The main conjectures of non-commutative Iwasawa theory

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1 Introduction

The lecture reported on joint work with T. Fukaya, K. Kato, R. Sujatha, and O. Venjakob [1] on the formulation of the main conjectures of non-commutative Iwasawa theory. The general methods developed in [1] were inspired by the Heidelberg Habilitation Thesis of Venjakob [2].

Let $G$ be a compact $p$-adic Lie group. We assume throughout that $G$ has no element of order $p$, so that $G$ has finite $p$-homological dimension. Let $\Lambda(G)$ denote the Iwasawa algebra of $G$. Let $M$ be a finitely generated torsion $\Lambda(G)$-module. How can we define a characteristic element for $M$, and relate it to the Euler characteristic of $M$ and its twists? In the classical case, when $G = \mathbb{Z}_p^d$ for some integer $d \geq 1$, such characteristic elements are defined via the structure theory of such modules up to pseudo-isomorphism. In fact, an analogue of the structure theorem is proven in [3] for all non-commutative $G$ which are $p$-valued. However, in the non-commutative theory this does not seem to yield characteristic elements, both because reflexive ideals of $\Lambda(G)$ are not, in general, principal, and because pseudo-null modules with finite $G$-Euler characteristic do not, in general, have Euler characteristic $1$[4]. The goal of [1] is to use localization techniques to find a way out of this dilemma for an important class of $p$-adic Lie groups $G$ and a class of finitely generated torsion $L(G)$-modules which we optimistically hope includes all modules which occur in arithmetic at ordinary primes.
2 Algebraic theory

From now on, we assume that $G$ satisfies the following:

**Hypothesis on $G$** There is no element of order $p$ in $G$, and $G$ has a closed normal subgroup $H$ such that $\Gamma = G/H$ is isomorphic to $\mathbb{Z}_p$.

For example, if $G$ is the Galois group of a $p$-adic Lie extension of a number field $F$ which contains the cyclotomic $\mathbb{Z}_p$-extension of $F$, then $G$ satisfies the second part of our hypotheses. We do not consider the category of all finitely generated torsion $\Lambda(G)$-modules, but rather the full subcategory $\mathfrak{M}_H(G)$ consisting of all finitely generated $\Lambda(G)$-modules $M$ such that $M/M(p)$ is finitely generated over $\Lambda(H)$; here $M(p)$ denotes the $p$-primary submodule of $M$. In the special case when $H = 1$, $\mathfrak{M}_H(G)$ is indeed the category of all finitely generated torsion $\Lambda(G)$-modules. We define $S$ to be the set of all $f$ in $\Lambda(G)$ such that $\Lambda(G)/\Lambda(G)f$ is a finitely generated $\Lambda(H)$-module, and put

$$S^* = \bigcup_{n\geq 0} p^n S.$$

**Theorem 2.1** The set $S^*$ is a multiplicatively closed left and right Ore set in $\Lambda(G)$, all of whose elements are non-zero divisors. A finitely generated $\Lambda(G)$-module $M$ is $S^*$-torsion if and only if it belongs to the category $\mathfrak{M}_H(G)$.

Thus $S^*$ is a canonical Ore set in $\Lambda(G)$, and we write $\Lambda(G)_{S^*}$ for the localization of $\Lambda(G)$ at $S^*$. If $R$ is any ring with unit, we write $K_m R$ ($m = 0, 1$) for the $m$-th $K$-group of $R$, and $R^\times$ for the group of units of $R$.

**Theorem 2.2** The natural map

$$\Lambda(G)_{S^*}^\times \to K_1(\Lambda(G)_{S^*})$$

is surjective.

Let $K_0(\mathfrak{M}_H(G))$ denote the Grothendieck group of the category $\mathfrak{M}_H(G)$. We recall that $\Lambda(G)$ has finite global dimension because $G$ has no element of order $p$.

**Theorem 2.3** We have an exact sequence of localization

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\delta_2} K_0(\mathfrak{M}_H(G)) \to 0.$$
If $M \in \mathfrak{M}_H(G)$, we write $[M]$ for the class of $M$ in $K_0(\mathfrak{M}_H(G))$. We then define a characteristic element of $M$ to be any element $\xi_M$ of $K_1(\Lambda(G)_{S^*})$ such that

$$\partial_G(\xi_M) = [M].$$

It is shown in [1] that $\xi_M$ has all the good properties we would expect of characteristic elements. Most important amongst these for arithmetic applications is its behaviour under twisting. Let

$$\rho : G \longrightarrow GL_n(O)$$

be any continuous homomorphism, where $O$ denotes the ring of integers of a finite extension of $\mathbb{Q}_p$. Of course, $\rho$ induces a ring homomorphism

$$\rho : \Lambda(G) \longrightarrow M_n(O),$$

where $M_n(O)$ denotes the ring of $n \times n$ matrices with entries in $O$. If $f$ is any element of $\Lambda(G)$, we define $f(\rho)$ to be the determinant of $\rho(f)$. Although it is far from obvious, it is shown in [1] that one can extend this notion to define $\xi_M(\rho)$ to be either $\infty$ or $a$. If $M$ is any module in $\mathfrak{M}_H(G)$, we can also define

$$tw_{\rho}(M) = M \otimes_{\mathbb{Z}_p} O^n$$

where $G$ acts on the second factor via $\rho$, and on the whole tensor product via the diagonal action. Again we have $tw_{\rho}(M)$ belongs to $\mathfrak{M}_H(G)$. We define

$$\chi(G, tw_{\rho}(M)) = \prod_{i \geq 0} \#(H_i(G, tw_{\rho}(M)))^{(-1)^i},$$

saying that the Euler characteristic is finite if all the homology groups $H_i(G, tw_{\rho}(M))$ are finite. We write $\hat{\rho}$ for the contragredient representation of $\rho$, i.e. $\hat{\rho}(g) = \rho(g^{-1})^t$, where the ' $t$' denotes the transpose matrix.

**Theorem 2.4** Let $M \in \mathfrak{M}_H(G)$, and let $\xi_M$ denote a characteristic element of $M$. For each continuous representation $\rho : G \rightarrow GL_n(\sigma)$ such that $\chi(G, tw_{\hat{\rho}}(M))$ is finite, we have $\xi_M(\rho) \neq 0, \infty$ and

$$\chi(G, tw_{\hat{\rho}}(M)) = |\xi_M(G)|_{\mathbb{Q}_p}^{-m_{\rho}},$$

where $m_{\rho}$ denotes the degree over $\mathbb{Q}_p$ of the quotient field of $O$. 
3 Connexion with $L$-values

We only briefly discuss the main conjecture when $E$ is an elliptic curve defined over $\mathbb{Q}$, $p > 5$ is a prime of good ordinary reduction, $F_\infty = \mathbb{Q}(E_{p^\infty})$, and $G$ is the Galois group of $F_\infty$ over $\mathbb{Q}$. Thus $G$ has dimension 2 or 4 according as $E$ does or does not have complex multiplication. Let $X(E/F_\infty)$ be the dual of the Selmer group of $E$ over $F_\infty$. Taking $H$ to be the subgroup of $G$ which fixes the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, the following conjecture (which can be proven in some cases) is made in [1].

Conjecture 3.1 $X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$.

Now let $\rho$ be a variable Artin representation of $G$, i.e. a representation which factors through a finite quotient of $G$. Let $L(\rho, s)$ denote the complex $L$-function of $\rho$, and $L(E, \rho, s)$ the complex $L$-function of $E$ twisted by $\rho$. The $L$-functions $L(E, \rho, s)$ appear to have many interesting properties, but they appear to have been somewhat neglected by the experts on automorphic forms. The point $s = 1$ is critical for $L(E, \rho, s)$, and we assume in what follows the analytic continuation is known at $s = 1$. We fix a minimal Weierstrass equation for $E$ over $\mathbb{Q}$, and let $\Omega_+(E)$ and $\Omega_-(E)$ denote generators of the groups of real and purely imaginary periods of the Néron differential of $E$. Let $d^+(\rho)$ (resp. $d^-(\rho)$) denote the dimension of the subspace of the realization of $\rho$ which is fixed by complex multiplication (resp. on which complex conjugation acts like -1). A special case of Deligne's conjecture asserts that

$$\frac{L(E, \rho, 1)}{\Omega_+(E)^{d^+(\rho)}\Omega_-(E)^{d^-(\rho)}} \in \overline{\mathbb{Q}}.$$

Let $p^\rho$ denote the $p$-part of the conductor of $\rho$. For each prime $q$, we let $P_q(\rho, X)$ be the polynomial such that the Euler factor of $L(\rho, s)$ at $q$ is $P_q(\rho, q^{-s})^{-1}$. Also, since $E$ is ordinary at $p$, we have

$$1 - a_pX + pX^2 = (1 - uX)(1 - wX),$$

where $u \in \mathbb{Z}_p^*$ and, as usual, $p + 1 - a_p$ is the number of points over $\mathbb{F}_p$ on the reduction of $E$ module $p$. Let $R$ be the finite set consisting of $p$ and all primes $q$ such that $\text{ord}_q(j_E) < 0$. We write $L_R(E, \rho, s)$ for the complex $L$-function obtained by suppressing in $L(E, \rho, s)$ the Euler factors at the primes in $R$. The following two conjectures are made in [1].
Conjecture 3.2 Assume that $p \geq 5$ and $E$ has good ordinary reduction at $p$. Then there exists $\mathcal{L}_E$ in $K_1(\Lambda(G)_{S^*})$ such that, for all Artin representations $\rho$ of $G$, we have $\mathcal{L}_E(\rho) \neq \infty$, and

$$\mathcal{L}_E(\rho) = \frac{L_R(E, \rho, 1)}{\Omega_+(E)^{d^+(\rho)}\Omega_-(E)^{d^{-}(ho)}} e_p(\rho) u^{-f_{\rho}},$$

where $e_p(\rho)$ denotes the local $\epsilon$-factor attached to $\rho$ at $p$.

Conjecture 3.3 (The main conjecture) Assume that $p \geq 5$, $E$ has good ordinary reduction at $p$, and $X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$. Granted Conjecture 2, the $p$-adic $L$-function $\mathcal{L}_E$ in $K_1(\Lambda(G)_{S^*})$ is a characteristic element of $X(E/F_\infty)$.

Of course, when $E$ does not admit complex multiplication, very little is known at present about Conjecture 3. However, when $E = X_1(11)$ and $p = 5$, some remarkable numerical calculations of T. Fisher and T. and V. Dokchitser provide fragmentary evidence in support of it.

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References


