-d}{d})\}t_0}}{\lambda + \mu_1\left(\frac{p-d}{d}\right)}\right\}. \end{aligned}$$

To derive the criteria for the existence and uniqueness of the optimal production time t_0^*, we define

$$\begin{aligned} q_1(t_0) = &\left[\lambda(C_1/\mu_1 - C_2/\mu_2) + \tfrac{C_i(p-d)pt_0}{d} + \tfrac{C_s d\lambda}{\mu_1} e^{-\mu_1(\frac{p-d}{d})t_0} - \tfrac{C_s d\{\lambda+\mu_2(\frac{p-d}{d})\}}{\mu_2} e^{-\mu_2(\frac{p-d}{d})t_0}\right] \\ &\times\ T_1(t_0) - \left[\tfrac{p}{d} + \tfrac{\lambda}{\mu_1} e^{-\mu_1(\frac{p-d}{d})t_0} - \tfrac{\lambda+\mu_2(\frac{p-d}{d})}{\mu_2} e^{-\mu_2(\frac{p-d}{d})t_0}\right] S_1(t_0), \end{aligned}$$

$$R = \left\{\lambda + \mu_2\left(\tfrac{p-d}{d}\right)\right\} e^{-\mu_2(\frac{p-d}{d})\overline{t_0}} - \lambda e^{-\mu_1(\frac{p-d}{d})\underline{t_0}}.$$

and make the following assumptions:

(A-4) $R \leq 0$ and $C_1(t_0)$ is bounded below by $C_s d$.

(A-5) $R \geq 0$ and $C_1(t_0)$ is bounded by $C_s d$ and $C_s d + C_i p/R$.

Theorem 2: Under assumption (A-4) or (A-5) (i) if $q_1(\underline{t_0}) < 0$ and $q_1(\overline{t_0}) > 0$ then the unique optimal production time t_0^* ($\underline{t_0} < t_0^* < \overline{t_0}$) which minimizes $C_1(t_0)$ is given by a positive real root of the non-linear equation $q_1(t_0) = 0$.

(ii) If $q_1(\overline{t_0}) \leq 0$ then the optimal production time is $t_0^* = \overline{t_0}$. On the other hand, if $q_1(\underline{t_0}) \geq 0$ then $t_0^* = \underline{t_0}$.

Proof: The necessary condition for a minimum of $C_1(t_0)$ gives $q_1(t_0) = 0$. Differentiation of $q_1(t_0)$ with respect to t_0 yields

$$\begin{aligned} \tfrac{dq_1(t_0)}{dt_0} = \tfrac{p-d}{d}&\left[C_i p\, T_1(t_0) + \left\{\lambda e^{-\mu_2(\frac{p-d}{d})t_0} + \mu_2\left(\tfrac{p-d}{d}\right) e^{-\mu_2(\frac{p-d}{d})t_0} - \lambda e^{-\mu_1(\frac{p-d}{d})t_0}\right\}\right. \\ &\left.\times \left\{C_s d T_1(t_0) - S_1(t_0)\right\}\right]. \end{aligned}$$

The first part of the theorem follows since $dq_1(t_0)/dt_0 > 0$ for all $t_0 \in [\underline{t_0}, \overline{t_0}]$, by assumption (A-4) or (A-5). The proof of the second part is similar to that of Theorem 1.

3. The net present value analysis

The present value of the expected inventory holding cost per cycle can be formulated as

$$\begin{aligned} H_c(t_0) &= C_i\left[\int_0^{t_0}\left\{\int_0^t (p-d)ye^{-\beta y}dy + \int_t^{pt/d} (pt - dy)\, e^{-\beta y}dy\right\} dF_X(t)\right. \\ &\quad \left. + \int_{t_0}^{\infty}\left\{\int_0^{t_0} (p-d)ze^{-\beta z}dz + \int_{t_0}^{pt_0/d} (pt_0 - dz)\, e^{-\beta z}dz\right\} dF_X(t)\right] \end{aligned}$$

Similarly, the present value of the expected shortage cost per cycle is

$$\begin{aligned} S_c(t_0) &= C_s d\Big[\int_0^{t_0}\int_{(p-d)t/d}^{\infty}\int_0^{l_1-(p-d)t/d} e^{-\beta(u+\frac{pt}{d})}\, dv\, dG_1(l_1)\, dF_X(t) \\ &\quad + \int_{t_0}^{\infty}\int_{(p-d)t_0/d}^{\infty}\int_0^{l_2-(p-d)pt_0/d} e^{-\beta(v+pt_0/d)}\, dv\, dG_2(l_2)\, dF_X(t)\Big] \end{aligned}$$

and the present value of the expected repair costs for one cycle is

$$\begin{aligned} R_c(t_0) &= C_1\int_0^{t_0}\int_0^{\infty}\int_0^{l_1} e^{-\beta(t+y_1)}dy_1\, dG_1(l_1)\, dF_X(t) \\ &\quad + C_2\int_{t_0}^{\infty}\int_0^{\infty}\int_0^{l_2} e^{-\beta(t_0+y_2)}dy_2\, dG_2(l_2)\, dF_X(t). \end{aligned}$$

Hence, the NPV of the expected total cost for one cycle is given by

$$S_\beta(t_0) = C_0 + H_c(t_0) + S_c(t_0) + R_c(t_0).$$

The NPV of mean unit cost after one cycle, can be obtained as

$$\begin{aligned} \delta_\beta(t_0) &= \int_0^{t_0}\Big\{\int_0^{(p-d)t/d} e^{-\frac{\beta pt}{d}}\, dG_1(l_1) + \int_{(p-d)t/d}^{\infty} e^{-\beta(t+l_1)}\, dG_1(l_1)\Big\}\, dF_X(t) \\ &\quad + \int_{t_0}^{\infty}\Big\{\int_0^{(p-d)t_0/d} e^{-\beta pt_0/d}\, dG_2(l_2) + \int_{(p-d)t_0/d}^{\infty} e^{-\beta(t_0+l_2)}\, dG_2(l_2)\Big\}\, dF_X(t). \end{aligned}$$

Hence, the NPV of the expected total cost over the time horizon $[0,\infty)$, when the initial point in time is taken to be the starting point of a production lot, is

$$TC_\beta(t_0) = \sum_{n=0}^{\infty} S_\beta(t_0)\Big[\delta_\beta(t_0)\Big]^n = \frac{S_\beta(t_0)}{1-\delta_\beta(t_0)}. \tag{4}$$

By perturbation of the instantaneous discount rate, we can easily establish the following relationship:

$$C(t_0) = \lim_{\beta\to 0} \beta\cdot TC_\beta(t_0), \tag{5}$$

where the evaluation of the limiting value in the right hand side of (5) is due to the l'Hospital's theorem. Our objective is to find the optimal production time t_0^* ($\underline{t_0} \le t_0^* \le \overline{t_0}$) which minimizes $TC_\beta(t_0)$.

The case of exponential failure and exponential repair :

For exponential failure and exponential repair time distributions as defined in the previous section, the NPV of the expected total cost per cycle can be obtained as

$$\begin{aligned} S_{1\beta}(t_0) &= C_0 + \frac{\lambda C_1}{\beta+\mu_1}\Big\{\frac{1-e^{-(\lambda+\beta)t_0}}{\lambda+\beta}\Big\} + \frac{C_2}{\beta+\mu_2}e^{-(\lambda+\beta)t_0} \\ &\quad + \frac{C_i}{\beta^2}\Big[(p-d)(1-e^{-\lambda t_0}) - \frac{\lambda p}{\lambda+\beta}\big\{1-e^{-(\lambda+\beta)t_0}\big\} + \frac{\lambda d}{\lambda+\beta p/d} \\ &\quad \times\Big\{1-e^{-(\lambda+\frac{\beta p}{d})t_0}\Big\}\Big] + \frac{C_i}{\beta^2}\Big[(p-d)e^{-\lambda t_0} - pe^{-(\lambda+\beta)t_0} + de^{-(\lambda+\frac{\beta p}{d})t_0}\Big] \\ &\quad + \frac{C_s d\lambda}{\beta+\mu_1}\Big[\frac{1-e^{-\left\{\lambda+\frac{\beta p}{d}+\frac{\mu_1(p-d)}{d}\right\}t_0}}{\lambda+\beta p/d+\mu_1(p-d)/d} + \frac{C_s d}{\beta+\mu_2}e^{-\{\lambda+\frac{\beta p}{d}+\frac{\mu_2(p-d)}{d}\}t_0}\Big]. \end{aligned}$$

On the other hand, the NPV of mean unit cost after one cycle is

$$\begin{aligned}\delta_{1\beta}(t_0) &\equiv \lambda\left[\frac{1-e^{-(\lambda+\frac{\beta p}{d})t_0}}{\lambda+\frac{\beta p}{d}}\right]-\frac{\lambda\beta}{\beta+\mu_1}\left[\frac{1-e^{-\{\lambda+\frac{\beta p}{d}+\frac{\mu_1(p-d)}{d}\}t_0}}{\lambda+\frac{\beta p}{d}+\frac{\mu_1(p-d)}{d}}\right]\\ &+e^{-\frac{\beta p t_0}{d}}\left[e^{-\lambda t_0}-\frac{\beta}{\beta+\mu_2}e^{-\{\lambda+\frac{\mu_2(p-d)}{d}\}t_0}\right].\end{aligned}$$

The necessary condition for a minimum of $TC_{1\beta}(t_0)=S_{1\beta}(t_0)/\ (1-\delta_{1\beta}(t_0))$ gives

$$\begin{aligned}W_{1\beta}(t_0) &= \left[\left\{\frac{C_1\lambda}{\beta+\mu_1}-\frac{C_2(\lambda+\beta)}{\beta+\mu_2}\right\}e^{\frac{\beta(p-d)t_0}{d}}-\frac{C_ip}{\beta}\left\{1-e^{\frac{\beta(p-d)t_0}{d}}\right\}\right.\\ &\left.+\frac{C_sd\lambda}{\beta+\mu_1}e^{-\frac{\mu_1(p-d)t_0}{d}}-\frac{C_s}{\beta+\mu_2}\left\{\beta p+d\lambda+\mu_2(p-d)\right\}e^{-\frac{\mu_2(p-d)t_0}{d}}\right]\\ &\times[1-\delta_{1\beta}(t_0)]-\left[\frac{\lambda\beta}{\beta+\mu_1}e^{-\frac{\mu_1(p-d)t_0}{d}}+\frac{\beta p}{d}-\frac{\beta^2p+\lambda\beta d+\mu_2(p-d)\beta}{(\beta+\mu_2)d}\right.\\ &\left.\times\ e^{-\frac{\mu_2(p-d)t_0}{d}}\right]S_{1\beta}(t_0)\ =0.\end{aligned}$$

It can be shown that $W_{1\beta}(t_0)$ behaves as an increasing function in $[\underline{t_0},\overline{t_0}]$ when the annualized cost is bounded by C_sd (lower bound) and $C_sd+\phi_\beta/\psi_\beta$ (upper bound)

where $\quad \phi_\beta=\left[\beta\left\{\frac{C_1\lambda}{\beta+\mu_1}-\frac{C_2(\lambda+\beta)}{\beta+\mu_2}\right\}+C_ip\right]e^{\frac{\beta(p-d)\overline{t_0}}{d}}\ >\ 0,$

$$\psi_\beta=\frac{\mu_2\{\beta p+\lambda d+\mu_2(p-d)\}}{d(\beta+\mu_2)}\ e^{-\frac{\mu_2(p-d)\overline{t_0}}{d}}\ -\frac{\lambda\mu_1}{\beta+\mu_1}\ e^{-\frac{\mu_1(p-d)t_0}{d}}\ \geq\ 0.$$

Therefore, if $W_{1\beta}(\underline{t_0})<0$ and $W_{1\beta}(\overline{t_0})>0$ then, as before, the unique optimal production time t_0^* ($\underline{t_0}\leq t_0^*\leq\overline{t_0}$) can be obtained by solving the non-linear equation $W_{1\beta}(t_0)=0$.

4. Numerical example

We take the following parameter values for the model with exponential failure and exponential repair (corrective and preventive) time distributions: $d=30$, $p=150$, $C_0=500$, $C_i=0.5$, $C_s=1.25$, $C_1=250, C_2=120$, $\mu_1=4$, $\mu_2=10, \beta=0.05$, $\underline{Q}=200$, $\overline{Q}=700$. The influence of the failure rate λ, repair rates μ_1 and μ_2 on the optimal production policy are shown in Tables 1 and 2.

Table 1 Influence of λ on the optimal production policy.

	Average cost model		NPV model	
λ	t_0^*	$C_1(t_0^*)$	t_0^*	$TC_{1\beta}(t_0^*)$
0.1	1.90597	115.368	1.80920	2640.98
0.2	1.96814	120.108	1.85634	2751.71
0.3	2.03427	125.086	1.90545	2867.25
0.4	2.10463	130.318	1.95654	2987.77
0.5	2.17949	135.819	2.00960	3113.41
0.6	2.25906	141.604	2.06459	3244.29
0.7	2.34357	147.684	2.12145	3380.46
0.8	2.43317	154.070	2.18010	3521.94
0.9	2.52799	160.769	2.24042	3668.68
1.0	2.62806	167.784	2.30227	3820.58

Table 2 Influence of μ_1 and μ_2 on the optimal production policy in the average cost model.

k	$(\mu_1,\ \mu_2) = (k, 10)$		$(\mu_1,\ \mu_2) = (4, k)$	
	t_0^*	$C_1(t_0^*)$	t_0^*	$C_1(t_0^*)$
1	2.08324	144.032	2.34871	136.330
2	2.10052	135.071	2.21523	133.114
3	2.10358	131.922	2.16958	131.975
4	2.10463	130.318	2.14653	131.392
5	2.10512	129.347	2.13262	131.037
6	2.10538	128.696	2.12332	130.799
7	2.10553	128.229	2.11666	130.628
8	2.10563	127.878	2.11165	130.499
8	2.10570	127.604	2.10776	130.399
10	2.10575	127.385	2.10463	130.318

From Table 1, it is to be noted that as the failure rate increases, the optimal production time, the expected cost rate and the NPV of the expected total cost over an infinite time horizon all increase gradually. Further, the optimal production lot size in average cost model is greater than the true optimum value based on discounting. Table 2 shows that the average cost rate has a decreasing trend with the increase in the corrective or preventive repair rate. For a meaningful comparison, we calculate the NPVs of the expected total cost over an infinite time span based on the optimal decisions of the NPV and average cost models. The results given in Table 3 indicates that the decision based on the NPV approach is more accuare than that of the long-run average cost approach.

Table 3 Comparison of the NPVs of the expected total cost based on optimal decisions of the NPV and average cost models when $\lambda = 0.3$.

β	t^*_{0-NPV}	t^*_{0-AVG}	$TC_{1\beta}(t^*_{0-NPV})$ (V_1)	$TC_{1\beta}(t^*_{0-AVG})$ (V_2)	$\left(\frac{V_2-V_1}{V_1}\right) \times 100$ (%)
0.05	1.90545	2.03427	2867.25	2871.30	0.14
0.10	1.79131	2.03427	1626.40	1634.36	0.49
0.15	1.69023	2.03427	1218.20	1229.60	0.93
0.20	1.60054	2.03427	1017.41	1031.60	1.39
0.25	1.52066	2.03427	899.08	915.36	1.81
0.30	1.44923	2.03427	821.68	839.35	2.15

t^*_{0-NPV} (t^*_{0-AVG}) : optimal production time in the NPV (average cost) model.

5. Concluding remarks

In this paper, we have presented an exact formulation of EMQ model with stochastic machine breakdown and repair under a general framework in which the time to machine failure, corrective and preventive repair times are assumed to be random variables. The long-run average cost in the steady state has been taken as a criterion for optimality. Moreover, for exact financial implications of the lot sizing decision, the proposed model has been studied based on the net present value (NPV) approach. To compare the performance of the traditional long-run average cost approach and the NPV approach, we have considered the NPVs of the expected total cost based on the optimal decisions of the two models. It has been observed from the numerical study that the decision based on the average cost model is inferior and can even be significantly worse than that of the NPV model.

References

[1] McCall JJ (1965). Maintenance policies for stochastically failing equipment: A survey. *Management Science* **11**: 493-524.

[2] Bielecki T and Kumar PR (1988). Optimality of zero-inventory policies for unreliable manufacturing systems. *Operations Research* **36**: 532-41.

[3] Posner MJM and Berg M (1989). Analysis of a production-inventory system with unreliable production facility. *Operaions Research Letter* **8**: 339-345.

[4] Groenevelt H, Pintelon L and Seidmann A (1992). Production lot sizing with machine breakdowns. *Management Science* **38**: 104-123.

[5] Groenevelt H, Pintelon L and Seidmann A (1992). Production batching with machine breakdowns and safety stocks. *Operations Research* **40**: 959-971.

[6] Kim CH and Hong, Y (1997). An extended EMQ model for a failure prone machine with general lifetime distribution.*International Journal of Production Economics* **49**: 215-223.

[7] Abboud NE (1997). A simple approximation of the EMQ model with Poisson machine failures. *Production Planning & Control* **8**, 385-397.

[8] Dohi T, Yamada Y, Kaio N and Osaki S (1997). The optimal lot sizing for unreliable economic manufacturing models. *International Journal of Reliability, Quality and Safety Engineering* **4**: 413-426.

[9] Dohi T, Okamura H and Osaki S (2001). Optimal control of preventive maintenance schedule and safety stocks in an unreliable manufacturing environment. *International Journal of Production Economics* **74**: 147-155.

[10] Makis V and Fung J (1998). An EMQ model with inspections and random machine failures. *Journal of the Operational Research Society* **49**: 66-75.

[11] Kim CH, Hong, Y and Kim, SY (1997). An extended optimal lot sizing model with an unreliable machine. *Production Planning & Control* **8**, 577-85.

[12] Ross SM (1993). *Introduction to Probability Models*, 5th ed., Academic Press, New York.