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G-FUNCTIONS, G-OPERATORS, AND DIOPHANTINE APPROXIMATIONS

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ABSTRACT. The aim of this artcle is to introduce an outline of topics on G-functions. For details, please refer to the REFERENCES.

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§0 THE DEFINITION OF G-FUNCTIONS, EXAMPLES

A G-function is a power series solution, with a certain arithmetical condition, of a linear differential equation over a rational function field.

Let K be a number field as the base field. (i.e., the degree of K over $\mathbb{Q}$ is finite. $[K:\mathbb{Q}]<\infty$.)
But for simplicity, we always assume that K is the rational number field $\mathbb{Q}$: $K=\mathbb{Q}$.

Definition. (G-functions)
We call $y=y(x)$ a G-function if
(0) y satisfies a linear differential equation $/K(x)$

(eq) $$y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0, \quad a_1,\cdots,a_n\in K(x)$$

and y is represented by a power series over K;

$$y = \sum_{i=0}^{\infty} \alpha_i x^i \quad \in K[[x]].$$

(1)[1] there exists a positive constant, C (which is independent of $i = 1, 2, \ldots$) such that

(G-1) $$|\alpha_i| \le C^i \quad \text{for } i = 1, 2, \ldots$$

For $i = 0, 1, 2, \ldots$, let $d_i \in \mathbb{N}$ be the minimum common denominator of $\alpha_0, \alpha_1, \ldots, \alpha_i$.
(i.e., d_i is the minimum positive integer such that $d_i\alpha_0, d_i\alpha_1, \ldots, d_i\alpha_i \in \mathbb{Z}$)
Then

(G-2) $$|d_i| \le C^i \quad \text{for } i = 1, 2, \ldots. \quad \square$$

This is the definition of G-functions.

By this definition, namely by this condition (G-2), in general, it is not so easy to determine whether a given function is a G-function or not.

Now, we show some examples of G-functions.

Examples of G-functions.

(1) The logarithmic function, polylogarithms: $\log(1-x) = -\sum_{i=1}^{\infty} \frac{x^i}{i}$, $L_k(x) := \sum_{i=1}^{\infty} \frac{x^i}{i^k}$; $k \in \mathbb{N}$.
These are G-functions.
To verify the condition (G-2) in the definition, we use "the prime number theorem."

(2) Algebraic functions $\in K[[x]]$ defined over $\mathbb{Q}$ (are also G-functions).
The second example (2) comes from so-called Eisenstein's theorem related to the radii of convergence.
For further information, see [BC].

(3) Gauss's hypergeometirc series with rational parameters (are also G-functions).

$${}_2F_1(\alpha, \beta, \gamma; x) := \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta)_i}{(\gamma)_i i!} x^i \quad \text{where } \alpha, \beta, \gamma \in \mathbb{Q}.$$

Here $(\alpha)_0 = 1$, $(\alpha)_i = \alpha(\alpha+1)\cdots(\alpha+i-1)$ for $i \in \mathbb{N}$. We note that the parameters α, β, γ are in $\mathbb{Q}$, not K. Of course, γ is assumed to not be a negative integer nor zero.

The reason of the third example (3) is similar to the first example (1).

We note that, in general, it is known that the rationality of the parameters α, β, γ is not unnecessary. For further infomation, see [A], and for the rationality of the parameters, see [Ba], [H].

Remark.

The exponential function $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ is not a G-function.

Proof of "log$(1-x)$ satisfies (G-2)*".*

Let d_n be the lowest common multiple of $1, 2, 3, \ldots, n$. Then

$$\begin{aligned} d_n &:= \prod_{\substack{p:\text{prime}\\ p \le n}} p^{e_p} \qquad (e_p := \max\{e \mid \exists\, m \in \{1, \ldots, n\} \text{ s.t. } p^{e_p} | m\}) \\ &\le \prod_{\substack{p:\text{prime}\\ p \le n}} p^{\log_p n} = \prod_{\substack{p:\text{prime}\\ p \le n}} n \sim n^{n/\log n} \end{aligned}$$

The last relation comes from "the prime number theorem."
Then

$$\log d_n \sim \frac{n}{\log n} \log n = n \quad \Rightarrow \quad d_n \sim e^n.$$

Therefore the condition (G-2) holds for $\log(1-x)$. $\square$

[1]We assume that $K = \mathbb{Q}$.

§1 G-FUNCTIONS AND DIOPHANTINE APPROXIMATIONS (Q1 AND Q2)

In this section, this word, Diophantine approximations, means Diophantine approximations about some values of G-functions at some points.

We will introduce a result, the original is as Galočhkin's result. This is similar to a result of E-functions, which are an object for the transcendental number theory.

After introducing both results, we propose two questions. Q1 and Q2. The aim of this section is to set down Q1 and Q2. In the other sections, we will consider these two questions.

1-0 E-FUNCTIONS

There is another class of solutions of linear differential equations, E-functions. The definition is similar to that of G-functions. An E-funciton is also a power series solution of (eq), with another certain aritmetical condition.

In short, the class of E-functions, in fact, looks like a generalization of the exponential function e^x.

This article is, of course, for G-functions, not for E-functions. So we will skip the definition of E-functions.[2] We will only introduce a property of E-functions.

But anyway, the exponential function e^x is a typical example of E-functions.[3] For details, see [Sh].

Before showing a property of E-functions, we introduce a linear differential equation of a matrix type.

We consider the linear differential equation of a matrix type (instead of (eq))

$$\text{(EQ)} \qquad \frac{d}{dx}m = Am, \quad A \in M_n(K(x)), \quad m : \text{a vector solution}$$

If every component of m satisfies the conditions (G-1) and (G-2) in the definition of G-functions, we also call it a G-function(, or m is a G-function vector).

As well as E-functions, if every component of m satisfies the arithmetical condition of the definition of E-funcitons, we also call it an E-function.

Now the next theorem is a property of E-functions.

Theorem E. (*Cf.* [Sh])

Assume that every components of $m = {}^t(f_1, \dots, f_n)$ in (EQ)*$(n \geq 2)$ is an E-function. And assume that $f_1, \dots, f_n$ are linearly independent over $\mathbb{C}(x)$. Then for any positive real $\forall\, \epsilon > 0$, and for $\forall\, \xi \in \mathbb{Q}$ but finite exceptions, there exists a constant $\exists\, c_1$ such that*

$$|H_1 f_1(\xi) + \cdots + H_n f_n(\xi)| > \frac{H}{H^{n+\epsilon}} \qquad \textit{holds}$$

for $\forall\, H_1, \dots, H_n \in \mathbb{Z}$ with $H := \max_i |H_i| \geq c_1$. (Here c_1 depends on A, m, ϵ, ξ.) □

One may come up with that this is an analogous to Lindemann's theorem in the transcendental number theory.

(Lindemann's theorem) Let $\alpha_1, \dots, \alpha_n \in K$ be distinct, then for $\forall\, H_1, \dots, H_n \in K$: not all zero, we have

$$H_1 e^{\alpha_1} + \cdots + H_n e^{\alpha_n} \neq 0. \quad \square$$

(That is to say, the numbers $e^{\alpha_1}, \dots, e^{\alpha_n}$ are linearly independent over K.)

We know that Lindemann's theorem allows the case $K = \overline{\mathbb{Q}}$.

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[2] An E-function is a power series solution of (eq), $y = \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i \in K[[x]]$ with (1) for any $\epsilon > 0$, $|\alpha_i| = O(i^{\epsilon i})$ for $i \to \infty$. (2) let d_i be the common denominator of $\alpha_0, \dots, \alpha_i$, then $|d_i| = O(i^{\epsilon i})$ for $i \to \infty$. Here $f(i) = O(g(i))$ for $i \to \infty$ means $\exists\, C$ s.t. $f(i) \leq Cg(i)$ for $i = 0, 1, 2, \dots$.

[3] Other examples of E-functions: (1) $\sum_{i=0}^{\infty} \frac{x^i}{(\lambda+1)\cdots(\lambda+i)}$ for $\lambda \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$. (2) $\sum_{i=0}^{\infty} \frac{(-1)^i}{i!(\lambda+1)_i} (\frac{x}{2})^{2i}$ for $\lambda \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$. See [Sh].

1-1 G-FUNCTIONS

It seems that the original of the following Theorem is as a result of Galočhkin.

In order to clarify our problems, we modify it slightly. But the essential is not modified. For details, see [G] and also [C].

Theorem 1. ([G]. *See also* [C])

Let $m = {}^t(f_1, \dots, f_n)$ *be a vector solution in* (EQ)$(n \geq 2)$.

Assume that $f_1, \dots, f_n$ *are G-functions, and assume that* $f_1, \dots, f_n$ *are linearly independent over* $\mathbb{C}(x)$.

If (EQ) *is a G-operator,*

(We will give the definition of G-operators in the next section.)
for any positive real $\forall\ \epsilon > 0$, *there exists constant* $\exists\ c_1$ *such that the following holds:*

for any $q \in \mathbb{Z}$ *with* $|q| \geq c_1$,

there exists constant $\exists\ c_2$ *such that*

$$|H_1 f_1(1/q) + \cdots + H_n f_n(1/q)| > \frac{H}{H^{n+\epsilon}} \quad \textit{holds}$$

for $\forall\ H_1, \dots, H_n \in \mathbb{Z}$ *with* $H := \max_i |H_i| \geq c_2$. *(Here* c_1 *depends on* m, A, ϵ, *and* c_2 *depends on* m, A, ϵ, q.*)* □

One might see that this Theorem 1 is similar to Theorem E.
But we have two questions.

1-2 TWO PROBLEMS

Problems.

(Q1) What is a G-operator?
Of course, we will introduce the definition of G-operators soon, in the next section.
But

What exactly does a G-operator mean?

(Q2) In Theorem 1, the value $1/q$ *in* $f_1(1/q), \dots, f_n(1/q)$ *looks artificial.*

Indeed, the radius of convergence of an E-function is infinite. On the other hand, the radius of convergence of a G-function is finite. There is a difference, of course.

But Theorem 1 is not for any points in the radius of convergence, but only for $1/q$. *It is the only case that the denominator q is (very) much larger than the numerator 1. We are afraid that it is unavailable for technical reasons.*

Here, we think "is it good that G-functions are regarded as an analogue to E-functions?"
So, the second question is

Find a better analogue.

From now on, we will consider the first question in the next section, Section 2, and we will attack the second question in Section 3.

§2 G-OPERATORS (AN ANSWER OF Q1)

In this section, we give the definition of G-operators. And we introduce two results: Chudnovsky's result and André's result. We are sure that these two results are one of the solutions of the first question in the previous section. In addition, we recall the definition of G-operators; it is an iteration. We will also introduce a slight generalization of G-operators. Then we will show some properties of them, and show an application. We think that it is a further consideration of the first question.

2-1 NOTATIONS AND DEFINITIONS

Let v be a normalized valuation of K, the base number field, with the product formula, and such that the absolute Height does not depend on the choice of the field K.

That is, if $K = \mathbb{Q}$,

$$\begin{cases} |p|_p := 1/p, & p\text{-adic valuation, (p: prime)} \\ |\xi|_\infty := |\xi| & \text{for } \xi \in \mathbb{Q}, \quad |\cdot| \text{ is the usual Archimedean valuation} \end{cases}$$

The p-adic valuation is this; for $m, n \in \mathbb{Z}$, $p^e || m$ (p^e is the maximum p-factor of m),(i.e., $p^e \mid m$ and $p^{e+1} \nmid m$), $p^f || n$, put $|m/n|_p := p^{f-e}$.

The usual Archimedean valuation means that for $m/n \in \mathbb{Q}$ with $m/n \geq 0$, $|m/n|_\infty = m/n$, otherwise, $|m/n|_\infty = -m/n$.

If we write $v \nmid \infty$, we mean that v is non-Archimedean.

The product formula means $\prod_v |\xi|_v = 1$ for $\xi \in K \setminus \{0\}$, where v in $\prod_v$ runs every normalized valuation of K.

For $M = (m_{i,j}) \in M_{n,n'}(K)$: $n \times n'$-matrix, every component is in K, we put $|M|_v := \max_{i,j} |m_{i,j}|_v$.

For any real number $a \in \mathbb{R}$, we put $\log^+ a := \log \max(1, a)$.

Definition. (sizes and global radii of power series)

For $Y = \sum_{i=0}^\infty Y_i x^i \in M_{n,n'}(K[[x]])$, we put 'the size of Y' $\sigma(Y)$ as

$$\sigma(Y) := \overline{\lim_{\mu\to\infty}} \sum_v \frac{1}{\mu} \max_{i\leq\mu} \log^+ |Y_i|_v \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We put 'the global radii' $\rho(Y)$ as

$$\rho(Y) := \sum_v \overline{\lim_{\mu\to\infty}} \frac{1}{\mu} \max_{i\leq\mu} \log^+ |Y_i|_v \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

where v in $\sum_v$ runs over every normalized valuation v in K. □

A paraphrase of the definition of G-functions.

Every component of $Y = \sum_{i=0}^\infty Y_i x^i \in M_{n,n'}(K[[x]])$ satisfies the conditions (G-1) *and* (G-2) *in the definition of G-functions, is equivalent to, $\sigma(Y) < \infty$.* □

That is to say, a G-function is a solution of a linear differential equation such that its size is finite.

We do not want to talk here about the radius of convergence, but we are sure that the radius of convergence is a usual notion, but the size is not so usual. So we remark ;

A relation between $\rho(Y)$ and the radius of convergence of Y. *(Cf.* [A]*)*

Put $R_v(Y) := \underline{\lim}_{i\to\infty} |Y_i|_v^{-1/i}$ (the radius of convergence at v), then

$$\rho(Y) = \sum_v \log^+ \frac{1}{R_v(Y)}. \quad □$$

This is the reason why we introduce the definition of the global radii.

It is obvious by the definitions, the difference between the size and the global radii is only the interchange of the limitation $\overline{\lim}_{\mu\to\infty}$ and the summation $\sum_v$.

Now, we have finished defining the size and the global radii for power series.
Next, we define the size and the global radii for the linear differential equation (EQ).

From now on, we consider only non-Archimedean cases $v \nmid \infty$.
For a rational function $f \in K(x)$, we write the so-called Gauss's norm of f at v as $|f|_v$. (That is, for $f = \sum_{i=0}^{N} f_i x^i \in K[x]$, put $|f|_v := \max_i |f_i|_v$, for $f, g \in K[x]$, $g \neq 0$, put $|f/g|_v := |f|_v/|g|_v$: well-defined.)

For $M = (m_{i,j}) \in M_n(K(x))$: $n\times n$-matrix over the rational function field $K(x)$, put $|M|_v := \max_{i,j} |m_{i,j}|_v$.

Let I be the identity matrix.
For the matrix A in (EQ), we consider the sequence $\{A_i\}_{i=0,1,\ldots}$

$$A_i := \frac{1}{i!}{}^{\mathrm{t}}\left(\left(\frac{d}{dx} + {}^{\mathrm{t}}A\right)^i I\right) \quad \in M_n(K(x)),$$

that is, if we put the map as

$$\varphi_A : M_n(K(x)) \to M_n(K(x)), \quad B \mapsto (d/dx + {}^{\mathrm{t}}A)B,$$

then

$${}^{\mathrm{t}}A_i = \frac{1}{i!}\varphi_A \circ \cdots \circ \varphi_A(I).$$

We note that this A_i satisfies

$$\frac{1}{i!}\left(\frac{d}{dx}\right)^i m = A_i m$$

for m in (EQ).

Definition. (sizes and global radii of linear differential equations)
We denote 'the size of (EQ) $\sigma(A)$' as

$$\sigma(A) := \overline{\lim_{\mu\to\infty}} \sum_{v\nmid\infty} \frac{1}{\mu} \max_{i\leq\mu} \log^+ |A_i|_v \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We denote 'the global radii of (EQ) $\rho(A)$' as

$$\rho(A) := \sum_{v\nmid\infty} \overline{\lim_{\mu\to\infty}} \frac{1}{\mu} \max_{i\leq\mu} \log^+ |A_i|_v \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

where v in $\sum_{v\nmid\infty}$ runs over every normalized non-Archimedean valuation $v \nmid \infty$ of K. □

Now we define G-operators.

Definition. (G-operators)
We call (EQ) a G-operator if $\sigma(A) < \infty$. □

(There is no way to call $\rho(A)$ as finite.)

The definitions of G-functions and G-operators are similar, but the difference is that G-operators use the Gauss' norms.

2-2 Chudnovsky's result

We think the following two theorems, Chudnovsky's and André's results, show the meaning of G-operators.

Theorem 2. ([C])

Under the assumptions of Theorem 1, that is, for $m = {}^t(f_1, \dots, f_n)$ *in* (EQ), $(n \geq 2)$, *assume that* $f_1, \dots, f_n$ *are* G*-functions and assume that* $f_1, \dots, f_n$ *are linearly independent over* $\mathbb{C}(x)$.

Then $\sigma(A) < \infty$. *that is,* (EQ) *is a* G*-operator.* □

It follows that the assumption, "(EQ) is a G-operator" in Theorem 1, is unnecessary.

2-3 André's result

Theorem 2 (Chudnovsky's result) says that "if components of $m = {}^t(f_1, \dots, f_n)$ are linearly independent over $\mathbb{C}(x)$, then $\sigma(m) < \infty$ implies $\sigma(A) < \infty$."

The next theorem, a result of André, says the opposite of this Theorem 2 in some sense.

We now consider the "matrix solution" of (EQ).

$$\frac{d}{dx}X = AX, \quad A \in M_n(K(x))$$

Here we assume that:

there exists $\exists\, C \in M_n(K)$, there exists $\exists\, Y = \sum_{i=0}^{\infty} Y_i x^i \in M_n(K[[x]])$ with $Y_0 = I$ (the initial condition) such that

$$X = Yx^C \quad \left(= Y \sum_{i=0}^{\infty} \frac{1}{i!} (C \log x)^i \right)$$

'plus' some suitable conditions.

Then

Theorem 3. ([A])

Under some conditions, $\sigma(A) < \infty$ *implies* $\sigma(Y) < \infty$. □

Finally, under "some conditions," G-functions and G-operators are equivalent (in the meaning).

This is exactly what a G-operator means. But we go on to a further consideration.

2-4 A generalization of G-operators

Let's recall the definition of G-operators. It comes from the iteration of the map:

$$\varphi_A \ : \ M_n(K(x)) \to M_n(K(x)).$$

This map defines a G-operator

So now we consider the following sequence:

Let $T, A, B \in M_n(K(x))$.

We define the sequence $\{(T, A, B)^{\langle i \rangle}\}_{i=0,1,\dots} \subset M_n(K(x))$ as

$$(T, A, B)^{\langle 0 \rangle} := T,$$

$$(T, A, B)^{\langle i+1 \rangle} := \frac{1}{i+1} \left(\frac{d}{dx}(T, A, B)^{\langle i \rangle} - A(T, A, B)^{\langle i \rangle} + (T, A, B)^{\langle i \rangle} B \right).$$

We put

$$\sigma(T, A, B) := \varlimsup_{\mu \to \infty} \sum_{v \nmid \infty} \frac{1}{\mu} \max_{i \leq \mu} \log^+ |(T, A, B)^{\langle i+1 \rangle}|_v, \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

$$\rho(T, A, B) := \sum_{v \nmid \infty} \varlimsup_{\mu \to \infty} \frac{1}{\mu} \max_{i \leq \mu} \log^+ |(T, A, B)^{\langle i+1 \rangle}|_v, \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Then we have ;

Proposition 4. ([N1, N2])
Let $T, T_1, T_2 \in \mathrm{GL}_n(K(x))$, and let $A, B, C \in M_n(K(x))$, Then
(0) $\sigma(A) = \sigma(I, 0, A)$.
(1) $\sigma(I, A, A) = 0$.
(2) $\sigma(T, A, B) = \sigma(I, A, T[B]) = \sigma(I, T^{-1}[A], B)$, where $T[B] := TBT^{-1} + ((d/dx)T)T^{-1}$.
(3) $\sigma(T_1, A, 0) = \sigma(T_2, 0, A)$.
(4) $\sigma(T_1T_2, A, B) \leq \sigma(T_1, A, C) + \sigma(T_2, C, B)$. (triangle inequality)
For ρ, the same (in)equalities hold. □

The idea of this proof is : first, show some equations for the sequences by induction. then, apply them to the definition to σ. The triangle inequality (4) comes from the triangle inequalities of valuations. Q.E.D.

Although it is a good opportunity to remark about Proposition 4 here, we would like to do it in the last section, Section 4.

As an application of Proposition 4, we obtain the next theorem.
Let's recall Theorem 2, Chudnovsky's result. Chudnovsky's result assumes the linearly independence of G-functions. And it is for a vector power series solution. Then, one might think it is not the strict contrary of André's result, Theorem 3. Here we obtain ;

Theorem 5. ([N2])
Under the assumptions of Theorem 3, $\sigma(Y) < \infty$ implies $\sigma(A) < \infty$. □

We can say that these results in this section is an answer of Question 1 in Section 1.

§3 G-functions and Diophantine approximations, revisited (an answer of Q2)

In this section, we want to consider the next question Q2: "Find a better analogue for G-functions"

The main idea in this section is that "G-functions are not so similar to E-functions, but algebraic functions". We would like to presume this a candidate of the solutions of the second question in Section 1. So we will introduce here a "circumstantial evidence."

First of all, the author has an easy thought.
For instance, the values of algebraic functions over rationals at algebraic points are, of course, algebraic numbers. Recall that algebraic functions $/\mathbb{Q}$ are also G-functions. And also recall, Theorem E is an analogue to Lindemann's Theorem, which is a point of view from the transcendental number theory. So, one may think, it is hard to regard the problems of special values of G-functions like as the ones of E-functions.
So, just in a private opinion, G-functions are not so similar to E-functions, but as algebraic functions.
This is a private solution of Question 2: "G-functions is analogue to algebraic functions."

In this section, we introduce a "circumstantial evidence" with this opinion.

3-0 Algebraic functions

We recall algebraic functions' case
But we don't know the "smart formulation", so we are afraid that we might not give a good explanation.

Proposition 6. (*Liouville's inequality*)
(We consider only the genus 0 case, and we restrict a special case.)
Let $f(x, y) := x - g(y) \in K(x, y)$, $g(y) \in K(y)$, $n := \deg_y g$.

Put

$$\begin{aligned} S_1 &:= \{x \in K \mid \exists\, y \in K \ s.t.\ f(x,y) = 0\} \\ &= \{g(y) \in K \mid y \in K\}. \end{aligned}$$

Let $t \in K$ with $\frac{d}{dy}g(t) \neq 0$. Put $a := g(t)$. Then there exists $\exists\, c > 0$ such that

$$|\alpha - a| > \frac{c}{H(\alpha)^{[K:\mathbb{Q}]/n}} \quad \textit{for } \forall\, \alpha \in S_1 \textit{ with } \alpha \neq a.$$

Here $H(\alpha)$ means the absolute Height of α. (i.e., $\log H(\alpha) = \sum_v \log^+ |\alpha|_v$.) (c is independent of α.) □

It is the so-called Liouville's inequality if $g(y) = y$.

We know also that "there exists an estimate of the number of rational points of irreducible algebraic curve $/\mathbb{Q}$ in general cases", but we will mention it soon.

Anyway, that is to say, for the "rational points" S_1, the inequality holds. This is Proposition 6.

3-1 G-FUNCTIONS

For G-functions, we obtain an analogue to Proposition 6.

Theorem 7. ([N4,N5])

Let $\zeta_0 \in K$ be given. Put $D \subset \mathbb{C}$: a closed disc centered ζ_0 with radius $< 1/2$. Let $d(x) \in \mathbb{Z}[x]$ be the common denominator of all components of A in (EQ).

We assume that

(1) (EQ) *is a G-operator. $n \geq 2$.*

(2) $m = {}^t(f_1, \cdots, f_n)$ is analytic on D.

(3) $f_1, \cdots, f_n$ are linearly independent over $\mathbb{C}(x)$.

(4) $\{z \in \mathbb{C} \mid d(z) = 0\} \cap D = \varnothing$.

Put

$$S_K := \{\zeta \in D \cap K \mid \exists\, \kappa_\zeta \in \mathbb{C} \setminus \{0\} \ s.t.\ \kappa_\zeta m(\zeta) \in K^n\}.$$

Then we have

(i) (For $\zeta_0 \in S_K$ and) for any small real $\forall\, \epsilon > 0$, there exists $\exists\, c < \infty$ such that

$$|\zeta - \zeta_0| > \frac{1}{H(\zeta)^{[K:\mathbb{Q}](\frac{1}{n}+\epsilon)}} \ \textit{for } \forall\, \zeta \in S_K \textit{ with } H(\zeta) \geq c.$$

Here c depends on $H(\zeta_0), A, \epsilon, D$, and is independent of ζ.

(ii)

$$\varlimsup_{B \to 0} \frac{\log \#\{\zeta \in S_K \mid H(\zeta) \leq B\}}{\log B} \leq \frac{4}{n}[K : \mathbb{Q}].$$

(the trivial estimate is $\leq 2[K : \mathbb{Q}]$) □

Remark 8.

(1) For algebraic functions' case, a similar estimate in (ii) with $\leq \frac{2}{n}[K : \mathbb{Q}]$ holds (for $K = \mathbb{Q}$ with different notations). See [Se] for the details.

(2) If $f_1, \cdots, f_n$ are algebraically independent over $\mathbb{C}(x)$, we have

· the left hand side in (i) $\geq 1/H(\zeta)^\epsilon$ holds.

· the left hand side in (ii) $= 0$ holds. □

One sees that Theorem 7 is similar to Proposition 6 and the first remark (1) in Remark 8.

Moreover, Theorem 7 does not contain artificial conditions like as $1/q$ in Question 2.

In conclusion, these might be an analogue of an appearance. But even so, one can think we may say that the similarity between Proposition 6 and Theorem 7 gives a candidate of the solutions of Question 2. It may mean that it is better that G-functions are not so similar to E-functions, but algebraic functions.

§4 FURTHER PROBLEMS

We would like to introduce some further problems.

First, as for G-operators:

(Q3) We know the triangle inequality in Proposition 4 (4).
"Find something interesting related to Proposition 4 and develop it." □

The author has spent much much time on it. But in vain. No results (except Theorem 5).
Here we would like to remark about Proposition 4 :

Remark.
(1) If we put

$$\tilde{\sigma}(A,B) := \frac{1}{2}(\sigma(I,A,B) + \sigma(I,B,A))$$

then from (1) and (4) in Proposition 4, the map

$$\tilde{\sigma} : \ M_n(K(x)) \times M_n(K(x)) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is a pseudodistance function, that is, $\tilde{\sigma}(A,A) = 0$, $\tilde{\sigma}(A,B) = \tilde{\sigma}(B,A)$, $\tilde{\sigma}(A,B) \leq \tilde{\sigma}(A,C) + \tilde{\sigma}(C,B)$.
But...
(2) we don't know any relation between A and B with $\sigma(I,A,B) = 0$.
(3) furthermore, for any A and B, we don't know whether $\sigma(I,A,B) = \sigma(I,B,A)$ holds or not.
It means that, a pseudodistance function $\tilde{\sigma}$ on $M_n(K(x))$ are defined by Proposition 4, but, that's all we can do. □

Next, as for Diophantine approximations and G-functions.

(Q4) Theorem 7 is, one may think, Liouville's inequality for G-functions.
So, "Does a Roth's type property on G-functions hold?" □

Here the original Roth's theorem is ;
(Roth's theorem) ([R]): For a given $\alpha \in \overline{\mathbb{Q}}$ with $\deg \alpha \geq 3$ and for any real $\epsilon > 0$,

$$|\alpha - \frac{p}{q}| > \frac{1}{q^{2+\epsilon}}$$

holds for any $p, q \in \mathbb{Z}, \neq 0$, with $(p,q) = 1$ but finite exceptions. □

First of all, the author has no idea even how to formulate it for G-functions. And he thinks that it is not so easy, probably very difficult.

For instance, the proof of the original Liouville's inequality requires only a few pages, probably, only one page, but the proof of the Roth's theorem requires 20 pages. This means, the number of pages of the proof of the Roth's theorem is 20 times than the one of Liouville's inequality.

Here, now, Theorem 7, Liouville's inequality for G-functions, needs more than 30 pages. Then IF one could prove any Roth's type properties for G-functions, it would need, probably, 20 times 30 equals to 600 pages!?

Of course, it is just a joke, a large story. But in this stuation, we can not believe that any proofs of Roth's type statements do not require any complicated arguments.

(Q5) It is well known that the number of $\mathbb{Q}$-rational points of any (projective, smooth) algebraic curve with large genus is finite.

How about G-functions? For example, how about the finiteness of the number of the set S_K in Theorem 7? "Especially, in Theorem 7, if $f_1, \ldots, f_n$ are algebraically independent over $\mathbb{C}(x)$, does $\#S_K < \infty$ hold?" □

We have no idea at all. The author cannot imagine if it is true or not.

(Q6) The special value of the polylogrithmic function $L_k(x)$ at $x = 1$, $L_k(1)$ equals to $\zeta(k)$ (the Riemann zeta-function) for $k \in \mathbb{Z}_{\geq 2}$.

"Apply G-function theory and deduce something about the special values of zeta-functions." □

This is "a big problem."

In the present situation, G-function theory tells NOTHING about special values at any points on the radius of convergence.

$\zeta(k)$ is the special value of the polylogarithmic function $L_k(x)$ at the point $x = 1$, it is on the radius of convergence of the polylogarithmic function $L_k(x)$. Therefore the present G-function theory can tell nothing about it in the present situaton.

The reader may think that every problem is, easy to say, but difficult to carry it out.

As for the number theory, the last question Q6 is very attractive, but we almost give up the Question 6 without some innovation.

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