A topological L-function for a threefold

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1 Introduction

In recent days, analogies between the number theory and the theory of threefolds are discussed by many mathematicians([1][6][7]). It will be Mazur who first pointed out analogies between primes and knots in the standard three dimensional sphere. Morishita([7]) has investigeted a similarity between the absolute Galois of \( \mathbb{Q} \) and a link group. (A link group is defined to be the fundamental group of a complement of a link in the standard three sphere.) Moreover he has interpreted various symbols (eg. Hilbert, Rédei) from a topological point of view. For an example, he has shown one may consider the Hilbert symbol of two primes as their “linking number”.

In this report, we will study a similarity between the number theory and the theory of topological threefold from a viewpoint of a representation theory. Namely an L-function associated to a topological threefold will be discussed. Since our definition of an L-function will be based on one of a local system on a curve defined over a finite field (i.e. the Hasse-Weil’s congruent L-function), we will recall the definition the L-function in the arithmetic case.

2 A brief review of the Hasse-Weil’s congruent L-function

In what follows, for an object \( Z \) over a finite field \( \mathbb{F}_q \), its base extension to \( \overline{\mathbb{F}}_q \) will be denoted by \( \overline{Z} \). We fix a rational prime \( l \) which is prime to \( q \).

Let \( C \) be a smooth curve over a finite field \( \mathbb{F}_q \) and let \( C \rightarrow \overline{C} \) be its compactification. Suppose we are given a \( \mathbb{Q}_l \)-smooth sheaf \( \mathcal{F} \) on \( C \). Then the \( q \)-th Frobenius \( \phi_q \) acts on \( H^1(\overline{C}, j_* \mathcal{F}) \) and the Hasse-Weil L-function is defined to be

\[
L(C, \mathcal{F}, T) = \det[1 - \phi_q^* T|H^1(\overline{C}, j_* \mathcal{F})].
\]

It has

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• an functional equation,
• an Euler product.

Suppose \( F \) is deduced from an abelian fibration. Namely let \( A \to C \) be an abelian fibration whose moduli is not a constant. We set \( F = R^1 f_* \mathbb{Q}_l \) and

\[
L(A, s) = L(C, F, q^{-s}).
\]

Then \( L(A, s) \) is an entire function and Artin and Tate ([11]) have given a detailed conjecture for a special value of \( L \)-function, which is a geometric analogue of the Birch and Swinnerton-Dyer conjecture. Their conjecture predicts that the order of \( L(A, s) \) at \( s = 1 \) should be equal to the rank of the Mordell-Weil group of the fibration. They have shown this is equivalent to the finiteness of \( l \)-primary part of the Brauer group of \( A \).

3 A definition of an \( L \)-function of a topological threefold

3.1 The definition

Let \( X \) be the complement of a knot \( K \) in the standard three dimensional sphere. By the Alexander duality, we know \( H_1(X, \mathbb{Z}) \cong \mathbb{Z} \) and therefore it admits an infinite cyclic covering

\[
Y \to X.
\]

Let \( S \) be a minimal Seifert surface of \( K \). Then its inverse image \( \pi^{-1}(S) \) is a disjoint union \( \sqcup_{n \in \mathbb{Z}} S_n \) of copies of \( S \) indexed by integers. We assume that the genus of \( S \) is greater than or equal to two and that the fundamental groups of \( S_0 = S \) and \( Y \) are isomorphic. Let \( L_X \) be a polarized local system on \( X \) and let \( L_S \) be its restriction to \( S \). The deck transformation of the covering may be considered as a diffeomorphism of \( S \) and it is easy to see it lifts to an isomorphism \( \phi \) of the local system \( L_S \).

Let us compare our situation to the arithmetic one. The covering \( Y \to X \) corresponds to \( \overline{C} \to C \) and the local system \( L_X \) is an analogy of \( F \). Let \( \rho_X \) be the representation of \( \pi_1(X) \) associated to \( L_X \). Since \( \pi^* L_X \) is the local system for the restriction of \( \rho_X \) to \( \pi_1(Y) \cong \pi_1(S) \), we may identify it with \( L_S \). Hence \( L_S \) is an analogy of \( F \) and \( \phi \) corresponds to the Frobenius.

According to the observation above, we will make the following set up.

Let \( \tilde{X} \) be a compact smooth threefold which may have smooth boundaries. Suppose it has an infinite cyclic covering

\[
\tilde{Y} \to \tilde{X}
\]

which satisfies the following properties.
1. There is a smoothly embedded connected surface

\[ \overline{S} \hookrightarrow \overline{X} \]

whose genus is greater than or equal to 2 and the boundaries are contained in \( \partial \overline{X} \) via \( i \).

2. Let \( \overline{T} \) be the inverse image of \( \overline{S} \) by \( \pi \), which is a disjoint union of copies of \( \overline{S} \) indexed by integers:

\[ \overline{T} = \sqcup_{n \in \mathbb{Z}} \overline{S}_{n}, \quad \overline{S}_{n} \cong \overline{S}. \]

Then the map \( i_{0} \) induces an isomorphism

\[ \pi_{1}(\overline{S}, s_{0}) \cong \pi_{1}(\overline{Y}, i_{0}(s_{0})). \]

We will refer such an infinite cyclic covering to be of a surface type. The following notations will be used.

**Notations 3.1.**

1. \( X \) (resp. \( S, Y \)) is the interior of \( \overline{X} \) (resp. \( \overline{S}, \overline{Y} \)).

2. \( \Pi_{S} \) (resp. \( \Pi_{Y}, \Pi_{X} \)) is the fundamental group of \( S \) (resp. \( Y, X \)) with respect to the base point \( s_{0} \) (resp. \( i_{0}(s_{0}), \pi(i_{0}(s_{0})) \)).

3. \( \Pi_{Y/X} \) is the covering transformation group of \( Y \to X \).

Let \( \Phi \) be a deck transformation generating \( \Pi_{Y/X} \). Identifying \( S \) with \( S_{0} \) (resp. \( S_{1} \)) via \( i_{0} \) (resp. \( i_{1} \)), \( \Phi \) induces a diffeomorphism \( \phi \) on \( S \) by restriction. Since the genus of \( S \) is greater than or equal to 2, it is diffeomorphic to a quotient of the Poincaré upper half plane \( \mathbb{H}^{2} \) by a discrete subgroup \( \Gamma \) of \( PSL_{2}(\mathbb{R}) \). Adding cusps \( \Sigma \) to the quotient, we get compactification \( S^{*} \). We will sometimes identify \( S \) with \( S^{*} \setminus \Sigma \).

**Remark 3.1.** Note that there is an exact sequence

\[ 1 \to \Pi_{S} \to \Pi_{X} \to \mathbb{Z} \to 1. \]

This is a geometric counterpart of the following situation in arithmetic geometry. Let \( C \) be a smooth curve defined over \( \mathbb{F}_{q} \) and let \( \overline{C} \) be its base extension to \( \overline{\mathbb{F}}_{q} \). Then their fundamental groups fit in the exact sequence

\[ 1 \to \pi_{1}(\overline{C}) \to \pi_{1}(C) \to \hat{\mathbb{Z}} \to 1. \]

Let \( F \) be a field of characteristic 0 and let \( L \) be a vector space over \( F \) of dimension \( 2g \) with a skew-symmetric nondegenerate pairing \( \alpha \). Suppose we are given a representation

\[ \Pi_{X} \xrightarrow{\rho_{S}} Aut(L, \alpha) \]

such that

\[ L^{\Pi_{S}} = 0, \quad (1) \]
Let $\rho_S$ be the restriction of $\rho_X$ to $\Pi_S$ and the local system associated to $\rho_X$ (resp. $\rho_S$) will be denoted by $L_X$ (resp. $L_S$). Then the diffeomorphism $\phi$ induces an isomorphism of a polarized local system:

$$
\begin{array}{c}
L_S \\
\phi \\
S
\end{array} \overset{\cong}{\longrightarrow} \begin{array}{c}
L_S \\
\phi \\
S
\end{array}
$$

Fig. 2.2

Let $j$ be the open immersion of $S$ into $S^*$ and let $i$ be the inclusion of $\Sigma$ into $S^*$. Then $\hat{\phi}$ acts on $H^1(S^*, j_*L_S)$, which is a geometric analogue of the Frobenius action. For a point $P$ in $\Sigma$, let $\Delta_P$ be a small disc centered at $P$ and we set $\Delta_P^* = \Delta_P \setminus \{P\}$. The parabolic cohomology $H^1_P$ is defined to be

$$
H^1_P(S, L_S) = \text{Ker}[H^1(S, L_S) \rightarrow \oplus_{P \in \Sigma} H^1(\Delta_P^*, Cs)].
$$

One can easily see that $H^1_P(S, L_S)$ admits an action of $\hat{\phi}$ and it is isomorphic to $H^1(S^*, j_*L_S)$ as a $F[\hat{\phi}]$-module. Also the nondegenerate skew symmetric pairing $\alpha$ and the Poincaré duality induce a perfect pairing on $H^1_P(S, L_S)$, which is invariant under the action of $\hat{\phi}$. Hence $H^1_P(S, L_S)$ is a semisimple $F[\hat{\phi}]$-module and it is isomorphic to its dual as a $F[\hat{\phi}]$-module.

Now we define the topological $L$-function $L(X, L_X)$ for the local system $L_X$ to be

$$
L(X, L_X) = \det[1 - \hat{\phi}^*T|H^1_P(S, L_S)].
$$

Here $T$ is an indeterminate.

Let $M_\phi(S)$ be the mapping torus of $\phi$ and let $M_\phi(L_S)$ be the local system on $X$ which is obtained by the same way as "mapping torus" from the isomorphism $L_S \overset{\cong}{\cong} L_S$. Note that by the definition we have

$$
L(X, L_X) = L(M_\phi(S), M_\phi(L_S)).
$$

3.2 Examples

Let $K$ be a knot embedded in the standard three dimensional sphere $S^3$ and let $N_K$ be its tubular neighborhood. Let $\bar{X}$ be the closure of the complement of
$N_K$ in $S^3$. Then $H_1(\bar{X}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ by the Alexander duality and $\bar{X}$ admits an infinite cyclic covering

$$\bar{Y} \xrightarrow{\mathbb{Z}} \bar{X}.$$ 

Let $X$ (resp. $Y$) be the interior of $\bar{X}$ (resp. $\bar{Y}$) (cf. Notations 3.1). Then the map induces an exact sequence

$$1 \to \Pi_Y \to \Pi_X \to \mathbb{Z} \to 1. \quad (2)$$

Let $\hat{S}$ be a minimal Seifert surface of $K$ and we set

$$\bar{S} = \hat{S} \cap \bar{X}.$$ 

It is known if $\Pi_Y$ is finitely generated, $S \xrightarrow{\mathbb{Z}} Y$ induces an isomorphism ([5])

$$\Pi_S \approx \Pi_Y.$$ 

Moreover Murasugi has shown if the absolute value of the Alexander polynomial $\Delta_K(t)$ of $K$ at $t = 0$ is equal to 1, then $\Pi_Y$ is finitely generated.

Fact 3.1. ([4/IV. Proposition 5) Suppose every closed incompressible surface in $X$ is boundary parallel. Then either

1. $X$ is Seifert fibred,

or

2. $X$ is hyperbolic. Namely there is the maximal order $O_F$ of an algebraic number field $F$ and a torsion free subgroup $\Gamma \subset PSL_2(O_F)$ such that $X$ is diffeomorphic to $\Gamma \backslash \mathbb{H}^3$. Here after fixing an embedding $F \rightarrow \mathbb{C}$, $\Gamma$ is regarded to be a subgroup of $PSL_2(\mathbb{C})$.

Now we assume that the infinite cyclic covering satisfies the following conditions.

Condition 3.1. 1. $\Pi_Y$ is finitely generated.

2. Either

(a) $X$ is Seifert fibred,

or

(b) there is the maximal order $O_F$ of an algebraic number field $F$ and a torsion free subgroup $\Gamma \subset SL_2(O_F)$ which freely acts on $\mathbb{H}^3$ so that $X$ is diffeomorphic to $\Gamma \backslash \mathbb{H}^3$. As before after fixing an embedding $F \rightarrow \mathbb{C}$, $\Gamma$ is regarded to be a subgroup of $SL_2(\mathbb{C})$.

Remark 3.2. Professor Fujii kindly informed us that if $X$ is hyperbolic, then the Condition 3.1. 2 (b) is always satisfied.
Suppose $X$ satisfies 1 and 2(b) of Condition 3.1. Then we have the canonical representation
\[ \Pi_X \simeq \Gamma_{\frac{\rho_X}{\phi}} \mathbb{S}L_2(\mathbb{C}). \]
We set
\[ L = \mathbb{C}^{\oplus 2}, \]
and let $\alpha$ be the standard symplectic form on $L$. Namely for elements $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of $L$, $\alpha(x, y)$ is defined as
\[ \alpha(x, y) = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}. \]
This invariant under the action of $\Pi_X$. The result of Neuwirth([5]) implies
\[ \Pi_S \simeq \Pi_Y \pi_1(i_{\mathbb{O}}), \]
and it is easy to see
\[ L^{\Pi_S} = 0. \]
Hence the conditions in §3.1 are satisfied.

Next suppose that $X$ satisfies 1 and 2(a) of Condition 3.1. Then under a mild condition, one can check its fundamental group has a linear representation
\[ \Pi_X \xrightarrow{\rho_X} \mathbb{S}L_2(\mathbb{C}) \]
such that
\[ L^{\Pi_S} = 0. \]
Details will be found in [9].

**Remark 3.3.** Even if we take the trivial representation, we can define an $L$-function for a knot complement. Note that this is nothing but the Alexander polynomial, which corresponds to the congruent zeta function of a curve. But contrary to the arithmetic case, as we have seen, we have a priori a two dimensional irreducible linear representation of $\pi_1(X)$. This is one of the main reasons to consider the $L$-function.

## 4 Properties of a topological L-function

In the present section, we will list up basic properties of our topological $L$-function. Proofs of the statements will be found in [9].
4.1 A functional equation

Let \( b(\mathcal{L}_S) \) be the dimension of \( H^1_P(S, \mathcal{L}_S) \).

**Theorem 4.1.** (The functional equation)

\[
L(X, \mathcal{L}_X)(T) = (-T)^{b(\mathcal{L}_S)} L(X, \mathcal{L}_X)(T^{-1}).
\]

**Corollary 4.1.** Suppose \( b(\mathcal{L}_S) \) is odd. Then \( L(X, \mathcal{L}_X)(1) \) vanishes and in particular the dimension of \( H^1_P(S, \mathcal{L}_S)^\phi \) is positive.

4.2 A geometric analogue of Birch and Swinnerton-Dyer conjecture

In the present section, we will work with the holomorphic category.

Let \( A^* \) be a smooth projective variety with a morphism

\[
A^* \xrightarrow{\mu} S^*
\]

such that its restriction to \( S \)

\[
A \xrightarrow{\overline{\mu}} S
\]

is a smooth fibration whose fibres are abelian varieties of dimension \( g \). Moreover we assume \( \overline{\mu} \) satisfies the following conditions.

**Condition 4.1.**
1. \( A^*/S^* \) is the Neron model of \( A/S \) and has a semistable reduction at each point \( s \in \Sigma \).
2. \( R^1\mu_*\mathbb{Q} \) is isomorphic to the local system \( \mathcal{L}_S \).
3.

\[
H^0(S^*, R^1\overline{\mu}_*\mathcal{O}_{A^*}) = 0.
\]

Suppose there is a commutative diagram

\[
\begin{array}{ccc}
A^* & \xrightarrow{\Phi} & A^* \\
\downarrow{\overline{\mu}} & \cong & \downarrow{\overline{\mu}} \\
S^* & \xrightarrow{\phi} & S^* \\
\end{array}
\]

Fig. 4.1
such that $\phi(\Sigma) = \Sigma$. Since $L_S = R^1 \mu_* Q$, this induces the diagram as Fig. 2.2. We define the Mordell-Weil group $MW_X(A)$ to be

$$MW_X(A) = A(S)^\phi$$

and its rank will be denoted by $r_X(A)$. Since the cycle map induces an imbedding

$$A(S) \otimes Q \hookrightarrow H^1_P(S, L_S),$$

$r_X(A)$ is less than or equal to the order of the topological L-function $L(X, L_X)$ at $T = 1$.

**Theorem 4.2.** Suppose $H^2(A^*, O_{A^*}) = 0$. Then $r_X(A)$ is equal to the order of the topological L-function $L(X, L_X)$ at $T = 1$.

We define the topological Brauer group $Br_{top}(A^*)$ to be

$$Br_{top}(A^*) = H^2(A^*, O_{A^*}^\times).$$

Then the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow O_{A^*} \rightarrow O_{A^*}^\times \rightarrow 0$$

implies the exact sequence

$$H^2(A^*, \mathbb{Z}) \rightarrow H^2(A^*, O_{A^*}) \rightarrow Br_{top}(A^*) \rightarrow H^3(A^*, \mathbb{Z}).$$

Since $A^*$ is compact, both $H^2(A^*, \mathbb{Z})$ and $H^3(A^*, \mathbb{Z})$ are finitely generated abelian groups. Hence $Br_{top}(A^*)$ is finitely generated if and only if $H^2(A^*, O_{A^*})$ vanish since the latter is a complex vector space.

**Corollary 4.2.** Suppose $Br_{top}(A^*)$ is finitely generated. Then the rank of the Mordell-Weil group $r_X(A)$ is equal to the order of the topological L-function $L(X, L_X)$ at $T = 1$.

Note that the corollary above is a geometric analogue of the theorem of Artin and Tate. ([10][11])

### 4.3 An Euler product and an Euler system

Suppose the map $\phi$ in Fig. 2.2 satisfies the following condition.

**Condition 4.2.** There exists a diffeomorphism $\phi_0$ of $S$ such that

1. $\phi_0$ is homotopic to $\phi$,

and

2. every fixed point of $\phi_0^n$ is non-degenerate and is isolated for any positive integer $n$. 
Because of Condition 4.2(1), Fig. 2.2 may be replaced by:

\[
\begin{array}{ccc}
\mathcal{L}_S & \xrightarrow{\phi_0} & \mathcal{L}_S \\
S & \xrightarrow{\phi_0} & S \\
\end{array}
\]

Fig. 7.1

We prepare some notations. Let us fix a positive integer \( n \). The set of fixed points of \( \phi_0^n \) will be denoted by \( S^{\phi_0^n} \). We define \( \Phi_0(n) \) to be the orbit space of the action of \( \phi_0 \) on

\[
\{ s \in S | \phi_0^n(s) = s \text{ and } \phi_0^m(s) \neq s \text{ for } 1 \leq m \leq n - 1 \}
\]

and we set

\[
\Phi_0 = \bigcup_{n=1}^{\infty} \Phi_0(n).
\]

For an element \( \gamma \) of \( \Phi_0(n) \), we call the integer \( n \) its length and we will denote it by \( l(\gamma) \). Let \( x \in S^{\phi_0^{l(\gamma)}} \) be a representative of \( \gamma \in \Phi_0 \). Then \( \hat{\phi}_0^{l(\gamma)} \) defines an automorphism of the fibre of \( \mathcal{L}_S \otimes \mathbb{Q} \) at \( x \) and the polynomial

\[
\det[1 - \hat{\phi}_0^{l(\gamma)}T|_{(\mathcal{L}_S \otimes \mathbb{Q})_x}]
\]

is independent of the choice of \( x \), which will be written as \( P_\gamma(T) \).

Let \( V^P \) is the invariant subspace of \( L \otimes \mathbb{Q} \) under the action of \( \pi_1(\Delta_P^*) \). It is easy to see \( \oplus_{P \in \Sigma} V^P \) has an action of \( \hat{\phi} \). Now the Grothendieck-Lefshetz trace formula implies the following theorem.

**Theorem 4.3. (Euler product formula)** Suppose the map \( \phi \) in Fig. 2.2 satisfies the Condition 4.2. Then

\[
L(X, \mathcal{L}_X) = (\det[1 - \hat{\phi}^*T|_{\oplus_{P \in \Sigma} V^P}]^{-1} \prod_{\gamma \in \Phi_0} P_\gamma(T^{l(\gamma)})^{-1}.
\]

Our L-function has a Euler system, which has been considered by Kolyvagin in the Iwasawa theory of an elliptic curve ([8]). Let \( \phi_0 \) be a diffeomorphism
of \( S \) satisfying the Condition 4.2 and let us fix a generator \( t \) of \( \Pi_{Y/X} \simeq \mathbb{Z} \). Then \( \mathbb{Q}[\Pi_{Y/X}] \) may be identified with \( P = \mathbb{Q}[t, t^{-1}] \) and defining the action of \( t \) by \((\phi_{0})^{-1}\), the compact supported cohomology group \( H_{c}^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q}) \) may be regarded as a \( P \)-module. In general, the Fitting ideal of a finitely generated \( P \)-module \( M \) will be denoted by \( \text{Fitt}_{P}(M) \). The following lemma directly follows from the definition of our \( L \) function.

**Lemma 4.1.**

\[
\text{Fitt}_{P}(H_{c}^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q})) = (L_{c}(X, \mathcal{L}_{X})),
\]

where \( L_{c}(X, \mathcal{L}_{X}) \) is defined to be

\[
L_{c}(X, \mathcal{L}_{X}) = \text{det}[1 - \hat{\phi}_{0}^{*}t|H_{c}^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q})].
\]

For \( \gamma \in \Phi_{0}(n) \), let \( O_{\gamma} \subset S \) be the corresponding orbit of \( \phi_{0} \) and let \( S_{\gamma} \) be its complement. The corestriction map

\[
H_{c}^{1}(S_{\gamma}, \mathcal{L}_{S} \otimes \mathbb{Q}) \xrightarrow{C_{\mathbb{Q}}} H_{c}^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q})
\]

is defined to be the Poincaré dual of the restriction map

\[
H^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q}) \xrightarrow{\text{Res}} H^{1}(S_{\gamma}, \mathcal{L}_{S} \otimes \mathbb{Q}).
\]

Observe that both of them are homomorphism of \( P \)-modules. The Thom-Gysin exact sequence implies

\[
0 \rightarrow H^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q}) \xrightarrow{\text{Res}} H^{1}(S_{\gamma}, \mathcal{L}_{S} \otimes \mathbb{Q}) \rightarrow \bigoplus_{x \in O_{\gamma}}(\mathcal{L}_{S} \otimes \mathbb{Q})_{x} \rightarrow 0,
\]

and let

\[
0 \rightarrow \bigoplus_{x \in O_{\gamma}}(\mathcal{L}_{S} \otimes \mathbb{Q})_{x} \rightarrow H_{c}^{1}(S_{\gamma}, \mathcal{L}_{S} \otimes \mathbb{Q}) \xrightarrow{C_{\mathbb{Q}}} H_{c}^{1}(S, \mathcal{L}_{S} \otimes \mathbb{Q}) \rightarrow 0 \quad (3)
\]

be its dual sequence. The following lemma follows from the observation:

\[
\text{Fitt}_{P}(\bigoplus_{x \in O_{\gamma}}(\mathcal{L}_{S} \otimes \mathbb{Q})_{x}) = (P_{\gamma}(t^{l(\gamma)})).
\]

**Lemma 4.2.**

\[
\text{Fitt}_{P}(H_{c}^{1}(S_{\gamma}, \mathcal{L}_{S} \otimes \mathbb{Q})) = (L_{c}(X, \mathcal{L}_{X}) \cdot P_{\gamma}(t^{l(\gamma)})).
\]

In general for an \( N \)-tuples of distinct elements \( \{\gamma_{1}, \cdots, \gamma_{N}\} \) of \( \Phi_{0} \), we set

\[
S_{\gamma} = S \setminus (O_{\gamma_{1}} \cup \cdots \cup O_{\gamma_{N}}).
\]

The induction on \( N \) shows the following proposition.

**Proposition 4.1.**

\[
\text{Fitt}_{P}(H_{c}^{1}(S_{\gamma}, \mathcal{L}_{S} \otimes \mathbb{Q})) = (L_{c}(X, \mathcal{L}_{X}) \cdot \prod_{i=1}^{N} P_{\gamma_{i}}(t^{l(\gamma)})).
\]
Definition 4.1. (An Euler system of a topological $L$-function) Let $\gamma$ be the empty set or an $N$-tuples of distinct elements of $\Phi_0$. Suppose a finitely generated $P$-modules $V_\gamma$ is given for such $\gamma$. If $\{V_\gamma\}_\gamma$ satisfy the following conditions, they will be refered as Euler system of the topological $L$-function.

1. 
   \[ \text{Fitt}_P(V_\phi) = (L_c(X, L_X)), \]

2. Suppose 
   \[ \gamma' = \gamma \cup \{\gamma_{N+1}\}, \quad \gamma_{N+1} \notin \gamma. \]
   Then there is a surjection as $P$-modules 
   \[ V_{\gamma'} \rightarrow V_\gamma \]
   and their Fitting ideals satisfy the relation 
   \[ \text{Fitt}_P(V_{\gamma'}) = \text{Fitt}_P(V_\gamma) \cdot (P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})). \]

We set 
\[ V_\phi = H^1_c(S, L_S \otimes \mathbb{Q}) \]
and for an $N$-tuples of distinct elements $\gamma$ of $\Phi_0$ we define 
\[ V_\gamma = H^1_c(S_\gamma, L_S \otimes \mathbb{Q}). \]

Then $\{V_\gamma\}_\gamma$ is an Euler system by Proposition 4.1.

Next we will show how Kolyvagin’s Euler system appears in our geometric situation. We assume any two of $\{P_\gamma(t^{l(\gamma)})\}_{\gamma \in \Phi_0}$ are relatively prime. Let $\gamma$ and $\gamma'$ be as 2. of Definition 4.1. The same arguments of to obtain (3) shows 
\[ 0 \rightarrow \oplus_{x \in O_{\gamma_{N+1}}} (L_S \otimes \mathbb{Q})_x \rightarrow H^1_c(S_{\gamma'}, L_S \otimes \mathbb{Q}) \xrightarrow{Cor} H^1_c(S_\gamma, L_S \otimes \mathbb{Q}) \rightarrow 0. \]

Note that $P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})$ annihilates $\oplus_{x \in O_{\gamma_{N+1}}} (L_S \otimes \mathbb{Q})_x$ and by the assumption its multiplication on $H^1_c(S_\gamma, L_S \otimes \mathbb{Q})$ is an isomorphism. These observations imply the following lemma.

Lemma 4.3. Let us fix $x_\gamma \in H^1_c(S_\gamma, L_S \otimes \mathbb{Q})$. If we take $y_{\gamma'} \in H^1_c(S_{\gamma'}, L_S \otimes \mathbb{Q})$ so that 
\[ Cor(y_{\gamma'}) = x_\gamma. \]

Then we have 
\[ Cor(P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})y_{\gamma'}) = P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})x_\gamma. \]
Moreover $P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})y_{\gamma'}$ is independent of the choice of $y_{\gamma'}$. 

Now we fix a non-zero element $c_\phi$ of $H^1_c(S, \mathcal{L}S \otimes \mathbb{Q})$. For an $N$-tuples of distinct elements $\gamma$ of $\Phi_0$ we will inductively define an element $c_\gamma$ of $H^1_c(S_\gamma, \mathcal{L}S \otimes \mathbb{Q})$. Let $\gamma' = \gamma \cup \{\gamma_{N+1}\}$ be as before. We take any $d_{\gamma'} \in H^1_c(S_{\gamma'}, \mathcal{L}S \otimes \mathbb{Q})$ to be

$$\text{Cor}(d_{\gamma'}) = c_\gamma,$$

and we set

$$c_\gamma = P_{\gamma_{N+1}}(t^{h(\gamma_{N+1})})d_{\gamma'}.$$  

Then the system $\{c_\gamma\}_\gamma$ is well-defined by Lemma 4.3 and they satisfy

$$\text{Cor}(c_\gamma) = P_{\gamma_{N+1}}(t^{h(\gamma_{N+1})})c_\gamma,$$

which is the same relation as Kolyvagin's Euler system ([8]). One may realize an Euler system is another appearance of Euler product.

4.4 The Franz-Reidemeister torsion and a special value

We will briefly recall the theory of densities and the Franz-Reidemeister torsion ([2] [3] [9]). Throughout the subsection, let $F$ be equal to $\mathbb{R}$ or $\mathbb{C}$. Let $V$ be a vector space over $F$ of dimension $r > 0$ and let $\{v_1, \cdots, v_r\}$ be its basis. We set

$$(\Lambda^r V)^\times = \{a \cdot v_1 \wedge \cdots \wedge v_r | a \in F^\times\}$$

and

$$|\Lambda^r V| = (\Lambda^r V)^\times / \{\pm 1\}.$$ 

Then $|\Lambda^r V|$ is isomorphic to $F^\times / \{\pm 1\}$ and will be mentioned as the space of densities on $V$. Let

$$(\Lambda^r V)^\times \xrightarrow{\pi} |\Lambda^r V|$$

be the canonical projection and the image $\pi(f)$ of $f \in (\Lambda^r V)^\times$ will be denoted by $|f|$. For the 0 dimensional vector space $0$, we define

$$\Lambda^0 0 = F; \quad (\Lambda^0 0)^\times = F^\times$$

and

$$|\Lambda^0 0| = F^\times / \{\pm 1\}.$$ 

Moreover for $f \in \Lambda^0 0 = F^\times$, its image in $|\Lambda^0 0| = F^\times / \{\pm 1\}$ will be denoted by $|f|$. If $F$ is $\mathbb{R}$, the canonical projection

$$(\Lambda^0 0)^\times \xrightarrow{\pi} |\Lambda^0 0| \simeq \mathbb{R}_{>0}$$

is nothing but the map of taking absolute value. In the followings, we always assume the 0 dimensional vector space $0$ has the density $1 \in |\Lambda^0 0| = F^\times / \{\pm 1\}$. Also we always assume every complex is bounded and consists of finite dimensional vector spaces over $F$. 
Definition 4.2. If a complex

\[ C' = [C^0 \rightarrow \cdots \rightarrow C^n] \]

has a density on each \( C^i \) and \( H^i \), we say the complex \( C' \) is given a density.

Remark 4.1. When \( C^i = H^i \), we assume \( H^i \) is given the same density as \( C^i \).

For a complex with a density

\[ C' = [C^0 \rightarrow \cdots \rightarrow C^n], \]

we can associate an element \( \tau_{FR}(C') \) of \( F^x/\{\pm 1\} \), which is called as the Franz-Reidemeister torsion (the FR-torsion for simplicity). Let \( |C^i| \) (resp. \( |H^i| \)) be the density on \( C^i \) (resp. \( H^i \)). Then one may intuitively think of \( \tau_{FR}(C') \) as

\[ \tau_{FR}(C') = \prod_{i=1}^{n} \frac{|C^i|}{|H^i|}^{(-1)^i}. \]

Let us take a finite triangulation of \( S \) which is preserved by \( \phi \). Then by a parallel transformation of the symplectic form \( \alpha \), we obtain a complex with a density \( C'_\phi \) such that its cohomology groups are isomorphic to \( H'(M_\phi(S), M_\overline{\phi}(\mathcal{L}_S)) \). Using the previous observation:

\[ L(X, \mathcal{L}_X) = L(M_\phi(S), M_\overline{\phi}(\mathcal{L}_S)), \]

we can show the following theorem.

Theorem 4.4. Suppose \( \hat{\phi}^* - 1 \) is isomorphic on \( H^1_{\phi}(S, \mathcal{L}_S) \). Then we have

\[ |L(X, \mathcal{L}_X)(1)| = \tau_{FR}(C'_\phi). \]

Remark 4.2. In general, we can show the following statement:

Let \( r \) be the dimension of \( \text{Ker}[\hat{\phi}^* - 1|H^1_{\phi}] \). Then we have

\[ \lim_{T \to 1} |(T - 1)^{-r}L(X, \mathcal{L}_X)(T)| = R((H^1_{\phi}')) \cdot \tau_{FR}(C'_\phi). \]

Here \( R((H^1_{\phi}')) \) is the regulator of the local system. Note that this is quite similar to the formula which is predicted by the Birch and Swinnerton-Dyer conjecture.

References


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