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範囲基準下の最適停止について

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1 Introduction

In this paper, we consider the optimal stopping problem for compoud criteria, whose counterpart is simple criteria such as terminal, additive and minimum. We introduce a new notion of gain process, which is evaluated at stopped state. Some of gain processes are terminal, additive, minimum, range, ratio, sample variance. The former three are simple. The latter three are compound. In this paper we discuss the compond criterion such as range, mid-range, ratio, average and sample variance.

2 General Process

We consider a class of finite-stage optimal stopping problems from a view point of reward accumulation. An N-stage problem has to stop by time N at the latest. Each stage allows eiher stop or continue. When a decision maker stops on a state at n-th stage, she gets a reward which is closely related to all the states she has experienced.

Let $\{X_n\}_0^N$ be a Markov chain on a finite state space X with a transition law $p = \{p(\cdot|\cdot)\}$. Letting $X^k := X \times X \times \cdots \times X$($k$ times) be the direct product of k state spaces X, we take $H_n := X^{n+1}$; the set of all *subpaths* $h_n = x_0x_1\cdots x_n$ up to stage n :

$$H_n = \{h_n = x_0x_1\cdots x_n \,|\, x_m \in X,\ 0 \le m \le n\} \quad 0 \le n \le N.$$

In particular, we set

$$\Omega := H_N.$$

Its element $\omega = h_N = x_0x_1\cdots x_N$ is called a *path.*

Let $\mathfrak{F}_m^n$ be the set of all subsets in Ω which are determined by random variables $\{X_m, X_{m+1}, \ldots, X_n\}$, where $X_k : \Omega \to X$ is the projection, $X_k(\omega) = x_k$. Strictly, $\mathfrak{F}_m^n$ is the σ-field on Ω generated by the set of all subsets of the form

$$\{X_m = x_m, X_{m+1} = x_{m+1}, \ldots, X_n = x_n\}\ (\subset \Omega)$$

where $x_m, x_{m+1}, \ldots, x_n$ are all elements in state space X. Let us take $\mathbb{N} = \{0, 1, \ldots, N\}$. A mapping $\tau : \Omega \to \mathbb{N}$ is called a *stopping time* if

$$\{\tau = n\} \in \mathfrak{F}_0^n \qquad \forall n \in \mathbb{N}.$$

where $\{\tau = n\} = \{x_0x_1\ldots x_N \,|\, \tau(x_0x_1\ldots x_N) = n\}$. The stopping time τ is called $\{\mathfrak{F}_0^n\}_0^N$-*adapted.* Let $\mathfrak{T}_0^N$ be the set of all such stopping times. Any stopping time $\tau \in \mathfrak{T}_0^N$ generates a *stopped subhistory* (random variable) $(X_0, X_1, \ldots, X_{\tau-1}, X_\tau)$ on Ω through

$$X_{\tau-n}(\omega) := X_{\tau(\omega)-n}(\omega) \qquad 0 \le n \le \tau(\omega).$$

Let $\{g_n\}_0^N$ be a sequence of *gain functions*

$$g_n : H_n \to R^1 \qquad 0 \le n \le N.$$

Then a *gain process* $\{G_n\}_0^N$ is defined by

$$G_n := g_n(X_0, X_1, \ldots, X_n).$$

Then any stopping time τ yields a *stopped reward* (random variable) $G_\tau : \Omega \to R^1$:

$$G_\tau(\omega) = G_{\tau(\omega)}(X_0(\omega), X_1(\omega), \ldots, X_{\tau-1}(\omega), X_\tau(\omega)).$$

We remark that the expected value $E_{x_0}[G_\tau]$ is expressed by sum of multiple sums :

$$\begin{aligned} E_{x_0}[G_\tau] &= \sum_{n=0}^{N} \sum_{\{\tau=n\}} G_n(h_n) P_{x_0}(X_0 = x_0, \ldots, X_n = x_n) \\ &= \sum_{n=0}^{N} \sum_{\{\tau=n\}} G_n(h_n) p(x_1|x_0) p(x_2|x_1) \cdots p(x_n|x_{n-1}). \end{aligned}$$

Now we consider the optimal stopping problem for the gain process :

$$\mathrm{G}_0(x_0) \qquad \text{Max} \quad E_{x_0}\left[G_\tau\right] \quad \text{s.t.} \quad \tau \in \mathcal{T}_0^N. \tag{1}$$

Then we have the corresponding recursive equation and optimal stopping time ([4]) :

Theorem 2.1

$$\begin{cases} v_N(h) = g_N(h) \qquad h \in H_N \\ v_n(h) = \text{Max}\left[\, g_n(h),\, E_x[v_{n+1}(h, X_{n+1})] \,\right] \\ \qquad\qquad h = (x_0, \ldots, x_{n-1}, x) \in H_n, \quad 0 \le n \le N-1. \end{cases} \tag{2}$$

Theorem 2.2 *The stopping time τ^* :*

$$\tau^*(\omega) = \min\{n \ge 0 : v_n(h_n) = G_n(h_n)\} \qquad \omega = x_0 x_1 \cdots x_N$$

is optimal :

$$E_{x_0}[G_{\tau^*}] \ge E_{x_0}[G_\tau] \qquad \forall \tau \in \mathcal{T}_0^N.$$

3 Expanded Control Chain

Now, in this section, let us discuss a general result for range process. We consider a maximization problem of expected value for stopped process under *range* criterion (As for nonstopping but control problems, see [11–16, 21]).

Let $\{X_n\}_0^N$ be the Markov chain on the finite state space X with the transition law $p = \{p(\cdot|\cdot)\}$ (Section 2). Let $g_n : X \to R^1$ be a *stop reward* for $0 \le n \le N$ and $r_n : X \to R^1$ be a *continue reward* for $0 \le n \le N-1$. Then an accumulation process is constructed as follows. When a decision-maker stops at stage x_n on stage n through a subhistory $(x_0, x_1, \ldots, x_{n-1})$, he or she will incur the range of reward up to stage n :

$$R_{an}(h_n) := r_0 \vee r_1 \vee \cdots \vee r_{n-1} \vee g_n - r_0 \wedge r_1 \wedge \cdots \wedge r_{n-1} \wedge g_n$$

where

$$h_n = (x_0, x_1, \dots, x_n), \quad r_m = r_m(x_m), \quad g_n = g_n(x_n).$$

The accumulation process $\{R_{an}\}_0^N$ is called a *range process.* Thus a stopped reward by adopting stopping time τ for range process is

$$R_{a\tau} = r_0 \vee r_1 \vee \cdots \vee r_{\tau-1} \vee g_\tau - r_0 \wedge r_1 \wedge \cdots \vee r_{\tau-1} \wedge g_\tau .$$

Now we consider the optimal stopping problem for range process :

$$\mathrm{Ra}_0(x_0) \qquad \mathrm{Max} \;\; E_{x_0}[R_{a\tau}] \quad \text{s.t.} \quad \tau \in \mathcal{T}_0^N .$$

The expected value of range is the sum of multiple sums :

$$E_{x_0}[R_{a\tau}] = \sum_{n=0}^{N} \sum_{\{\tau=n\}} \{R_{an}(h_n) \times p(x_1|x_0)p(x_2|x_1)\cdots p(x_n|x_{n-1})\} .$$

Let us now imbed $\mathrm{Ra}_0(x_0)$ into a new class of additional parametric subproblems [2, 17]. First we define the *past-valued (cumulative) random variables* $\{\widetilde{\Lambda}_n\}_0^N$, $\{\widetilde{\Xi}_n\}_0^N$ up to n-th stage and the *past-value sets* $\{\Lambda_n\}_0^N$, $\{\Xi_n\}_0^N$ they take :

$$\begin{aligned}
\widetilde{\Lambda}_0 &:= \tilde{\lambda}_0 \quad \text{where } \tilde{\lambda}_0 \text{ is smaller than or equal to } g_n(x), r_n(x) \\
\widetilde{\Xi}_0 &:= \tilde{\xi}_0 \quad \text{where } \tilde{\xi}_0 \text{ is larger than or equal to } g_n(x), r_n(x) \\
\widetilde{\Lambda}_n &:= r_0(X_0) \vee \cdots \vee r_{n-1}(X_{n-1}) \\
\widetilde{\Xi}_n &:= r_0(X_0) \wedge \cdots \wedge r_{n-1}(X_{n-1})
\end{aligned}$$

$$\begin{aligned}
\Gamma_0 &:= \{(\tilde{\lambda}_0, \tilde{\xi}_0)\} \\
\Gamma_n &:= \left\{ (\lambda_n, \xi_n) \;\middle|\; \begin{array}{l} \lambda_n = r_0(x_0) \vee \cdots \vee r_{n-1}(x_{n-1}), \\ \xi_n = r_0(x_0) \wedge \cdots \wedge r_{n-1}(x_{n-1}), \\ (x_0, \dots, x_{n-1}) \in X \times \cdots \times X \end{array} \right\} .
\end{aligned}$$

We have

Lemma 3.1 *(Forward recursive formulae)*

$$\begin{aligned}
&\widetilde{\Lambda}_0 = \tilde{\lambda}_0 \\
&\widetilde{\Lambda}_{n+1} = \widetilde{\Lambda}_n \vee r_n(X_n) \qquad 0 \le n \le N\text{–}1, \\
&\widetilde{\Xi}_0 = \tilde{\xi}_0 \\
&\widetilde{\Xi}_{n+1} = \widetilde{\Xi}_n \wedge r_n(X_n) \qquad 0 \le n \le N\text{–}1, \\
&\Gamma_0 = \{(\tilde{\lambda}_0, \tilde{\xi}_0)\} \\
&\Gamma_{n+1} = \{(\lambda \vee r_n(x),\, \xi \wedge r_n(x)) \,|\, (\lambda, \xi) \in \Gamma_n,\; x \in X\} \qquad 0 \le n \le N\text{–}1.
\end{aligned}$$

Let us now expand the original state space X to a direct product space :

$$Y_n := X \times \Gamma_n \qquad 0 \le n \le N.$$

We define a sequence of *stop-reward functions* $\{G_n\}_0^N$ by

$$G_n(x; \lambda, \xi) := \lambda \vee g_n(x) - \xi \wedge g_n(x) \qquad (x; \lambda, \xi) \in Y_n$$

and a *nonstationary Markov transition law* $q = \{q_n\}_0^{N-1}$ by

$$q_n(y;\mu,\nu\,|\,x;\lambda,\xi\,) := \begin{cases} p(y|x) & \text{if } \lambda \vee r_n(x) = \mu,\ \ \xi \wedge r_n(x) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Let us define $\widetilde{\Gamma}_n$ through

$$\widetilde{\Gamma}_n := (\widetilde{\Lambda}_n, \widetilde{\Xi}_n).$$

Then $\{(X_n, \widetilde{\Gamma}_n)\}_0^N$ is a Markov chain on state spaces $\{Y_n\}$ with transition law q. We consider the *terminal* criterion $\{G_n\}_0^N$ on the expanded process :

$$\overline{\mathrm{T}}_0(y_0) \qquad \text{Max} \ \ \mathbb{E}_{y_0}[G_\tau] \ \ \text{s.t.} \ \ \tau \in \widetilde{\mathfrak{T}}_0^N$$

where $y_0 = (x_0; \tilde{\lambda}_0, \tilde{\mu}_0)$, and $\widetilde{\mathfrak{T}}_n^N$ is the set of all stopping times which take values in $\{n, n+1, \ldots, N\}$ on the new Markov chain.

Now we conside a subprocess which starts at state $y_n = (x_n; \lambda_n, \xi_n)(\in Y_n)$ on n-th stage :

$$\overline{\mathrm{T}}_n(y_n) \qquad \text{Max} \ \ \mathbb{E}_{y_n}[G_\tau] \ \ \text{s.t.} \ \ \tau \in \widetilde{\mathfrak{T}}_n^N.$$

Let $v_n(y_n)$ be the maximum value of $\overline{\mathrm{T}}_n(y_n)$, where

$$v_N(y_N) \triangleq G_N(y_N) \qquad y_N \in Y_N.$$

Then we have the the backward recursive equation :

Corollary 3.1

$$\begin{cases} v_N(y) = G_N(y) & y \in Y_N \\ v_n(y) = \text{Max}\left[\, G_n(y),\, \mathbb{E}_y[v_{n+1}(Y_{n+1})]\,\right] & y \in Y_n, \quad 0 \le n \le N-1 \end{cases}$$

where $\mathbb{E}_y$ *is the one-step expectation operator induced from the Markov transition probabilities* $q_n(\cdot|\cdot)$ *:*

$$\mathbb{E}_y[h(Y_{n+1})] = \sum_{z \in Y_{n+1}} h(y) q_n(z|y).$$

Corollary 3.2 *The stopping time* τ^* *:*

$$\tau^*(\omega) = \min\{n \ge 0 : v_n(y_n) = G_n(y_n)\} \qquad \omega = y_0 y_1 \cdots y_N$$

is optimal :

$$\mathbb{E}_{y_0}[G_{\tau^*}] \ge \mathbb{E}_{y_0}[G_\tau] \qquad \forall \tau \in \widetilde{\mathfrak{T}}_0^N.$$

Then we have the corresponding recursive equation for the original process with maximum reward :

Theorem 3.1

$$\begin{cases} v_N(x;\lambda,\xi) = \lambda \vee g_N(x) - \xi \wedge g_N(x) & x \in X, \quad (\lambda,\xi) \in \Gamma_N \\ v_n(x;\lambda,\xi) = \text{Max}\left[\, \lambda \vee g_n(x) - \xi \wedge g_n(x), E_x[v_{n+1}(X_{n+1}; \lambda \vee r_n(x), \xi \wedge r_n(x))]\,\right] & \\ \qquad\qquad x \in X, \quad (\lambda,\xi) \in \Gamma_n, \quad 0 \le n \le N-1. \end{cases} \tag{3}$$

Here we consider a family of subprocesses which start at x_n ($\in X$) with a pair of accumulated maximum and maximum up to there (λ_n, ξ_n) :

$$\mathrm{Ra}_n(x_n; \lambda_n, \xi_n) \quad \begin{array}{ll} \text{Max} & E_{x_n}[\,\lambda_n \wedge r_n \vee \cdots \vee r_{\tau-1} \vee g_\tau - \xi_n \wedge r_n \wedge \cdots \wedge r_{\tau-1} \wedge g_\tau\,] \\ \text{s.t.} & \tau \in \mathcal{T}_n^N \\ & \qquad x_n \in X,\ (\lambda_n, \xi_n) \in \Gamma_n,\ 0 \le n \le N-1 \end{array}$$

where

$$\begin{aligned} & E_{x_n}[\,\lambda_n \vee r_n \vee \cdots \vee r_{\tau-1} \vee g_\tau - \xi_n \wedge r_n \wedge \cdots \wedge r_{\tau-1} \wedge g_\tau\,] \\ = & \sum_{m=n}^{N} \sum_{\{\tau=m\}} \{\,[\,\lambda_n \vee r_n(x_n) \vee \cdots \vee r_{m-1}(x_{m-1}) \vee g_m(x_m) \\ & - \xi_n \wedge r_n(x_n) \wedge \cdots \wedge r_{m-1}(x_{m-1}) \wedge g_m(x_m)\,] \times p(x_{n+1}|x_n)p(x_{n+2}|x_{n+1}) \cdots p(x_m|x_{m-1})\}. \end{aligned}$$

Let $v_n(x_n; \lambda_n, \xi_n)$ be the maximum value for $\mathrm{Ra}_n(x_n; \lambda_n, \xi_n)$, where

$$v_N(x_N; \lambda_N, \xi_N) = \lambda_N \vee g_N(x_N) - \xi_N \wedge g_N(x_N).$$

Then the maximum value functions satisfy the recursive equation (3).

Theorem 3.2 *The stopping time τ^* :*

$$\begin{aligned} \tau^*(\omega) &= \min\{n \ge 0 : v_n(x_n; \lambda_n, \xi_n) = \lambda_n \vee g_n(x_n) - \xi_n \wedge g_n(x_n)\} \\ \omega &= (x_0; \tilde{\lambda}_0, \tilde{\xi}_0)(x_1; \lambda_1, \xi_1) \cdots (x_N; \lambda_N, \xi_N) \end{aligned}$$

is optimal :

$$E_{x_0}[R_{a\tau^*}] \ge E_{x_0}[R_{a\tau}] \quad \forall \tau \in \mathcal{T}_0^N.$$

3.1 DP solution

Let us illustrate a two-state four-stage model, which is specified by an optimal stopping problem:

$$\begin{array}{ll} \text{Max} & E_{x_0}\,[\,r_0(X_0) \vee \cdots \vee r_{\tau-1}(X_{\tau-1}) \vee g_\tau(X_\tau) - r_0(X_0) \wedge \cdots \wedge r_{\tau-1}(X_{\tau-1}) \wedge g_\tau(X_\tau)\,] \\ \text{s.t.} & \text{(i) } \tau \in \mathcal{T}_0^4 \end{array} \tag{4}$$

where the stop/continue-reward $\{g_0, g_1, g_2, g_3, g_4;\ r_0, r_1, r_2, r_3\}$ is given in Table 1, and the transition matrix is symmetric ($p = q = 1/2$). Let us find an optimal stopping time by solving recursive equation.

Table 1 stop/continue-reward

x_n		s_1		s_2	
$g_0(x_0)$	$r_0(x_0)$	5	*4*	4	*5*
$g_1(x_1)$	$r_1(x_1)$	6	*5*	4	*6*
$g_2(x_1)$	$r_2(x_1)$	3	*4*	7	*6*
$g_3(x_1)$	$r_3(x_1)$	5	*7*	8	*3*
$g_4(x_4)$		6		7	

It is shown that the total number of stopping times $\{f_m(n)\}$ for m-state n-stage model satisfies the recursive relation ([6])

$$\begin{aligned} f_m(n+1) &= 1+(f_m(n))^m \\ f_2(m) &= 1+2^m. \end{aligned}$$

There exist $f_2(4)=677$ stopping times for two-state ($m=2$) four-stage ($n=4$) model. Among them, let us find an optimal stopping time by solving dynamic programming recursive equation (3).

First, the forward recursion in Lemma 3.1 generates the following past-value sets :

$$\Gamma_0=\{(-\infty,\infty)\},\quad \Gamma_1=\{(4,4)\},\quad \Gamma_2=\{(5,4),(6,4)\}$$
$$\Gamma_3=\{(5,4),(6,4)\},\quad \Gamma_4=\{(7,4),(5,3),(6,3)\}.$$

Second, the backward recursion (3) yields an optimal solution in expanded Markov class $\tilde{\Pi}$; optimal value functions

$$v_0,\ v_1,\ v_2,\ v_3,\ v_4\,;\quad v_n=v_n(x_n;\lambda_n,\xi_n)$$

and an optimal policy

$$\pi^*=\{\pi_0^*,\ \pi_1^*,\ \pi_2^*,\ \pi_3^*\}\,;\quad \pi_n^*=\pi_n^*(x_n;\lambda_n,\xi_n).$$

In fact the optimal solution is calculated as follows:

$$\begin{aligned}
v_4(s_1;7,4) &= 7\vee g_4(s_1)-4\wedge g_4(s_1)=7\vee 6-4\wedge 6=7-4=3\\
v_4(s_1;5,3) &= 5\vee g_4(s_1)-3\wedge g_4(s_1)=5\vee 6-3\wedge 6=6-3=3\\
v_4(s_1;6,3) &= 6\vee g_4(s_1)-3\wedge g_4(s_1)=6\vee 6-3\wedge 6=6-3=3\\
v_4(s_2;7,4) &= 7\vee g_4(s_2)-4\wedge g_4(s_2)=7\vee 7-4\wedge 7=7-4=3\\
v_4(s_2;5,3) &= 5\vee g_4(s_2)-3\wedge g_4(s_2)=5\vee 7-3\wedge 7=7-3=4\\
v_4(s_2;6,3) &= 6\vee g_4(s_2)-3\wedge g_4(s_2)=6\vee 7-3\wedge 7=7-3=4
\end{aligned}$$

$$\begin{aligned}
v_3(s_1;5,4) &= \mathrm{Max}\begin{cases} 5\vee g_3(s_1)-4\wedge g_3(s_1)\\ v_4(s_1;5\vee r_3(s_1),4\wedge r_3(s_1))\cdot\dfrac{1}{2}+v_4(s_1;5\vee r_3(s_1),4\wedge r_3(s_1))\cdot\dfrac{1}{2}\end{cases}\\
&= \mathrm{Max}\begin{cases} 5\vee g_3(s_1)-4\wedge g_3(s_1)\\ v_4(s_1;5\vee 7,4\wedge 7)\cdot\dfrac{1}{2}+v_4(s_1;5\vee 7,4\wedge 7)\cdot\dfrac{1}{2}\end{cases}\\
&= \mathrm{Max}\begin{cases} 5\vee 5-4\wedge 5\\ v_4(s_1;7,4)\cdot\dfrac{1}{2}+v_4(s_1;7,4)\cdot\dfrac{1}{2}\end{cases}\\
&= \mathrm{Max}\begin{cases} 5-4\\ 3\cdot\dfrac{1}{2}+3\cdot\dfrac{1}{2}\end{cases} = \mathrm{Max}\begin{cases} 1\\ 3\end{cases} = 3 \qquad \pi_3^*(s_1;5,4)=\text{continue}
\end{aligned}$$

$$
\begin{aligned}
v_3(s_1;6,4) &= \operatorname{Max}\begin{cases} 6 \vee g_3(s_1) - 4 \wedge g_3(s_1) \\ v_4(s_1; 6 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} + v_4(s_1; 6 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} \end{cases} \\
&= \operatorname{Max}\begin{cases} 6 \vee 5 - 4 \wedge 5 \\ v_4(s_1;7,4) \cdot \frac{1}{2} + v_4(s_1;7,4) \cdot \frac{1}{2} \end{cases} \\
&= \operatorname{Max}\begin{cases} 6 - 4 \\ 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \end{cases} = \operatorname{Max}\begin{cases} 2 \\ 3 \end{cases} = 3 \qquad \pi_3^*(s_1;6,4) = \text{continue}
\end{aligned}
$$

$$v_3(s_2;5,4) = \operatorname{Max}\begin{cases} 8-4 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{cases} = 4 \qquad \pi_3^*(s_2;5,4) = \text{s}$$

$$v_3(s_2;6,4) = \operatorname{Max}\begin{cases} 8-4 \\ 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \end{cases} = 4 \qquad \pi_3^*(s_2;6,4) = \text{s}$$

$$v_2(s_1;5,4) = \operatorname{Max}\begin{cases} 5-3 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_2^*(s_1;5,4) = \text{c}$$

$$v_2(s_1;6,4) = \operatorname{Max}\begin{cases} 6-3 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_2^*(s_1;6,4) = \text{c}$$

$$v_2(s_2;5,4) = \operatorname{Max}\begin{cases} 7-4 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_2^*(s_2;5,4) = \text{c}$$

$$v_2(s_2;6,4) = \operatorname{Max}\begin{cases} 7-4 \\ 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_2^*(s_2;6,4) = \text{c}$$

$$v_1(s_1;4,4) = \operatorname{Max}\begin{cases} 6-4 \\ 3.5 \cdot \frac{1}{2} + 3.5 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_1^*(s_1;4,4) = \text{c}$$

$$v_1(s_2;4,4) = \operatorname{Max}\begin{cases} 4-4 \\ 3.5 \cdot \frac{1}{2} + 3.5 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_1^*(s_2;4,4) = \text{c}$$

$$v_0(s_1;-\infty,\infty) = \operatorname{Max}\begin{cases} 5-5 \\ 3.5 \cdot \frac{1}{2} + 3.5 \cdot \frac{1}{2} \end{cases} = 3.5 \qquad \pi_0^*(s_1;-\infty,\infty) = \text{c}$$

The optimal solution is tabulated in Table 2:

Table 2 optimal solution

	$v_4(x_4;\lambda_4,\xi_4)$		
$x_4\backslash(\lambda_4,\xi_4)$	(7,4)	(5,3)	(6,3)
s_1	3	3	3
s_2	3	4	4

	$v_3(x_3;\lambda_3,\xi_3)$ $\pi_3^*(x_3;\lambda_3,\xi_3)$				$v_2(x_2;\lambda_2,\xi_2)$ $\pi_2^*(x_2;\lambda_2,\xi_2)$			
$x_n\backslash(\lambda_n,\xi_n)$	(5,4)		(6,4)		(5,4)		(6,4)	
s_1	3	c	3	c	3.5	c	3.5	c
s_2	4	s	4	s	3.5	c	3.5	c

$x_n\backslash(\lambda_n,\xi_n)$	$v_1(x_1;\lambda_1,\xi_1)$	$\pi_1^*(x_1;\lambda_1,\xi_1)$	$v_0(x_0;1)$	$\pi_0^*(x_0;\lambda_0,\xi_0)$
	(4,4)		$(-\infty,\infty)$	
s_1	3.5	c	3.5	c
s_2	3.5	c	—	–

Third, we see that an optimal stpping rule τ^* is to stop at state s_2 on stage 3. The rule implies that an optimal decison-maker should continue at any state on stages 0, 1, 2 and at state s_1 on stage 3 (Figure 1). The maximum expected value of range is $v_0(s_0;-\infty,\infty)=3.5$. As is directly verified at the bottom line, the optimal expected value is equal to $E_{s_1}[R_{\tau^*}]$. The stopping time τ^* has

$$E_{s_1}[R_{\tau^*}]=3\cdot\frac{1}{16}+3\cdot\frac{1}{16}+\cdots+4\cdot\frac{1}{8}=\frac{7}{2}=3.5.$$

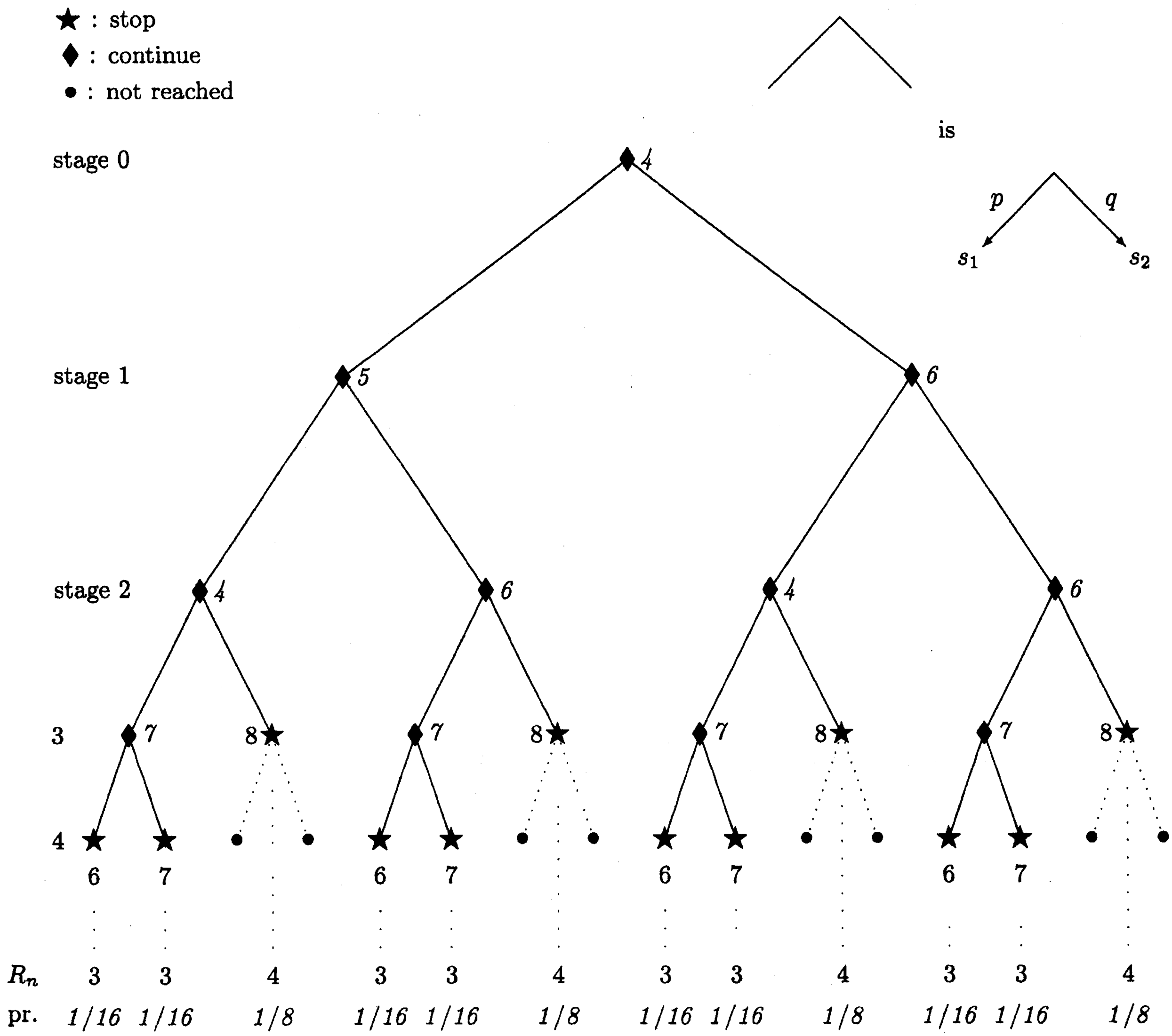

Figure 1 It is optimal to stop at state s_2 on stage 3 : τ^*

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