数理解析研究所講究録 1376 巻 2004 年 42-49

A spectral sequence for Iwasawa adjoints

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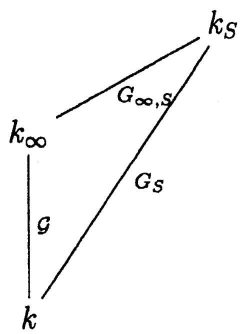
Abstract

This article is, apart from some small corrections, an unchanged version of a note written in 1994. It was circulated as a manuscript and quoted in several articles. Its aim was to provide a purely algebraic tool for the Iwasawa theory of arbitrary p-adic Lie groups, by constructing a spectral sequence which relates the generalized 'Iwasawa adjoints' $E^i(M)$ of certain Iwasawa modules M to projective limits often encountered in the applications. A somewhat extended version will be published elsewhere.

Let k be a number field, fix a prime p, and let k_∞ be some Galois extension of k such that $\mathcal{G} = \mathrm{Gal}\,(k_\infty/k)$ is a p-adic Lie-group (e.g., $\mathcal{G} \cong \mathbb{Z}_p^r$ for some $r \geqslant 1$). Let S be a finite set of primes containing all primes above p and ∞, and all primes ramified in k_∞/k, and let k_S be the maximal S-ramified extension of k; by assumption, $k_\infty \subseteq k_S$. Let $G_S = \mathrm{Gal}\,(k_S/k)$ and $G_{\infty,S} = \mathrm{Gal}\,(k_S/k_\infty)$.



Let A be a discrete (left) G_S-module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geqslant 1$ as an abelian group (e.g., $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, or $A = E[p^\infty]$, the group of p-power torsion points of an elliptic curve E/k with good reduction outside S). We are **not** assuming that $G_{\infty,S}$ acts trivially.

Let $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ be the completed group ring. For a finitely generated Λ-module M we put

$$E^i(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda)\ .$$

Hence $E^0(M) = \mathrm{Hom}_\Lambda(M, \Lambda) =: M^+$ is just the Λ-dual of M. This has a natural structure of a Λ-module, by letting $\sigma \in \mathcal{G}$ act via

$$\sigma f(m) \;=\; f(m) \cdot \sigma^{-1}$$

for $f \in M^+$, $m \in M$. It is known that Λ is a noetherian ring (here we use that $\mathcal{G}$ is a p-adic Lie group), by results of Lazard [La]. Hence M^+ is a finitely generated Λ-module again (choose a projection $\Lambda^r \twoheadrightarrow M$; then we have an injection $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$). By standard homological algebra, the $E^i(M)$ are finitely generated Λ-modules for all $i \geqslant 0$.

Examples a) If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$ is the classical Iwasawa algebra, and, for a Λ-torsion module M, $E^1(M)$ is isomorphic to the Iwasawa adjoint, which can be defined as

$$\text{ad}\,(M) = \varprojlim_n\,(M/\alpha_n M)^\vee$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of elements in Λ such that $\lim_{n\to\infty} \alpha_n = 0$ and (α_n) is prime to the support of M for every $n \geqslant 1$, and where

$$N^\vee = \text{Hom}\,(N,\, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a compact $\mathbb{Z}_p$-module N. For any finitely generated Λ-module M, $E^1(M)$is quasi-isomorphic to $\text{Tor}_\Lambda(M)^\sim$, where $\text{Tor}_\Lambda(M)$ is the Λ-torsion submodule of M, and $M^\sim$ is the "Iwasawa twist" of a Λ-module M: the action of $\gamma \in \mathcal{G}$ is changed to the action of γ^{-1}.

b) If $\mathcal{G} = \mathbb{Z}_p^r$, $r \geqslant 1$, then the $E^i(M)$ are the standard groups considered in local duality. By duality for the ring $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \dots, x_r]]$, they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

Theorem 1 *There is a spectral sequence*

$$E_2^{p,q} = E^p(H^q(G_{\infty,S},\, A)^\vee) \Rightarrow \varprojlim_{k',m}\; H^{p+q}(G_S(k'),\, A[p^m])\,.$$

Here the limit runs through the natural numbers m and the finite extensions k'/k contained in k_∞, via the natural maps

$$H^n(G_S(k'),\, A[p^{m+1}]) \to H^n(G_S(k'),\, A[p^m]) \quad (m \geqslant 1)$$

and the corestrictions.

Before we give the proof, we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$\begin{aligned} 0 \;\to\; & E^1(H^0(G_{\infty,S},\, A)^\vee) \xrightarrow{inf^1} \varprojlim_{k'}\; H^1(G_S(k'),\, T_pA) \\ \to\; & (H^1(G_{\infty,S},\, A)^\vee)^+ \;\longrightarrow\; E^2(H^0(G_{\infty,S},\, A)^\vee) \xrightarrow{inf^2} \varprojlim_{k'} H^2(G_S(k'),\, T_pA)\,. \end{aligned}$$

To say more, we consider some assumptions.

A.1 Assume that $p > 2$ or that k_∞ is totally imaginary. Then

$$H^r(G_{\infty,S},\, A) = 0 = \varprojlim_{k'} H^r(G_S(k'),\, T_pA)$$

for all $r > 2$.

Corollary 2 *Assume that $H^2(G_{\infty,S},\, A) = 0$ (This is the so-called "weak Leopoldt conjecture" for A. It is stated classically for $A = \mathbb{Q}_p/\mathbb{Z}_p$, and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja2]).*

Then the cokernel of $\inf^2$ *is*

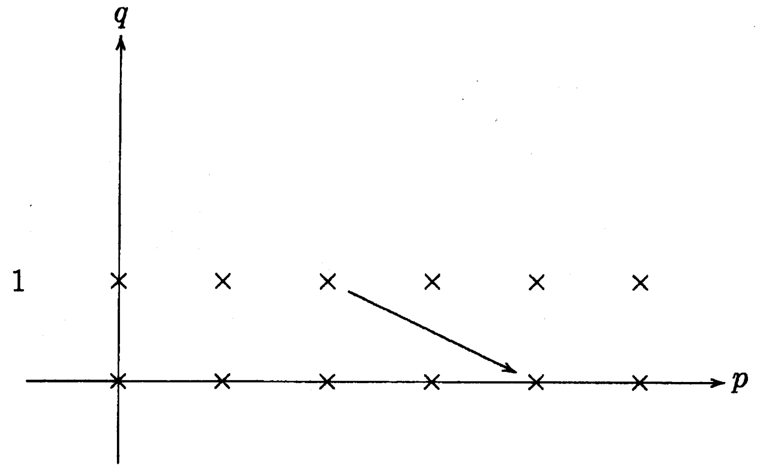
$$\ker\left(E^1(H^1(G_{\infty,S},\,A)^\vee) \twoheadrightarrow E^3(H^0(G_{\infty,S},\,A)^\vee)\right),$$

and there are isomorphisms

$$E^i(H^1(G_{\infty,S},\,A)^\vee) \xrightarrow{\sim} E^{i+2}(H^0(G_{\infty,S},\,A)^\vee)$$

for $i \geqslant 2$.

Proof This comes from A.1 and the following picture of the spectral sequence



Corollary 3 *Assume that* $H^0(G_{\infty,S},\,A) = 0$. *Then*

(a)

$$\varprojlim_{k'} H^1(G_S(k'),\,T_pA) \xrightarrow{\sim} H^1(G_{\infty,S},\,A)^\vee)^+$$

(b) *There is an exact sequence*

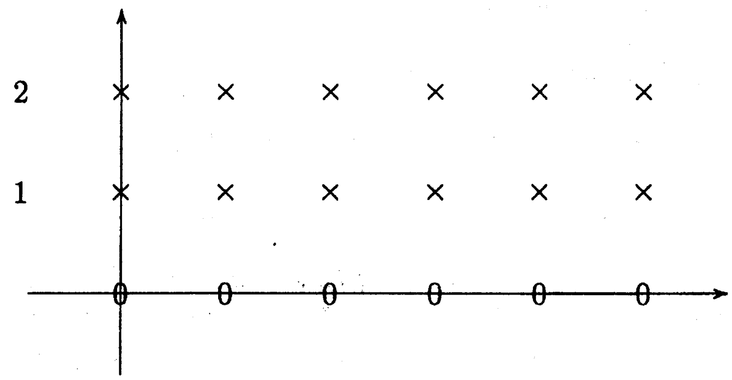
$$0 \to E^1(H^1(G_{\infty,S},\,A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\,T_pA)$$
$$\to (H^2(G_{\infty,S},\,A)^\vee)^+ \to E^2(H^1(G_{\infty,S},\,A)^\vee) \to 0$$

(c) *There are isomorphisms*

$$E^i(H^2(G_{\infty,S},\,A)^\vee) \xrightarrow{\sim} E^{i+2}(H^1(G_{\infty,S},\,A)^\vee)$$

for $i \geqslant 1$.

Proof In this case, the spectral sequence looks like



Corollary 4 *Assume that $\mathcal{G}$ is a p-adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(-) = 0$ for $i \geqslant 3$. Let*

$$B = \mathrm{im}\ (\inf{}^2 : E^2(H^0(G_{\infty,S},\ A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\ T_pA))$$

Then B is finite, and there is an exact sequence

$$0 \to E^1(H^1(G_{\infty,S},\ A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\ T_pA)/B \to (H^2(G_{\infty,S},\ A)^\vee)^+$$

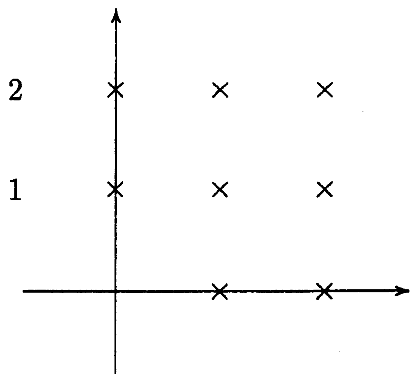
$$\to E^2(H^1(G_{\infty,S},\ A)^\vee) \to 0\,,$$

and

$$E^1(H^2(G_{\infty,S},\ A)^\vee) = 0 = E^2(H^2(G_{\infty,S},\ A)^\vee)\,,$$

i.e., $(H^2(G_{\infty,S},\ A)^\vee$ is a projective Λ-module.

Proof Quite generally, for a p-adic Lie group $\mathcal{G}$ of dimension n one has $vcd_p(\mathcal{G}) = n$ for the virtual cohomological p-dimension of $\mathcal{G}$, and hence $E^i(-) = 0$ for $i > n+1$, cf. [Ja3]. The finiteness of $E^2(M)$ (for a finitely generated Λ-module M) in our case is well-known, cf. [Ja3]. The remaining claims follow from the following shape of the spectral sequence:



Lemma 5 *Assume that $\mathcal{G}$ is a p-adic Lie group of dimension n (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$). Then $E^i(H^0(G_{\infty,S},\ A)^\vee) = 0$ for $i \neq n,\ n+1$.*

(a) *If $H^0(G_{\infty,S},\ A)$ is divisible (e.g., if $G_{\infty,S}$ acts trivially on A), then*

$$E^i(H^0(G_{\infty,S},\ A)^\vee) = \begin{cases} 0 & \text{for } \ i \neq n \\ \mathrm{Hom}\ (D,\ H^0(G_{\infty,S},\ A)), & \text{for } \ i = n\,, \end{cases}$$

where D is the dualising module for $\mathcal{G}(D = \mathbb{Q}_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n)$.

(b) *If $H^0(G_{\infty,S},\ A)$ is finite, then*

$$E^i(H^0(G_{\infty,S},\ A)^\vee) = \begin{cases} 0 & \text{for } \ i \neq n+1 \\ \mathrm{Hom}\ (H^0(G_{\infty,S},\ A),\ D)^\vee & \text{for } \ i = n+1 \end{cases}$$

Proof This is well-known, see [Ja3].

Corollary 6 *Let $\mathcal{G}$ is a p-adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty,S}$ acts trivially on A, then there are exact sequences*

$$0 \to \varprojlim_{k'} H^1(G_S(k'),\, T_pA) \to (H^1(G_{\infty,S},\, A)^\vee)^+$$

$$\to T_pA \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'),\, T_pA)$$

and

$$0 \to E^1(H^1(G_{\infty,S},\, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\, T_pA)_{/\operatorname{im}\inf^2}$$

$$\to (H^1(G_{\infty,S},\, A)^\vee)^+ \to E^2(H^1(G_{\infty,S},\, A)^\vee) \to 0\,,$$

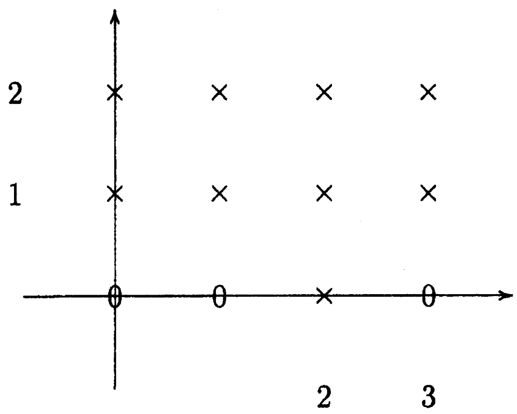
an isomorphism

$$E^1(H^2(G_{\infty,S},\, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S},\, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty,S},\, A)^\vee) \;=\; 0 \;=\; E^3(H^2(G_{\infty,S},\, A)^\vee)\,.$$

Proof The spectral sequence looks like



Corollary 7 *Let $\mathcal{G}$ be a p-adic Lie group of dimension 2 (So $E^i(-) = 0$ for $i \geqslant 4$). If $H^0(G_{\infty,S},\, A)$ is finite, then*

$$\varprojlim_{k'} H^1(G_S(k'),\, T_pA) \cong (H^1(G_{\infty,S},\, A)^\vee)^+\,.$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty,S},\, A)^\vee) \;\to\; E^3(H^0(G_{\infty,S},\, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$0 \;\to\; \ker d_2^{1,1} \;\to\; \varprojlim_{k'} H^2(G_S(k'),\, T_pA) \;\to$$

$$\to\; \ker(d_2^{0,2}\colon (H^2(G_{\infty,S},\, A)^\vee)^+ \twoheadrightarrow E^2(H^1(G_{\infty,S},\, A)^\vee)) \to\; \operatorname{coker}\, d_2^{1,1} \to 0\,,$$

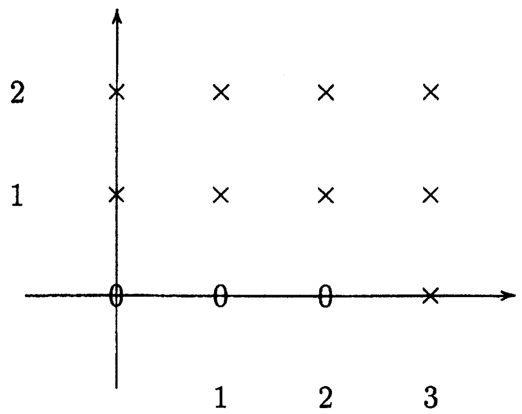
an isomorphism

$$E^1(H^2(G_{\infty,S},\, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S},\, A)^\vee)\,,$$

and the vanishing

$$E^2(H^2(G_{\infty,S},\, A)^\vee) \;=\; 0 \;=\; E^3(H^2(G_{\infty,S},\, A)^\vee)\,.$$

Proof The spectral sequence looks like



Remark In the situation of Corollary 5, one has an exact sequence up to *finite modules*:

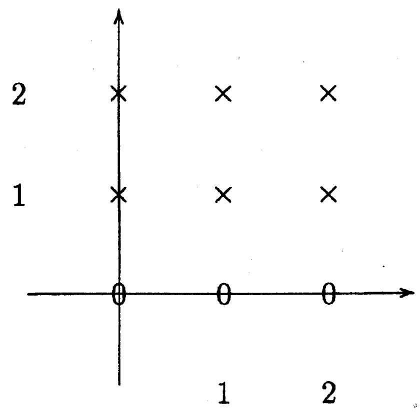
$$0 \to E^1(H^1(G_{\infty,S},\, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\, T_pA)$$

$$\to (H^2(G_{\infty,S},\, A)^\vee)^+ \to E^2(H^1(G_{\infty,S},\, A)^\vee) \to 0\,.$$

Corollary 8 *Let $\mathcal{G}$ be a p-adic Lie group of dimension > 2. Then*

$$(H^1(G_{\infty,S},\, A)^\vee)^+ \;\cong\; \varprojlim_{k'} H^1(G_S(k'),\, T_pA)$$

Proof The first three columns of the spectral sequence look like



We now turn to the **proof of Theorem 1**:

Let G be any profinite group, let $H \leq G$ be a closed normal subgroup, and denote $\mathcal{G} = G/H$. Let M_G be the category of discrete G-modules which are torsion as abelian groups, and let $M_G^{\mathbb{N}}$ be the category of inverse systems (A_n) in M_G as in [Ja1]. For any A in M_G, one gets an inverse system $T_pA := (A[p^n])$, where the transition maps $A[p^{n+1}] \to A[p^n]$ are induced by multiplication with p in A. For reasons explained later, denote by $H^m_{\mathrm{cont}}(G, H; RT_pA)$ the value at A of the m-th derived functor of the left exact functor

$$F : A \mapsto \varprojlim_n \varprojlim_U H^0(U, A[p^n])$$

where U runs through all open subgroups $U \subset G$ containing H, and the transition maps are the corestriction maps and these coming from $A[p^{n+1}] \to A[p^n]$, respectively. We can write F as the composition of the functors

$$\begin{array}{rcl} T_p : M_G & \to & M_G^{\mathbb{N}} \\ A & \mapsto & (A[p^n]) \end{array}$$

and

$$\begin{array}{rcl} H^0_{\mathrm{cont}}(G, H; -) : M_G^{\mathbb{N}} & \to & Ab \\ (A_n) & \mapsto & \varprojlim_n \varprojlim_U H^0(U, A_n) \end{array}$$

where Ab is the category of abelian groups. Because T_p maps injectives to $H^0_{\mathrm{cont}}(G, H; -)$-acyclics we get a spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{cont}}(G, H; R^qT_pA) \Rightarrow H^{p+q}_{\mathrm{cont}}(G, H; RT_pA) .$$

One has

$$R^qT_pA = \begin{cases} (A/p^mA) & q = 1 \\ 0 & q > 1 \end{cases}$$

and hence short exact sequences

$$0 \to H^n_{\mathrm{cont}}(G, H; T_pA) \to H^n_{\mathrm{cont}}(G, H; RT_pA) \to H^{n-1}_{\mathrm{cont}}(G, H; R^1T_pA) \to 0$$

explaining the notation for R^nF. In fact, $H^n_{\mathrm{cont}}(G, H; RT_pA)$ is the hypercohomology with respect to $H^0_{\mathrm{cont}}(G, H; -)$ of a complex RT_pA in $M_G^{\mathbb{N}}$ computing the R^iT_pA. If A is p-divisible, then $R^qT_pA = 0$ for all $q > 0$, the spectral sequence degenerates and gives isomorphisms $H^n_{\mathrm{cont}}(G, H; RT_pA) \cong H^n_{\mathrm{cont}}(G, H; T_pA)$. There is another spectral sequence from deriving the inverse limit, viz.

$$E_2^{p,q} = R^p \varprojlim_{n,U} H^q(U, A_n) \Rightarrow H^{p+q}_{\mathrm{cont}}(G, H; (A_n))$$

If G/H has a countable basis of neighbourhoods of identity, i.e., if there is a countable family U_ν of open subgroup, $H \leq U_\nu \leq G$, with $\bigcap_\nu U_\nu = H$, then $R^p \varprojlim_{n,U} = 0$ for $p > 1$, and $R^1 \varprojlim_{n,U}$ has the usual description ([Ja1]). If, in addition, all $H^q(U, A_n)$ are finite, then $R^1 \varprojlim_{n,U} H^q(U, A_n) = 0$, and we get isomorphisms

$$H^n_{\mathrm{cont}}(G, H; (A_n)) = \varprojlim_n \varprojlim_U H^n(U, A_n) .$$

All this applies in the situation of the theorem, so that

$$H^n_{\mathrm{cont}}(G_S,\, G_{\infty,S};\, RT_pA) = \varprojlim_n \varprojlim_U H^n(G_S(k'),\, A[p^m])$$

where k' runs through the finite extensions k'/k contained in k_∞.

On the other hand, we can write F as the composition of the left exact functors

$$\begin{array}{rcl} H^0(H,\, -) : M_G & \to & M_{\mathcal{G}} \\ A & \mapsto & A^H \end{array}$$

and

$$\begin{array}{rcl} M_{\mathcal{G}} & \to & Ab \\ B & \mapsto & \mathrm{Hom}_{\Lambda}(B^\vee,\, \Lambda) \end{array}$$

where $\Lambda = \mathbb{Z}_p[|\mathcal{G}|]$ is the completed $\mathbb{Z}_p$-group ring of $\mathcal{G}$ and $B^\vee$ is the Pontrjagin dual of B, which is a compact Λ-module. In fact, one has (cf. [Ja3] p. 179).

$$\begin{aligned} \mathrm{Hom}_{\Lambda}(B^\vee,\, \Lambda) &= \varprojlim_U \mathrm{Hom}_{\mathcal{G}}(B^\vee,\, \mathbb{Z}_p[G/U]) \\ &= \varprojlim_n \varprojlim_U \mathrm{Hom}(H^0(U/H, B)^\vee,\, \mathbb{Z}/p^n\mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U \mathrm{Hom}(H^0(U/H, B[p^n])^\vee,\, \mathbb{Z}/p^n\mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U H^0(U/H, B[p^n]) \end{aligned}$$

and hence

$$\mathrm{Hom}_{\Lambda}(H^0(H,\, A)^\vee,\, \Lambda) = \varprojlim_n \varprojlim_U (H^0(U,\, A[p^n]) = F(A)\,.$$

Since taking Pontrjagin duals is an exact functor

$$M_{\mathcal{G}} \to (\text{compact } \Lambda\text{-modules})$$

taking injectives to projectives, the derived functors of the functor $B \mapsto \mathrm{Hom}_{\Lambda}(B^\vee,\, \Lambda)$ are the functors $B \mapsto \mathrm{Ext}^i_{\Lambda}(B^\vee,\, \Lambda) = E^i(B^\vee)$, and we get a spectral sequence

$$E_2^{p,q} = E^p(H^q(H,\, A)^\vee) \Rightarrow R^{p+q}F(A) = H^{p+q}_{\mathrm{cont}}(G,\, H;\, RT_pA)\,.$$

The theorem follows.

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