数理解析研究所講究録 1373 巻 2004 年 167-174

The Quiet Accumulation Game on a Linear Graph
— A Special Case —

William H.Ruckle
Department of Mathematical Science
Clemson University

神戸商科大学管理科学科　菊田　健作 (Kensaku Kikuta)
Department of Management Science
Kobe University of Commerce

Abstract. A quiet accumulation game on a linear graph is formulated as a two-person zero-sum game. Upper and lower bounds for the value of the game are given as well as pure strategies which assure those bounds by mixing them. Numerical examples are solved when the number of nodes is small.

Keywords: Search Theory, Zero-sum Games, Search Games

1. The Model.

The type of search game in this paper models a game-theoretic situation between an organization and the law enforcement agency. For example, a polluter attempts to illegally conceal a quantity of waste, and an enforcement agency tries to uncover this attempt. Another example is : An illicit organization, such as terrorist organization, attempts to accumulate a certain minimum amount of material and a law enforcement agency attempts to prevent this by means of a limited number of inspections.

Now, there are two players, called the hider and the seeker. There is a linear graph (N, E) where $N = \{1, ..., n\}$ is a finite set of nodes and $E = \{(1,2), ..., (n-1, n)\}$ is the set of arcs. At each turn the hider chooses only one of the empty nodes in V, and hides an (immobile) object there, and then the seeker examines only one node, without knowing the hider's choice. The object hidden is left there unless either the game ends or the seeker finds it. The seeker will find an object with certainty and remove it if it is at a node examined. At the beginning, all nodes are empty and the seeker is at Node 1, and the hider knows the seeker's initial location. At each turn the seeker can either move to one of the adjacent nodes along an arc and examine it or stay there and examine the node at which the seeker stays. The hider knows a node examined at each turn only when the seeker finds an object there. The hider can use this information to choose nodes on the following turns. The game ends and the hider wins (payoff 1) if at the end of any turn there are k objects remaining in k nodes (i.e., one object at each of k nodes). The hider loses (payoff 0) if after t turns the hider has failed to accumulate k objects. By definition, $k \leq min\{n, t\}$. The seeker is the minimizer of the hider's payoff. This game is called the quiet accumulation game on a linear graph (abbreviated as QAGLG) with n nodes, k locations and t turns.

In this paper we analyse the case of

$$k = t. \tag{1.1}$$

This assumption implies that the game ends as soon as the seeker finds an object. We give upper and lower bounds for the value of the game as well as special pure strategies which assure those bounds by mixing them. We solve numerical examples when k's are small, in which mixes of special pure strategies are optimal. For the reader the following diagram is helpful to understand the game clearly.

Kikuta/Ruckle[3] is the first of the study on accumulation games by these authors. It treats the noisy case, i.e., the case that the hider can know a node examined at every turn. Ruckle/kikuta [9] studies special cases of the quiet accumulation games. Ruckle [7] is a good survey on accumulation games. In the discrete case in [3], [5], [8] and [9], we can regard the games are on complete graphs. So this paper is the first on the games on special graphs by the authors.

Proposition 1.1. If $n \geq 2t+1$, the hider wins by choosing Nodes $n, n-1, ..., n-t+1$ in this order. If $n = t$, the seeker wins by examining any node at each turn.

From Proposition 1.1, it suffices to consider the case of

$$t+1 \leq n \leq 2t. \tag{1.2}$$

2. Payoff-Matrix and Pure Strategies.

In this section we define pure strategies of both players, give elementary properties of them, and then pay attention to undominated strategies.

By the assumption (1.1), the game ends once the seeker finds an object. So each player does not know previous choices of the opposite player. Hence for each player it suffices to consider the choices of his(her) own in the previous turns. Consequently, pure strategies of both players can be expressed as maps h and s from $T \equiv \{1, \ldots, t\}$ to N. So let $h(x)$ and $s(x)$ be the choices of the hider and the seeker at the x-th turn respectively. As usual, for any $T' \subseteq T$, we let $h(T') \equiv \{h(x) : x \in T'\}$ and $s(T') \equiv \{s(x) : x \in T'\}$. These are images of the functions h and s restricted to T' respectively, and means sets of previous choices in the turns in T'. Frequently we express as $h = (h(1), \ldots, h(t))$ and $s = (s(1), \ldots, s(t))$. Denote by $\mathcal{H}$ and $\mathcal{S}$ the sets of pure strategies of the hider and the seeker respectively. For $y \in N$, let

$$\mathcal{S}_y \equiv \{s \in \mathcal{S} : s(y) = y\} \text{ and } \mathcal{H}_{-y} \equiv \{h \in \mathcal{H} : y \notin h(T)\}.$$

Let $a(h, s)$ is the payoff to the hider, in which the hider and the seeker use h and s respectively. Then the game is characterized by a payoff-(zero-one-)matrix $A = (a(h, s))$ of the hider. The value of this game is denoted by $v \equiv v(n, k)$. Domination relations between pure strategies can be considered, based on this payoff-matrix. For $s, s' \in \mathcal{S}$, we say s (weakly) dominates $s\prime$ if and only if $a(h, s) \geq a(h, s')$ for all $h \in \mathcal{H}$ and there exists $h \in \mathcal{H}$ such that $a(h, s) > a(h, s')$. A strategy $s \in \mathcal{S}$ is called undominated if it is not (weakly) dominated by any other strategy. For $s \in \mathcal{S}$ and $y \in N$, we let $t_s(y) \equiv max\{x : s(x) = y\}$ if $y \in s(T)$ and $= 0$ if $y \notin s(T)$. $t_s(y)$ is the latest seeking time of the node y under the pure strategy s. Since the function h is one-to-one from T to N, the inverse h^{-1} is well-defined, by letting $h^{-1}(y) = t+1$ if $y \notin h(T)$.

Proposition 2.1. If $n = 2t$, the value of the game is $t/(t+1)$. An optimal strategy for the hider is to choose one of

$$(2t, 2t-1, \ldots, t+2, y), \quad 1 \leq y \leq t+1,$$

with probability $1/(t+1)$. An optimal strategy for the seeker is to choose one in $\mathcal{S}_y$ (say s_y), $1 \leq y \leq t+1$ with probability $1/(t+1)$.

From Proposition 2.1 and the condition (1.2), it suffices to consider the case

$$t+1 \leq n \leq 2t-1. \tag{2.1}$$

By the assumption (1.1), we have straightforwardly, for $h \in \mathcal{H}$ and $s \in \mathcal{S}$,

$$a(h,s) = 1 \text{ if and only if } t_s(y) < h^{-1}(y) \text{ for all } y \in N. \tag{2.2}$$

It is helpful for our analysis to express pure strategies for both players in a two-dimensional plane. So, for $s \in \mathcal{S}$ and $h \in \mathcal{H}$, we consider graphs of s and h in the turn-node plane (xy-plane), that is, $G(s) \equiv \{(x, s(x)) | 1 \leq x \leq t\}$ and $G(h) \equiv \{(x, h(x)) | 1 \leq x \leq t\}$. Here x means the x-th turn. Then we let

$$D(s) \equiv \bigcup_{t'=1}^{t} \{(x, s(t')) | 1 \leq x \leq t'\} = \bigcup_{y=1}^{n} \{(x,y) | x \leq t_s(y)\}. \tag{2.3}$$

If $(x, h(x)) \in D(s)$ for some x then $h(x) = s(x')$ for some $x' \geq x$, and so $a(h,s) = 0$. If $(x, h(x)) \notin D(s)$ for all x, then $t_s(y) < h^{-1}(y)$ for all y, and so $a(h,s) = 1$. So we have, for $h \in \mathcal{H}$ and $s \in \mathcal{S}$,

$$a(h,s) = 1 \text{ if and only if } D(s) \cap G(h) = \emptyset. \tag{2.4}$$

Lemma 2.2. For any pure strategy $s \in \mathcal{S}$ of the seeker, there exists a pure strategy $h \in \mathcal{H}$ of the hider such that $G(h) \cap D(s) = \emptyset$.

For a pure strategy $s \in \mathcal{S}$ of the seeker we define a pure strategy $h_s \in \mathcal{H}$ of the hider by Lemma 2.2.

3. Dominance Relations in Pure Strategies.

In this section we examine dominance relations in pure strategies for the seeker.

Proposition 3.1. For $s, s' \in \mathcal{S}$, s' (weakly) dominates s if and only if the set $D(s')$ strictly includes the set $D(s)$.

Remark. For any $s, s' \in \mathcal{S}$, $a(h_s, s) = 1 \geq a(h_s, s')$ and $a(h_{s'}, s') = 1 \geq a(h_{s'}, s)$. So there is no strict domination between pure strategies of the seeker.

Lemma 3.2. Suppose $s \in \mathcal{S}$ is undominated. Then there exists a undominated strategy $s' \in \mathcal{S}$ such that $D(s) = D(s')$ and

$$s'(\ell) \neq s'(\ell+1) \text{ for all } 1 \leq \ell \leq k-1. \tag{3.1}$$

Since the seeker moves to an adjacent node or stays at a node, it holds $|s(j) - s(j-1)| \leq 1$, for all $j = 2, \ldots, t$.. From this and Lemma 3.2, for the seeker it suffices to consider strategies which satisfy

$$|s(j) - s(j-1)| = 1, \text{ for all } j = 2, \ldots, t. \tag{3.2}$$

We give here extreme strategies which satisfy the condition (3.2). In Section 4.2, we will see that a mixed combination of these extreme strategies will be an optimal strategy for the seeker when $n = t+1$ and n is small.

$$s^1 \equiv (2,3,\dots,t+1), \quad s^2 \equiv (1,2,\dots,t),$$

$$s^{2t-1} \equiv \begin{cases} (1,2,\dots,\frac{t+1}{2},\frac{t+1}{2}-1,\dots,2,1), & \text{if } t \text{ is odd;} \\ (2,3,\dots,\frac{t}{2}+1,\frac{t}{2},\dots,2,1), & \text{if } t \text{ is even,} \end{cases}$$

and for $y = 2,\dots,t-1$, if $t+y$ is even,

$$s^{y1}(j) = \begin{cases} j, & \text{if } 1 \le j \le \frac{t-y}{2}+1; \\ s^{y1}(j-1)-1, & \text{if } \frac{t-y}{2}+2 \le j \le t-y+1; \\ j+y-t, & \text{if } t-y+2 \le j \le t, \end{cases} \tag{3.3}$$

and if $t+y$ is odd,

$$s^{y1}(j) = \begin{cases} j+1, & \text{if } 1 \le j \le \frac{t-y+1}{2}; \\ s^{y1}(j-1)-1, & \text{if } \frac{t-y+1}{2}+1 \le j \le t-y+1; \\ j+y-t, & \text{if } t-y+2 \le j \le t, \end{cases} \tag{3.4}$$

and

$$s^{y2} = \begin{cases} (1,2,\dots,\frac{t+y}{2},\frac{t+y}{2}-1,\dots,y), & \text{if } t+y \text{ is even;} \\ (2,3,\dots,\frac{t+y+1}{2},\frac{t+y-1}{2},\dots,y), & \text{if } t+y \text{ is odd.} \end{cases} \tag{3.5}$$

4. A Special Case $n = t+1$.

In this section we analyse the case $n = k+1 = t+1$. Without loss of generality, we assume that undominated strategies of the seeker satisfiy the condition (3.1) in Lemma 3.2.

For $y \in N$, we define a reduced game G_y where the payoff-matrix of it is a submatrix of A whose rows and columns are corresponding to $\mathcal{H}_{-y}$ and $\mathcal{S}_y$ respectively. It is easy to see that $\{\mathcal{S}_y\}_{y\in N}$ and $\{\mathcal{H}_{-y}\}_{y\in N}$ are partitions of $\mathcal{S}$ and $\mathcal{H}$ respectively. By Lemma 3.2,

$$a(h,s) \neq 0 \Longrightarrow \exists y \in N \text{ such that } (h,s) \in \mathcal{H}_{-y} \times \mathcal{S}_y.$$

We can apply Lemma A.1 and we see that it suffices to solve the reduced games $\{G_y\}_{y\in N}$.

4.1. Analysis of Reduced Games.

For $y \in N$, let $\mathcal{T} \subseteq \mathcal{S}_y$ be a set of the seeker's pure strategies which satisfies

$$\begin{cases} \text{(i)} & \forall s, s' \in \mathcal{T}, s \neq s', \quad (D(s) \cup D(s')) \cap G(h) \neq \emptyset, \forall h \in \mathcal{H}_{-y}, \\ \text{(ii)} & s \in \mathcal{T}_y \Longrightarrow D(s) \not\subset D(s'), \forall s' \in \mathcal{S}_y \\ \text{(iii)} & |\mathcal{T}| \text{ is the maximum among those of the sets satisfying (i) and (ii).} \end{cases} \tag{4.1}$$

Lemma 4.1. Assume a set $\mathcal{T} \subseteq \mathcal{S}_y$ satisfies the condition (4.1). The value v_y of the game G_y is less than or equal to $\frac{1}{|\mathcal{T}|}$.

Remark. The condition (4.1) implies that

$$\forall s, s' \in \mathcal{T}, s \neq s', \quad D(s) \setminus D(s') \neq \emptyset \text{ and } D(s') \setminus D(s) \neq \emptyset. \tag{4.2}$$

Indeed, assume that $\mathcal{T}$ satisfies (4.1) and for $s, s' \in \mathcal{T}$, it holds $D(s') \subseteq D(s)$. Then $D(s) \cap G(h) = \emptyset$ for all $h \in \mathcal{H}$. But, this is impossible because of Lemma 2.1.

Remark. The condition (i) in (4.1) is alternatively stated as

$$\forall s, s' \in \mathcal{T}, s \neq s', \exists \ell, 1 \leq \ell \leq t. |s(\ell) - s'(\ell)| > t - \ell.$$

Lemma 4.2. For each $y \in N$, assume a set $\mathcal{T}_y \subseteq \mathcal{S}_y$ satisfies the condition (4.1). Then $|\mathcal{T}_y| \geq 2$ for $2 \leq y \leq t-1$. Furthermore, $s^1 \in \mathcal{T}_n, s^2 \in \mathcal{T}_t$, and $s^{2k-1} \in \mathcal{T}_1$.

Theorem 4.3. Assume $t \geq 2$. The seeker can expect to hold the hider's payoff to no more than $1/(2t-1)$ by using $\{s^{y1}, s^{y2} : 2 \leq y \leq t-1\} \cup \{s^1, s^2, s^{2t-1}\}$ with equal probability.

Now we define a class $\mathcal{H}^e$ of pure strategies for the hider as follows:

For each $h \in \mathcal{H}^e$, there are positive integers $\ell_1, \ldots, \ell_m$ such that

$$\text{either} \quad \text{(I)} \quad \ell_1 = 0, \quad \ell_2 = k \quad \text{or} \quad \text{(II)} \quad 0 < \ell_1 < \cdots < \ell_m = k,$$

and

$$h(j) = \begin{cases} n+1-j, & \text{if } 1 \leq j \leq \ell_1; \\ j - \ell_1, & \text{if } \ell_1 + 1 \leq j \leq \ell_2; \\ n+1-\ell_1+\ell_2-j, & \text{if } \ell_2 + 1 \leq j \leq \ell_3; \\ \ell_2 - \ell_1 + j - \ell_3, & \text{if } \ell_3 + 1 \leq \ell_4; \\ \cdots. \end{cases}$$

In the case (II) this strategy h indicates that first the hider locates at each node from $h(1) = n$ to $h(\ell_1) = n+1-\ell_1$, then locates from $h(\ell_1+1) = 1$ to $h(\ell_2) = \ell_2 - \ell_1$, then locates from $h(\ell_2+1) = h(\ell_1) - 1$ to $h(\ell_3) = h(\ell_1) - (\ell_3 - \ell_2)$, then locates from $h(\ell_3+1) = \ell_2 - \ell_1 + 1$ to $h(\ell_4) = \ell_2 - \ell_1 + \ell_4 - \ell_3$, then $\cdots$.

Theorem 4.4. Assume $k \geq 2$. The hider can expect to hold his/her average payoff to no less than $1/|\mathcal{H}^e|$ by using all strategies in $\mathcal{H}^e$ with equal probability.

Remark. It is interesting to compare the bounds given in Theorems 4.3 and 4.4 with the value $1/(k+1)!$ of the game on the complete graph given in Ruckle/Kikuta [8]. It is easy to see

$$2k-1 \leq \frac{1}{\text{the value of QAGLG}} \leq |\mathcal{H}^e| \leq (k+1)!.$$

There are big gaps between $|\mathcal{H}^e|$ and $(k+1)!$ and between $2k-1$ and $|\mathcal{H}^e|$. The former gap depends on differences of underlying graphs.

4.2. Solutions to Games with Small Number of Nodes.

Theorems 4.3 and 4.4 suggest us to consider some strategies for the seeker and the hider, as is illustrated in Table 1 and Example 1 below for $k = 6$ and $n = 7$. Note that each strategy is expressed as $s = (s(1), \ldots, s(k))$ and $h = (h(1), \ldots, h(k))$.

Table 1.

Seeker	Hider
$s^1 = (2,3,4,5,6,7)$	$h^1 = (1,2,3,4,5,6)$
$s^2 = (1,2,3,4,5,6)$	$h^2 = (7,1,2,3,4,5)$
$s^3 = (2,3,4,5,6,5)$	$h^3 = (7,1,2,3,4,6)$
$s^4 = (1,1,2,3,4,5)$	$h^4 = (7,6,1,2,3,4)$
$s^5 = (1,2,3,4,5,4)$	$h^5 = (7,6,1,2,3,5)$
$s^6 = (1,1,1,2,3,4)$	$h^6 = (7,6,5,1,2,3)$
$s^7 = (2,3,4,5,4,3)$	$h^7 = (7,6,1,2,5,4)$
$s^8 = (1,1,1,1,2,3)$	$h^8 = (7,6,5,4,1,2)$
$s^9 = (1,2,3,4,3,2)$	$h^9 = (7,6,5,1,4,3)$
$s^{10} = (1,1,1,1,1,2)$	$h^{10} = (7,6,5,4,3,1)$
$s^{11} = (2,3,4,3,2,1)$	$h^{11} = (7,6,5,4,3,2)$

Example 1. Suppose $k = 6$ and $n = 7$. An optimal strategy for the seeker is to use $s^i, (1 \le i \le 11)$ in Table 1 with probability $1/11$ each. An optimal strategy for the hider is to use $h^i, (1 \le i \le 11)$ in Table 1 with probability $1/11$ each. The value of the game is $1/11$.

The following Table 2 gives pure strategies used in optimal mixed strategies for $k = 7$. Some pure strategies for the seeker in Table 2 are not represented by any of (3.3) – (3.5).

Table 2.

Seeker	Hider
$s^1 = (2,3,4,5,6,7,8)$	$h^1 = (1,2,3,4,5,6,7)$
$s^2 = (1,2,3,4,5,6,7)$	$h^2 = (8,1,2,3,4,5,6)$
$s^3 = (2,3,4,5,6,7,6)$	$h^3 = (8,1,2,3,4,5,7)$
$s^4 = (1,1,2,3,4,5,6)$	$h^4 = (8,7,1,2,3,4,5)$
$s^5 = (1,2,3,4,5,6,5)$	$h^5 = (8,7,1,2,3,4,6)$
$s^6 = (1,1,1,2,3,4,5)$	$h^6 = (8,7,6,1,2,3,4)$
$s^7 = (2,3,4,5,6,5,4)$	$h^7 = (8,7,1,2,3,6,5)$
$s^8 = (1,1,1,1,2,3,4)$	$h^8 = (8,7,6,5,1,2,3)$
$s^9 = (2,3,4,5,4,3,4)$	$h^9 = (8,7,6,1,2,5,3)$
$s^{10} = (1,2,3,4,5,4,3)$	$h^{10} = (8,7,6,1,2,5,4)$
$s^{11} = (1,1,1,1,1,2,3)$	$h^{11} = (8,7,6,5,4,1,2)$
$s^{12} = (2,3,4,5,4,3,2)$	$h^{12} = (8,7,6,1,5,4,3)$
$s^{13} = (1,1,1,1,1,1,2)$	$h^{13} = (8,7,6,5,4,3,1)$
$s^{14} = (1,1,1,1,2,3,2)$	$h^{14} = (8,7,6,5,4,1,3)$
$s^{15} = (1,2,3,4,3,2,1)$	$h^{15} = (8,7,6,5,4,3,2)$

Every pure strategy for the hider in Tables 1 and 2 is in $\mathcal{H}^e$ for $k = 6$ and 7 respectively.

Example 2. Suppose $k = 7$ and $n = 8$. An optimal strategy for the seeker is to use $s^i, (1 \le i \le 15)$ in Table 2 with probability $1/15$ each. An optimal strategy for the hider is to use $h^i, (1 \le i \le 15)$ in Table 2 with probability $1/15$ each. The value of the game is $1/15$.

Example 3. When $k = 5$ and $n = 6$, the value is $1/9$. When $k = 4$ and $n = 5$, the value is $1/7$. When $k = 3$ and $n = 4$, the value is $1/5$. For all of these cases, an optimal strategy for the seeker is to mix strategies in (3.3) – (3.5) with equal probability. Optimal strategies for both players are to mix pure strategies listed in the table below.

Table 3.

Seeker	Hider
k=5	
$s^1 = (2,3,4,5,6)$	$h^1 = (1,2,3,4,5)$
$s^2 = (1,2,3,4,5)$	$h^2 = (6,1,2,3,4)$
$s^3 = (2,3,4,5,4)$	$h^3 = (6,1,2,3,5)$
$s^4 = (1,1,2,3,4)$	$h^4 = (6,5,1,2,3)$
$s^5 = (1,2,3,4,3)$	$h^5 = (6,5,1,2,4)$
$s^6 = (1,1,1,2,3)$	$h^6 = (6,5,4,1,2)$
$s^7 = (2,3,4,3,2)$	$h^7 = (6,5,1,4,3)$
$s^8 = (1,1,1,1,2)$	$h^8 = (6,5,4,3,1)$
$s^9 = (1,2,3,2,1)$	$h^9 = (6,5,4,3,2)$
k=4	
$s^1 = (2,3,4,5)$	$h^1 = (1,2,3,4)$
$s^2 = (1,2,3,4)$	$h^2 = (5,1,2,3)$
$s^3 = (2,3,4,3)$	$h^3 = (5,1,2,4)$
$s^4 = (1,1,2,3)$	$h^4 = (5,4,1,2)$
$s^5 = (1,2,3,2)$	$h^5 = (5,4,1,3)$
$s^6 = (1,1,1,2)$	$h^6 = (5,4,3,1)$
$s^7 = (2,3,2,1)$	$h^7 = (5,4,3,2)$
k=3	
$s^1 = (2,3,4)$	$h^1 = (1,2,3)$
$s^2 = (1,2,3)$	$h^2 = (4,1,2)$
$s^3 = (2,3,2)$	$h^3 = (4,1,3)$
$s^4 = (1,1,2)$	$h^4 = (4,3,1)$
$s^5 = (1,2,1)$	$h^5 = (4,3,2)$

References.

[1]Alpern, S. and S.Gal: *The Theory of Search Games and Rendezvous*, Kluwer Acadenic Publishers, Boston,2003.

[2]S.Gal: *Search Games*, Math.in Sci. and Eng., 149, Academic Press, 1980.

[3]Kikuta, K. and W.H.Ruckle:Accumulation Games I-Noisy Search. *Journal of Optimization Theory and Applications*, Vol.94, No.2, August 1997, 395-408.

[4] —:Continuous Accumulation Games in Continuous regions. *Journal of Optimization Theory and Applications*, Vol.106, No.3, 2000, 581-601.

[5] —:Continuous Accumulation Games on Discrete Locations. *Naval Research Logistics*, Vol.49, 2002,pp.60-77.

[6] W.H.Ruckle: *Geometric Games and Their Applications*, Pitman, Boston, 1983.

[7] W.H.Ruckle: Accumulation Games. *Scientiae Mathematicae Japonica*,Vol.54,No.1,

2001,173-203.

[8] Ruckle, W.H. and K.Kikuta:Very Quiet Discrete Accumulation Search. mimeo. 2000.

[9] ——:Quiet Accumulation Games. WP No.176,Institute of Economic Research, Kobe Univ. Commerce, August 1999.

Appendix.

Lemma A.1. Let $A = (a_{xy})$ be a $r \times c$ matrix. Let $R = \{1, \ldots, r\}$ and $C = \{1, \ldots, c\}$. Assume that $\{R_1, \ldots, R_m\}$ and $\{C_1, \ldots, C_m\}$ are partitions of R and C respectively, satisfying

$$a_{xy} \neq 0 \Longrightarrow \exists \ell \text{ such that } (x, y) \in R_\ell \times C_\ell. \tag{A.1}$$

For every ℓ, let A_ℓ be a $|R_\ell| \times |C_\ell|$ submatrix of A whose rows and columns are R_ℓ and C_ℓ respectively. For every ℓ, let p^ℓ, q^ℓ and v_ℓ be optimal strategies and the value fot the matrix game A_ℓ. Then

$$p = (\alpha_1 p^1, \ldots, \alpha_m p^m) \text{ and } q = (\alpha_1 q^1, \ldots, \alpha_m q^m)$$

are optimal strategies and the value of the matrix game A is

$$v = \frac{1}{\sum_{\ell=1}^{m} \frac{1}{v_\ell}}$$

where

$$\alpha_\ell = \frac{1}{v_\ell \sum_{\ell=1}^{m} \frac{1}{v_\ell}}.$$