


1 Introduction

In this paper, we consider the optimal stopping problem for compound criteria, whose counterpart is simple criteria such as terminal, additive and minimum. We introduce a new notion of gain process, which is evaluated at stopped state. Some of gain processes are terminal, additive, minimum, range, ratio, sample variance. The former three are simple. The latter three are compound. In this paper we discuss the compound criterion such as range, mid-range, ratio, average and sample variance.

2 General Process

We consider a class of finite-stage optimal stopping problems from a view point of reward accumulation. An $N$-stage problem has to stop by time $N$ at the latest. Each stage allows either stop or continue. When a decision maker stops on a state at $n$-th stage, she gets a reward which is closely related to all the states she has experienced.

Let $\{X_n\}_{0}^{N}$ be a Markov chain on a finite state space $X$ with a transition law $p = \{p(\cdot|\cdot)\}$. Letting $X^k := X \times X \times \cdots \times X (k \text{ times})$ be the direct product of $k$ state spaces $X$, we take $H_n := X^{n+1}$; the set of all subpaths $h_n = x_0x_1 \cdots x_n$ up to stage $n$:

$$H_n = \{h_n = x_0x_1 \cdots x_n \mid x_m \in X, 0 \leq m \leq n\} \quad 0 \leq n \leq N.$$  

In particular, we set

$$\Omega := H_N.$$  

Its element $\omega = h_N = x_0x_1 \cdots x_N$ is called a path.

Let $\mathcal{T}_m^n$ be the set of all subsets in $\Omega$ which are determined by random variables $\{X_m, X_{m+1}, \ldots, X_n\}$, where $X_k : \Omega \to X$ is the projection, $X_k(\omega) = x_k$. Strictly, $\mathcal{T}_m^n$ is the $\sigma$-field on $\Omega$ generated by the set of all subsets of the form

$$\{X_m = x_m, X_{m+1} = x_{m+1}, \ldots, X_n = x_n\} (\subset \Omega)$$

where $x_m, x_{m+1}, \ldots, x_n$ are all elements in state space $X$. Let us take $N = \{0, 1, \ldots, N\}$. A mapping $\tau : \Omega \to N$ is called a stopping time if

$$\{\tau = n\} \in \mathcal{T}_0^n \quad \forall n \in N.$$  

where $\{\tau = n\} = \{x_0x_1 \ldots x_N \mid \tau(x_0x_1 \ldots x_N) = n\}$. The stopping time $\tau$ is called $\{\mathcal{T}_0^n\}_0^N$-adapted. Let $\mathcal{T}_0^N$ be the set of all such stopping times. Any stopping time $\tau \in \mathcal{T}_0^N$ generates a stopped subhistory (random variable) $(X_0, X_1, \ldots, X_{\tau-1}, X_\tau)$ on $\Omega$ through

$$X_{\tau-n}(\omega) := X_{\tau(\omega)-n}(\omega) \quad 0 \leq n \leq \tau(\omega).$$
Let \( \{g_n\}_{0}^{N} \) be a sequence of gain functions
\[
g_n : H_n \rightarrow R^1 \quad 0 \leq n \leq N.
\]
Then a gain process \( \{G_n\}_{0}^{N} \) is defined by
\[
G_n := g_n(X_0, X_1, \ldots, X_n).
\]
Then any stopping time \( \tau \) yields a stopped reward (random variable) \( G_\tau : \Omega \rightarrow R^1 \):
\[
G_\tau(\omega) = G_{\tau(\omega)}(X_0(\omega), X_1(\omega), \ldots, X_{\tau-1}(\omega), X_\tau(\omega)).
\]
We remark that the expected value \( E_{x_0}[G_\tau] \) is expressed by sum of multiple sums:
\[
E_{x_0}[G_\tau] = \sum_{n=0}^{N} \sum_{\{\tau=n\}} G_n(h_n) P_{x_0}(X_0=x_0, \ldots, X_n=x_n)
\]
Now we consider the optimal stopping problem for the gain process:
\[
G_0(x_0) \quad \text{Max} \quad E_{x_0}[G_\tau] \quad \text{s.t.} \quad \tau \in \mathcal{T}_0^N.
\]
Then we have the corresponding recursive equation and optimal stopping time ([4]):

\[\text{Theorem 2.1} \]
\[
\begin{cases} 
v_N(h) = g_N(h) & h \in H_N \\
v_n(h) = \text{Max} \left[ g_n(h), E_x[v_{n+1}(h, X_{n+1})] \right] & h = (x_0, \ldots, x_{n-1}, x) \in H_n, \quad 0 \leq n \leq N-1.
\end{cases}
\]

\[\text{Theorem 2.2} \quad \text{The stopping time } \tau^*: \]
\[
\tau^*(\omega) = \min\{n \geq 0 : v_n(h_n) = G_n(h_n)\} \quad \omega = x_0 x_1 \cdots x_N
\]
is optimal:
\[
E_{x_0}[G_{\tau^*}] \geq E_{x_0}[G_\tau] \quad \forall \tau \in \mathcal{T}_0^N.
\]

3 Expanded Control Chain

Now, in this section, let us discuss a general result for range process. We consider a maximization problem of expected value for stopped process under range criterion (As for nonstopping but control problems, see [11-16, 21]).

Let \( \{X_n\}_{0}^{N} \) be the Markov chain on the finite state space \( X \) with the transition law \( p = \{p(\cdot|\cdot)\} \) (Section 2). Let \( g_n : X \rightarrow R^1 \) be a stop reward for \( 0 \leq n \leq N \) and \( r_n : X \rightarrow R^1 \) be a continue reward for \( 0 \leq n \leq N-1 \). Then an accumulation process is constructed as follows. When a decision-maker stops at stage \( x_n \) on stage \( n \) through a subhistory \( (x_0, x_1, \ldots, x_{n-1}) \), he or she will incur the range of reward up to stage \( n \):
\[
R_{\text{an}}(h_n) := r_0 \lor r_1 \lor \cdots \lor r_{n-1} \lor g_n - r_0 \land r_1 \land \cdots \land r_{n-1} \land g_n
\]
where
\[ h_n = (x_0, x_1, \ldots, x_n), \quad r_m = r_m(x_m), \quad g_n = g_n(x_n). \]

The accumulation process \( \{R_{an}\}_{0}^{N} \) is called a \textit{range process}. Thus a stopped reward by adopting stopping time \( \tau \) for range process is
\[ R_{a\tau} = r_0 \lor r_1 \lor \cdots \lor r_{\tau-1} \lor g_\tau - r_0 \land r_1 \land \cdots \lor r_{\tau-1} \land g_\tau. \]

Now we consider the optimal stopping problem for range process:
\[ R_{a0}(x_0) \quad \text{Max } \mathbb{E}_{x_0}[R_{a\tau}] \quad \text{s.t. } \tau \in \mathcal{T}_{0}^{N}. \]

The expected value of range is the sum of multiple sums:
\[
\mathbb{E}_{x_0}[R_{a\tau}] = \sum_{n=0}^{N} \sum_{\{\tau=n\}} \{R_{an}(h_n) \times p(x_1|x_0)p(x_2|x_1)\cdots p(x_n|x_{n-1})\}. 
\]

Let us now imbed \( R_{a0}(x_0) \) into a new class of additional parametric subproblems [2, 17]. First we define the \textit{past-valued (cumulative) random variables} \( \{\widetilde{\Lambda}_{n}\}_{0}^{N}, \{\widetilde{\Xi}_{n}\}_{0}^{N} \) up to \( n \)-th stage and the \textit{past-value sets} \( \{\Lambda_{n}\}_{0}^{N}, \{\Xi_{n}\}_{0}^{N} \) they take:
\begin{align*}
\widetilde{\Lambda}_0 &:= \tilde{\lambda}_0 \quad \text{where } \tilde{\lambda}_0 \text{ is smaller than or equal to } g_n(x), r_n(x) \\
\widetilde{\Xi}_0 &:= \tilde{\xi}_0 \quad \text{where } \tilde{\xi}_0 \text{ is larger than or equal to } g_n(x), r_n(x) \\
\widetilde{\Lambda}_n &:= r_0(X_0) \lor \cdots \lor r_{n-1}(X_{n-1}) \\
\widetilde{\Xi}_n &:= r_0(X_0) \land \cdots \land r_{n-1}(X_{n-1}) \\
\Gamma_0 &:= \{(\tilde{\lambda}_0, \tilde{\xi}_0)\} \\
\Gamma_n &:= \left\{ (\lambda_n, \xi_n) \mid \lambda_n = r_0(x_0) \lor \cdots \lor r_{n-1}(x_{n-1}), \right. \\
& \left. \quad \xi_n = r_0(x_0) \land \cdots \land r_{n-1}(x_{n-1}), \right. \\
& \left. \quad (x_0, \ldots, x_{n-1}) \in X \times \cdots \times X \right\}. 
\end{align*}

We have

\textbf{Lemma 3.1 (Forward recursive formulae)}
\begin{align*}
\widetilde{\Lambda}_0 &= \tilde{\lambda}_0 \\
\widetilde{\Lambda}_{n+1} &= \widetilde{\Lambda}_n \lor r_n(X_n) \quad 0 \leq n \leq N-1, \\
\widetilde{\Xi}_0 &= \tilde{\xi}_0 \\
\widetilde{\Xi}_{n+1} &= \widetilde{\Xi}_n \land r_n(X_n) \quad 0 \leq n \leq N-1, \\
\Gamma_0 &= \{(\tilde{\lambda}_0, \tilde{\xi}_0)\} \\
\Gamma_{n+1} &= \{(\lambda \lor r_n(x), \xi \land r_n(x)) \mid (\lambda, \xi) \in \Gamma_n, \ x \in X\} \quad 0 \leq n \leq N-1.
\end{align*}

Let us now expand the original state space \( X \) to a direct product space:
\[ Y_n := X \times \Gamma_n \quad 0 \leq n \leq N. \]

We define a sequence of \textit{stop-reward functions} \( \{G_n\}_{0}^{N} \) by
\[ G_n(x; \lambda, \xi) := \lambda \lor g_n(x) - \xi \land g_n(x) \quad (x; \lambda, \xi) \in Y_n \]
and a nonstationary Markov transition law $q = \{q_n\}_{0}^{N-1}$ by

$$q_n(y; \mu, \nu | x; \lambda, \xi) := \begin{cases} p(y|x) & \text{if } \lambda \vee r_n(x) = \mu, \xi \wedge r_n(x) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Let us define $\bar{\Gamma}_n$ through

$$\bar{\Gamma}_n := (\bar{\lambda}_n, \bar{\Xi}_n).$$

Then $\{(X_n, \bar{\Gamma}_n)\}_{0}^{N}$ is a Markov chain on state spaces $\{Y_n\}$ with transition law $q$. We consider the terminal criterion $\{G_n\}_{0}^{N}$ on the expanded process:

$$\bar{T}_0(y_0) \quad \text{Max} \quad \mathbb{E}_y[G_\tau] \quad \text{s.t. } \tau \in \bar{T}_n^N$$

where $y_0 = (x_0; \tilde{\lambda}_0, \tilde{\mu}_0)$, and $\bar{T}_n^N$ is the set of all stopping times which take values in $\{n, n + 1, \ldots, N\}$ on the new Markov chain.

Now we consider a subprocess which starts at state $y_n = (x_n; \lambda_n, \xi_n)(\in Y_n)$ on $n$-th stage:

$$\bar{T}_n(y_n) \quad \text{Max} \quad \mathbb{E}_{y_n}[G_\tau] \quad \text{s.t. } \tau \in \bar{T}_n^N.$$ 

Let $u_n(y_n)$ be the maximum value of $\bar{T}_n(y_n)$, where

$$u_N(y_N) \triangleq G_N(y_N) \quad y_N \in Y_N.$$ 

Then we have the backward recursive equation:

**Corollary 3.1**

$$\begin{cases} u_N(y) = G_N(y) & y \in Y_N \\ u_n(y) = \text{Max} \left[ G_n(y), \mathbb{E}_y[u_{n+1}(Y_{n+1})] \right] & y \in Y_n, \ 0 \leq n \leq N - 1 \end{cases}$$

where $\mathbb{E}_y$ is the one-step expectation operator induced from the Markov transition probabilities $q_n(\cdot | \cdot)$:

$$\mathbb{E}_y[h(Y_{n+1})] = \sum_{z \in Y_{n+1}} h(y)q_n(z|y).$$

**Corollary 3.2** The stopping time $\tau^*$:

$$\tau^*(\omega) = \min\{n \geq 0 : u_n(y_n) = G_n(y_n)\} \quad \omega = y_0y_1 \cdots y_N$$

is optimal:

$$\mathbb{E}_{y_0}[G_{\tau^*}] \geq \mathbb{E}_{y_0}[G_\tau] \quad \forall \tau \in \bar{T}_0^N.$$ 

Then we have the corresponding recursive equation for the original process with maximum reward:

**Theorem 3.1**

$$\begin{cases} u_N(x; \lambda, \xi) = \lambda \vee g_N(x) - \xi \wedge g_N(x) & x \in X, \ (\lambda, \xi) \in \Gamma_N \\ u_n(x; \lambda, \xi) = \text{Max} \left[ \lambda \vee g_n(x) - \xi \wedge g_n(x), \mathbb{E}_x[u_{n+1}(X_{n+1}; \lambda \vee r_n(x), \xi \wedge r_n(x))] \right] & x \in X, \ (\lambda, \xi) \in \Gamma_n, \ 0 \leq n \leq N - 1. \end{cases}$$
Here we consider a family of subprocesses which start at $x_n (\in X)$ with a pair of accumulated maximum and maximum up to there $(\lambda_n, \xi_n)$:

$$\text{Max } E_{x_n} \left[ \lambda_n \wedge r_n \lor \cdots \lor r_{\tau-1} \lor g_{\tau} - \xi_n \lor r_n \land \cdots \land r_{\tau-1} \land g_{\tau} \right]$$

subject to $\tau \in \mathcal{T}_n^N$

$$x_n \in X, \ (\lambda_n, \xi_n) \in \Gamma_n, \ 0 \leq n \leq N - 1$$

where

$$E_{x_n} \left[ \lambda_n \lor r_n \lor \cdots \lor r_{\tau-1} \lor g_{\tau} - \xi_n \lor r_n \land \cdots \land r_{\tau-1} \land g_{\tau} \right]$$

$$= \sum_{m=n}^{N} \sum_{\{\tau=m\}} \{ [\lambda_n \lor r_n(x_n) \lor \cdots \lor r_{m-1}(x_{m-1}) \lor g_m(x_m) - \xi_n \lor r_n(x_n) \land \cdots \land r_{m-1}(x_{m-1}) \land g_m(x_m)] \times p(x_{n+1}|x_n)p(x_{n+2}|x_{n+1}) \cdots p(x_{m}|x_{m-1}) \}. $$

Let $v_n(x_n; \lambda_n, \xi_n)$ be the maximum value for $R_{a\tau}(x_n; \lambda_n, \xi_n)$, where

$$v_N(x_N; \lambda_N, \xi_N) = \lambda_N \lor g_N(x_N) - \xi_N \land g_N(x_N).$$

Then the maximum value functions satisfy the recursive equation (3).

**Theorem 3.2** The stopping time $\tau^*$:

$$\tau^*(\omega) = \min \{ n \geq 0 : v_n(x_n; \lambda_n, \xi_n) = \lambda_n \lor g_n(x_n) - \xi_n \land g_n(x_n) \}$$

is optimal:

$$E_{x_0}[R_{a\tau^*}] \geq E_{x_0}[R_{a\tau}] \ \forall \tau \in \mathcal{T}_0^N.$$ 

### 3.1 DP solution

Let us illustrate a two-state four-stage model, which is specified by an optimal stopping problem:

$$\text{Max } E_{x_0} \left[ r_0(X_0) \lor \cdots \lor r_{\tau-1}(X_{\tau-1}) \lor g_{\tau}(X_{\tau}) - r_0(X_0) \land \cdots \land r_{\tau-1}(X_{\tau-1}) \land g_{\tau}(X_{\tau}) \right]$$

subject to (i) $\tau \in \mathcal{T}_0^4$  \hspace{1cm} (4)

where the stop/continue-reward $\{g_0, g_1, g_2, g_3, g_4; r_0, r_1, r_2, r_3\}$ is given in Table 1, and the transition matrix is symmetric ($p = q = 1/2$). Let us find an optimal stopping time by solving recursive equation.

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_0(x_0)$</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$g_1(x_1)$</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>$g_2(x_2)$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$g_3(x_2)$</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>$g_4(x_4)$</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>$g_5(x_5)$</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>
It is shown that the total number of stopping times \( \{ f_m(n) \} \) for \( m \)-state \( n \)-stage model satisfies the recursive relation ([6])

\[
f_m(n + 1) = 1 + (f_m(n))^m
f_2(m) = 1 + 2^m.
\]

There exist \( f_2(4) = 677 \) stopping times for two-state \((m = 2)\) four-stage \((n = 4)\) model. Among them, let us find an optimal stopping time by solving dynamic programming recursive equation (3).

First, the forward recursion in Lemma 3.1 generates the following past-value sets:

\[
\Gamma_0 = \{ (-\infty, \infty) \}, \quad \Gamma_1 = \{ (4, 4) \}, \quad \Gamma_2 = \{ (5, 4), (6, 4) \}
\Gamma_3 = \{ (5, 4), (6, 4) \}, \quad \Gamma_4 = \{ (7, 4), (5, 3), (6, 3) \}.
\]

Second, the backward recursion (3) yields an optimal solution in expanded Markov class \( \tilde{\Pi} \); optimal value functions

\[
v_0, v_1, v_2, v_3, v_4; \quad v_n = v_n(x_n; \lambda_n, \xi_n)
\]

and an optimal policy

\[
\pi^* = \{ \pi_0^*, \pi_1^*, \pi_2^*, \pi_3^* \}; \quad \pi_n^* = \pi_n^*(x_n; \lambda_n, \xi_n).
\]

In fact the optimal solution is calculated as follows:

\[
v_3(s_1; 5, 4) = \text{Max} \left\{ \begin{array}{l}
5 \vee g_3(s_1) - 4 \wedge g_3(s_1) \\
v_4(s_1; 5 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2} + v_4(s_1; 5 \vee r_3(s_1), 4 \wedge r_3(s_1)) \cdot \frac{1}{2}
\end{array} \right\}
\]

\[
= \text{Max} \left\{ \begin{array}{l}
5 \vee g_3(s_1) - 4 \wedge g_3(s_1) \\
v_4(s_1; 5 \vee 7, 4 \wedge 7) \cdot \frac{1}{2} + v_4(s_1; 5 \vee 7, 4 \wedge 7) \cdot \frac{1}{2}
\end{array} \right\}
\]

\[
= \text{Max} \left\{ \begin{array}{l}
5 \vee 5 - 4 \wedge 5 \\
v_4(s_1; 7, 4) \cdot \frac{1}{2} + v_4(s_1; 7, 4) \cdot \frac{1}{2}
\end{array} \right\}
\]

\[
= \text{Max} \left\{ \begin{array}{l}
5 - 4 \\
3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2}
\end{array} \right\} = \text{Max} \left\{ \begin{array}{l}
1 \cdot \frac{1}{3} = 3
\end{array} \right\} = \pi_3^*(s_1; 5, 4) = \text{continue}
\]
\[ v_3(s_1; 6, 4) = \text{Max} \left\{ 6 \lor g_3(s_1) - 4 \land g_3(s_1) \right\} \]
\[ = \text{Max} \left\{ g_4(s_1; 6 \lor r_3(s_1), 4 \land r_3(s_1)) \cdot \frac{1}{2} + v_4(s_1; 6 \lor r_3(s_1), 4 \land r_3(s_1)) \cdot \frac{1}{2} \right\} \]
\[ = \text{Max} \left\{ 6 \lor 5 - 4 \land 5 \right\} \]
\[ = \text{Max} \left\{ g_4(s_1; 7, 4) \cdot \frac{1}{2} + v_4(s_1; 7, 4) \cdot \frac{1}{2} \right\} \]
\[ = \text{Max} \left\{ 6 - 4 \right\} \]
\[ = \frac{1}{2} + \frac{1}{2} = \text{Max} \left\{ \frac{2}{3} = 3 \right\} \]
\[ \pi_3^*(s_1; 6, 4) = \text{continue} \]

\[ v_3(s_2; 5, 4) = \text{Max} \left\{ 8 - 4 \right\} \]
\[ = \frac{3}{2} + \frac{4}{2} = 4 \]
\[ \pi_3^*(s_2; 5, 4) = s \]

\[ v_3(s_2; 6, 4) = \text{Max} \left\{ 8 - 4 \right\} \]
\[ = \frac{3}{2} + \frac{4}{2} = 4 \]
\[ \pi_3^*(s_2; 6, 4) = s \]

\[ \pi_3^*(s_1; 5, 4) = s \]
\[ \pi_3^*(s_1; 6, 4) = s \]

\[ \pi_2^*(s_1; 5, 4) = s \]
\[ \pi_2^*(s_1; 6, 4) = s \]

\[ \pi_2^*(s_2; 5, 4) = s \]
\[ \pi_2^*(s_2; 6, 4) = s \]

\[ \pi_1^*(s_1; 4, 4) = s \]
\[ \pi_1^*(s_1; 6, 4) = s \]

\[ \pi_1^*(s_2; 4, 4) = s \]
\[ \pi_1^*(s_2; 6, 4) = s \]

\[ \pi_0^*(s_1; -\infty, \infty) = c \]

The optimal solution is tabulated in Table 2:

<table>
<thead>
<tr>
<th>$x_4 \setminus (\lambda_4, \xi_4)$</th>
<th>$v_4(x_4; \lambda_4, \xi_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$(7, 4)$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x_n \setminus (\lambda_n, \xi_n)$</th>
<th>$v_3(x_3; \lambda_3, \xi_3)$</th>
<th>$v_2(x_2; \lambda_2, \xi_2)$</th>
<th>$\pi_3^*(x_3; \lambda_3, \xi_3)$</th>
<th>$\pi_2^*(x_2; \lambda_2, \xi_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$3$</td>
<td>$c$</td>
<td>$3$</td>
<td>$c$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$4$</td>
<td>$s$</td>
<td>$4$</td>
<td>$s$</td>
</tr>
</tbody>
</table>
Third, we see that an optimal stopping rule $\tau^*$ is to stop at state $s_2$ on stage 3. The rule implies that an optimal decision-maker should continue at any state on stages 0, 1, 2 and at state $s_1$ on stage 3 (Figure 1). The maximum expected value of range is $v_0(s_0; -\infty, \infty) = 3.5$. As is directly verified at the bottom line, the optimal expected value is equal to $E_{s_1}[R_{\tau^*}]$. The stopping time $\tau^*$ has

$$E_{s_1}[R_{\tau^*}] = 3 \cdot \frac{1}{16} + 3 \cdot \frac{1}{16} + \cdots + 4 \cdot \frac{1}{8} = \frac{7}{2} = 3.5.$$  

- : stop
- : continue
- : not reached

![Diagram](image)

Figure 1 It is optimal to stop at state $s_2$ on stage 3: $\tau^*$

References


