A spectral sequence for Iwasawa adjoints

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Abstract

This article is, apart from some small corrections, an unchanged version of a note written in 1994. It was circulated as a manuscript and quoted in several articles. Its aim was to provide a purely algebraic tool for the Iwasawa theory of arbitrary $p$-adic Lie groups, by constructing a spectral sequence which relates the generalized 'Iwasawa adjoints' $E^i(M)$ of certain Iwasawa modules $M$ to projective limits often encountered in the applications. A somewhat extended version will be published elsewhere.

Let $k$ be a number field, fix a prime $p$, and let $k_{\infty}$ be some Galois extension of $k$ such that $\mathcal{G} = \text{Gal}(k_{\infty}/k)$ is a $p$-adic Lie-group (e.g., $\mathcal{G} \cong \mathbb{Z}_p^r$ for some $r \geq 1$). Let $S$ be a finite set of primes containing all primes above $p$ and $\infty$, and all primes ramified in $k_{\infty}/k$, and let $k_S$ be the maximal $S$-ramified extension of $k$; by assumption, $k_{\infty} \subseteq k_S$. Let $G_S = \text{Gal}(k_S/k)$ and $G_{\infty,S} = \text{Gal}(k_{\infty}/k_S)$.

Let $A$ be a discrete (left) $G_S$-module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geq 1$ as an abelian group (e.g., $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, or $A = E[p^{\infty}]$, the group of $p$-power torsion points of an elliptic curve $E/k$ with good reduction outside $S$). We are not assuming that $G_{\infty,S}$ acts trivially.

Let $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ be the completed group ring. For a finitely generated $\Lambda$-module $M$ we put

$$E^i(M) = \text{Ext}_A^i(M, \Lambda).$$

Hence $E^0(M) = \text{Hom}_A(M, \Lambda) =: M^+$ is just the $\Lambda$-dual of $M$. This has a natural structure of a $\Lambda$-module, by letting $\sigma \in \mathcal{G}$ act via

$$\sigma f(m) = f(m) \cdot \sigma^{-1}$$

for $f \in M^+, m \in M$. It is known that $\Lambda$ is a noetherian ring (here we use that $\mathcal{G}$ is a $p$-adic Lie group), by results of Lazard [La]. Hence $M^+$ is a finitely generated $\Lambda$-module again (choose a projection $\Lambda^r \rightarrow M$; then we have an injection $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$). By standard homological algebra, the $E^i(M)$ are finitely generated $\Lambda$-modules for all $i \geq 0$. 
**Examples**  
a) If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$ is the classical Iwasawa algebra, and, for a $\Lambda$-torsion module $M$, $E^1(M)$ is isomorphic to the Iwasawa adjoint, which can be defined as

$$\text{ad}(M) = \lim_{n \to \infty} (M/\alpha_n M)$$

where $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of elements in $\Lambda$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $(\alpha_n)$ is prime to the support of $M$ for every $n \geq 1$, and where

$$N^\vee = \text{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a compact $\mathbb{Z}_p$-module $N$. For any finitely generated $\Lambda$-module $M$, $E^1(M)$ is quasi-isomorphic to $\text{Tor}_\Lambda(M)^\sim$, where $\text{Tor}_\Lambda(M)$ is the $\Lambda$-torsion submodule of $M$, and $M^\sim$ is the "Iwasawa twist" of a $\Lambda$-module $M$: the action of $\gamma \in \mathcal{G}$ is changed to the action of $\gamma^{-1}$.

b) If $\mathcal{G} = \mathbb{Z}_p^f$, $r \geq 1$, then the $E^i(M)$ are the standard groups considered in local duality. By duality for the ring $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \ldots, x_r]]$, they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

**Theorem 1** There is a spectral sequence

$$E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \lim_{k' \in k, m} H^{p+q}(G_{S}(k'), A[p^m]).$$

Here the limit runs through the natural numbers $m$ and the finite extensions $k'/k$ contained in $k_\infty$, via the natural maps

$$H^n(G_S(k'), A[p^{m+1}]) \to H^n(G_S(k'), A[p^m]) \quad (m \geq 1)$$

and the corestrictions.

Before we give the proof, we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$0 \to E^1(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}_1} \lim_{k'} H^1(G_S(k'), T_p A) \to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}_2} \lim_{k'} H^2(G_S(k'), T_p A).$$

To say more, we consider some assumptions.

A.1 Assume that $p > 2$ or that $k_\infty$ is totally imaginary. Then

$$H^r(G_{\infty,S}, A) = 0 = \lim_{k'} H^r(G_S(k'), T_p A)$$

for all $r > 2$.

**Corollary 2** Assume that $H^2(G_{\infty,S}, A) = 0$ (This is the so-called "weak Leopoldt conjecture" for $A$. It is stated classically for $A = \mathbb{Q}_p/\mathbb{Z}_p$, and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja2]).
Then the cokernel of $\inf^2$ is
\[ \ker (E^1(H^1(G_{\infty,S}, A)^\vee)) \to E^3(H^0(G_{\infty,S}, A)^\vee), \]
and there are isomorphisms
\[ E^i(H^1(G_{\infty,S}, A)^\vee) \simarrow E^{i+2}(H^0(G_{\infty,S}, A)^\vee) \]
for $i \geq 2$.

**Proof** This comes from A.1 and the following picture of the spectral sequence

![Spectral Sequence Diagram](image)

**Corollary 3** Assume that $H^0(G_{\infty,S}, A) = 0$. Then
(a) \[ \lim_{k'} H^1(G_S(k'), T_p A) \simarrow H^1(G_{\infty,S}, A)^\vee \]
(b) There is an exact sequence
\[ 0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \lim_{k'} H^2(G_S(k'), T_p A) \to (H^2(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0 \]
(c) There are isomorphisms
\[ E^i(H^2(G_{\infty,S}, A)^\vee) \simarrow E^{i+2}(H^1(G_{\infty,S}, A)^\vee) \]
for $i \geq 1$.

**Proof** In this case, the spectral sequence looks like

![Spectral Sequence Diagram](image)
Corollary 4 Assume that \( \mathcal{G} \) is a \( p \)-adic Lie group of dimension 1 (equivalently: an open subgroup is \( \cong \mathbb{Z}_p \)). Then \( E^i(-) = 0 \) for \( i \geq 3 \). Let

\[
B = \text{im} \left( \inf^2 : E^2(H^0(G_{\infty}, s, A)^\vee) \to \varprojlim_{k'} H^2(G_{S}(k'), T_pA) \right)
\]

Then \( B \) is finite, and there is an exact sequence

\[
0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \varprojlim_{k'} H^2(G_{S}(k'), T_pA)/B \to (H^2(G_{\infty,s}, A)^\vee)^+ \to E^2(H^1(G_{\infty,s}, A)^\vee) \to 0,
\]

and

\[
E^1(H^2(G_{\infty,s}, A)^\vee) = 0 = E^2(H^2(\mathbb{G}_{\infty,s}, A)^\vee),
\]

i.e., \( (H^2(G_{\infty,s}, A)^\vee \) is a projective \( \Lambda \)-module.

Proof Quite generally, for a \( p \)-adic Lie group \( \mathcal{G} \) of dimension \( n \) one has \( vcd_p(\mathcal{G}) = n \) for the virtual cohomological \( p \)-dimension of \( \mathcal{G} \), and hence \( E^i(-) = 0 \) for \( i > n + 1 \), cf. [Ja3]. The finiteness of \( E^2(M) \) (for a finitely generated \( \Lambda \)-module \( M \)) in our case is well-known, cf. [Ja3]. The remaining claims follow from the following shape of the spectral sequence:

\[
\begin{array}{cccc}
2 & \times & \times & \\
1 & \times & \times & \\
\end{array}
\]

Lemma 5 Assume that \( \mathcal{G} \) is a \( p \)-adic Lie group of dimension \( n \) (e.g., \( \mathcal{G} \) contains an open subgroup \( \cong \mathbb{Z}_p^n \)). Then \( E^i(H^0(G_{\infty,s}, A)^\vee) = 0 \) for \( i \neq n, n + 1 \).

(a) If \( H^0(G_{\infty,s}, A) \) is divisible (e.g., if \( G_{\infty,s} \) acts trivially on \( A \)), then

\[
E^i(H^0(G_{\infty,s}, A)^\vee) = \begin{cases} 
0 & \text{for } i \neq n \\
\text{Hom} \ (D, H^0(G_{\infty,s}, A)), & \text{for } i = n,
\end{cases}
\]

where \( D \) is the dualising module for \( \mathcal{G} \) (if \( \mathcal{G} = \mathbb{Q}_p/\mathbb{Z}_p^\infty \).

(b) If \( H^0(G_{\infty,s}, A) \) is finite, then

\[
E^i(H^0(G_{\infty,s}, A)^\vee) = \begin{cases} 
0 & \text{for } i \neq n + 1 \\
\text{Hom} \ (H^0(G_{\infty,s}, A), D)^\vee & \text{for } i = n + 1
\end{cases}
\]
Proof This is well-known, see [Ja3].

Corollary 6 Let $\mathcal{G}$ is a $p$-adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty,S}$ acts trivially on $A$, then there are exact sequences

$$0 \to \lim_{k'} H^1(G_S(k'), T_p A) \to (H^1(G_{\infty,S}, A)^\vee)^+$$

$$\to T_p A \xrightarrow{\text{inf}^2} \lim_{k'} H^2(G_S(k'), T_p A)$$

and

$$0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \lim_{k'} H^2(G_S(k'), T_p A) \to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0,$$

an isomorphism

$$E^1(H^2(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty,S}, A)^\vee) = 0 = E^3(H^2(G_{\infty,S}, A)^\vee).$$

Proof The spectral sequence looks like

```
\begin{array}{ccccccc}
 &  &  &  &  &  &  \\
& 2 & \times & \times & \times & \times & \\
1 & \times & \times & \times & \times & & \\
0 & 0 & \times & 0 & & & \\
\end{array}
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Corollary 7 Let $\mathcal{G}$ be a $p$-adic Lie group of dimension 2 (So $E^i(-) = 0$ for $i \geq 4$). If $H^0(G_{\infty,S}, A)$ is finite, then

$$\lim_{k'} H^1(G_S(k'), T_p A) \cong (H^1(G_{\infty,S}, A)^\vee)^+.$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty,S}, A)^\vee) \to E^3(H^0(G_{\infty,S}, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$0 \to \ker d_2^{1,1} \to \lim_{k'} H^2(G_S(k'), T_p A) \to$$

$$\to \ker(d_2^{0,2} : (H^2(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee)) \to \text{coker } d_2^{1,1} \to 0,$$
an isomorphism
\[ E^1(H^2(G_{\infty}, s, A)^\vee) \cong E^3(H^1(G_{\infty}, s, A)^\vee), \]
and the vanishing
\[ E^2(H^2(G_{\infty}, s, A)^\vee) = 0 = E^3(H^2(G_{\infty}, s, A)^\vee). \]

**Proof** The spectral sequence looks like

\[
\begin{array}{cccc}
2 & \times & \times & \times \\
1 & \times & \times & \times \\
0 & 0 & 0 & \times \\
1 & 2 & 3
\end{array}
\]

**Remark** In the situation of Corollary 5, one has an exact sequence up to finite modules:
\[
0 \to E^1(H^1(G_{\infty}, s, A)^\vee) \to \lim_{k'} H^2(G_{S}(k'), T_p A) \to (H^2(G_{\infty}, s, A)^\vee)^+ \to E^2(H^1(G_{\infty}, s, A)^\vee) \to 0.
\]

**Corollary 8** Let \( G \) be a \( p \)-adic Lie group of dimension \( > 2 \). Then
\[
(H^1(G_{\infty}, s, A)^\vee)^+ \cong \lim_{k'} H^1(G_{S}(k'), T_p A)
\]

**Proof** The first three columns of the spectral sequence look like

\[
\begin{array}{cccc}
2 & \times & \times & \times \\
1 & \times & \times & \times \\
0 & 0 & 0 & \\
1 & 2
\end{array}
\]
We now turn to the proof of Theorem 1:

Let \( G \) be any profinite group, let \( H \leq G \) be a closed normal subgroup, and denote \( G = G/H \). Let \( M_G \) be the category of discrete \( G \)-modules which are torsion as abelian groups, and let \( M_G^N \) be the category of inverse systems \( (A_n) \) in \( M_G \) as in [Ja1]. For any \( A \) in \( M_G \), one gets an inverse system \( T_p A := (A[p^n]) \), where the transition maps \( A[p^{n+1}] \to A[p^n] \) are induced by multiplication with \( p \) in \( A \). For reasons explained later, denote by \( H^m_{cont}(G, H; RT_p A) \) the value at \( A \) of the \( m \)-th derived functor of the left exact functor

\[
F : A \mapsto \lim_{\overset{\longleftarrow}{n}} \lim_{U} H^0(U, A[p^n])
\]

where \( U \) runs through all open subgroups \( U \subset G \) containing \( H \), and the transition maps are the corestriction maps and these coming from \( A[p^{n+1}] \to A[p^n] \), respectively. We can write \( F \) as the composition of the functors

\[
T_p : M_G \to M_G^N \quad A \mapsto (A[p^n])
\]

and

\[
H^0_{cont}(G, H; -) : M_G^N \to Ab \quad (A_n) \mapsto \lim_{\overset{\longleftarrow}{n}} \lim_{U} H^0(U, A_n)
\]

where \( Ab \) is the category of abelian groups. Because \( T_p \) maps injectives to \( H^0_{cont}(G, H; -) \)-acyclics we get a spectral sequence

\[
E^{p,q}_2 = H^p_{cont}(G, H; R^q T_p A) \Rightarrow H^{p+q}_{cont}(G, H; RT_p A).
\]

One has

\[
R^q T_p A = \begin{cases} 
(A/p^n A) & q = 1 \\
0 & q > 1
\end{cases}
\]

and hence short exact sequences

\[
0 \to H^n_{cont}(G, H; T_p A) \to H^n_{cont}(G, H; RT_p A) \to H^{n-1}_{cont}(G, H; R^1 T_p A) \to 0
\]

explaining the notation for \( R^n F \). In fact, \( H^n_{cont}(G, H; RT_p A) \) is the hypercohomology with respect to \( H^0_{cont}(G, H; -) \) of a complex \( RT_p A \) in \( M_G^N \) computing the \( R^q T_p A \). If \( A \) is \( p \)-divisible, then \( R^q T_p A = 0 \) for all \( q > 0 \), the spectral sequence degenerates and gives isomorphisms \( H^n_{cont}(G, H; RT_p A) \cong H^n_{cont}(G, H; T_p A) \). There is another spectral sequence from deriving the inverse limit, viz.

\[
E^{p,q}_2 = R^p \lim_{U} H^q(U, A_n) \Rightarrow H^{p+q}_{cont}(G, H; (A_n))
\]

If \( G/H \) has a countable basis of neighbourhoods of identity, i.e., if there is a countable family \( U_\nu \) of open subgroup, \( H \leq U_\nu \leq G \), with \( \bigcup U_\nu = H \), then \( R^p \lim_{U} = 0 \) for \( p > 1 \), and \( R^1 \lim_{U} \) has the usual description ([Ja1]). If, in addition, all \( H^q(U, A_n) \) are finite, then \( R^1 \lim_{U} H^q(U, A_n) = 0 \), and we get isomorphisms

\[
H^n_{cont}(G, H; (A_n)) = \lim_{\overset{\longleftarrow}{n}} \lim_{U} H^n(U, A_n)
\]
All this applies in the situation of the theorem, so that

\[ H^n_{\text{cont}}(G_S, G_\infty; RT_p A) = \lim \lim H^n(G_S(k'), A[p^m]) \]

where \( k' \) runs through the finite extensions \( k'/k \) contained in \( k_\infty \).

On the other hand, we can write \( F \) as the composition of the left exact functors

\[ H^0(H, -) : M_G \rightarrow M_G \quad A \mapsto A^H \]

and

\[ M_G \rightarrow Ab \quad B \mapsto \text{Hom}_A(B^\vee, \Lambda) \]

where \( \Lambda = \mathbb{Z}_p[[G]] \) is the completed \( \mathbb{Z}_p \)-group ring of \( G \) and \( B^\vee \) is the Pontrjagin dual of \( B \), which is a compact \( \Lambda \)-module. In fact, one has (cf. [Ja3] p. 179).

\[
\begin{align*}
\text{Hom}_A(B^\vee, \Lambda) &= \lim \lim \text{Hom}_G(B^\vee, \mathbb{Z}_p[G/U]) \\
&= \lim \lim \text{Hom}(H^0(U/H, B)^\vee, \mathbb{Z}/p^n\mathbb{Z}) \\
&= \lim \lim \text{Hom}(H^0(U/H, B[p^n])^\vee, \mathbb{Z}/p^n\mathbb{Z}) \\
&= \lim \lim H^0(U/H, B[p^n])
\end{align*}
\]

and hence

\[ \text{Hom}_A(H^0(H, A)^\vee, \Lambda) = \lim \lim (H^0(U, A[p^n])) = F(A). \]

Since taking Pontrjagin duals is an exact functor

\[ M_G \rightarrow (\text{compact } \Lambda\text{-modules}) \]

taking injectives to projectives, the derived functors of the functor \( B \mapsto \text{Hom}_A(B^\vee, \Lambda) \) are the functors \( B \mapsto \text{Ext}_A^i(B^\vee, \Lambda) = E^i(B^\vee) \), and we get a spectral sequence

\[ E_2^{p,q} = E^p(H^q(H, A)^\vee) \Rightarrow R^{p+q}F(A) = H_{\text{cont}}^{p+q}(G, H; RT_p A). \]

The theorem follows.

References


