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SYMBOLIC DYNAMICAL SYSTEMS AND ENDOMORPHISMS ON C^*-ALGEBRAS

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1. INTRODUCTION

This article is a survey of the author's recent preprint entitled " Actions of symbolic dynamical systems on C^*-algebras", that is written based on the talk at RIMS, Jan. 2004. Details are given in the preprint.

In [CK], J. Cuntz and W. Krieger have founded a close relationship between symbolic dynamics and C^*-algebras (cf.[C2]). They constructed purely infinite simple C^*-algebras from irreducible topological Markov shifts. They have proved that their stabilization with gauge action is invariant under topological conjugacy of topological Markov shifts, so that K-theoretic invariants of the C^*-algebras with gauge actions yield invariants of topological Markov shifts. The invariants are the dimension group introduced by W. Krieger [Kr] and the Bowen-Franks group [BF]. They play a crucial rôle in the classification theory of topological Markov shifts. R. F. Williams has classified topological Markov shifts in terms of an algebraic relation of underlying matrices [Wi]. The algebraic relation is called a strong shift equivalence. M. Nasu generalized Williams's classification result to sofic shifts, that are subshifts coming from finite labeled graphs [N].

In [Ma], the author introduced a notion of λ-graph system, whose matrix version is called symbolix matrix system. A λ-graph system is a generalization of a finite labeled graph and presents a subshift. Conversely any subshift is presented by a λ-graph system, and the topological conjugacy classes of the subshifts are exactly corresponding to the strong shift equivalence classes of the symbolic matrix systems of the canonical λ-graph systems. He constructed C^*-algebras from λ-graph systems [Ma3] as a generalization of the above Cuntz-Krieger algebras. It has been proved that the outer conjugacy class of the stabilized gauge action is invariant under strong shift equivalence of the symbolic matrix system of the λ-graph system [Ma4]. Hence K-theoretic invariants of the C^*-algebras with gauge actions constructed from λ-graph systems yield invariants of topological conjugacy classes of subshifts.

In this survey article, we will study and generalize the above discussions in purely C^*-algebra setting. We will introduce a notion of *C^*-symbolic dynamical system*, that is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a unital C^*-algebra $\mathcal{A}$ indexed by symbols Σ satisfying the condition $\sum_{\alpha\in\Sigma}\rho_\alpha(1)\geq 1$. A finite labeled graph gives rise to a C^*-symbolic dynamical system $(\mathcal{A},\rho,\Sigma)$ such that $\mathcal{A}$ is commutative

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and finite dimensional. Conversely, if the C^*-algebra $\mathcal{A}$ is commutative and finite dimensional, the C^*-symbolic dynamical system comes from a finite labeled graph. A λ-graph system gives rise to a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that $\mathcal{A}$ is commutative and AF. Conversely, if the C^*-algebra $\mathcal{A}$ is commutative and AF, the C^*-symbolic dynamical system comes from a λ-graph system ([Theorem 3.4]). We may prove that equivalence classes of the predecessor-separated λ-graph systems exactly correspond to the isomorphism classes of the predecessor-separated C^*-symbolic dynamical systems of the commutative AF-algebras ([Corollary 3.7]).

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a nontrivial subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ over Σ and a Hilbert C^*-right $\mathcal{A}$-module $\mathcal{H}^{\rho}_{\mathcal{A}}$. For $\alpha_1, \dots, \alpha_k \in \Sigma$, a word $(\alpha_1, \dots, \alpha_k)$ is admissible for the subshift if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$. The Hilbert C^*-right $\mathcal{A}$-module $\mathcal{H}^{\rho}_{\mathcal{A}}$ has an orthogonal finite basis $\{u_\alpha\}_{\alpha\in\Sigma}$ and a unital faithful diagonal left action $\phi_\rho : \mathcal{A} \to L(\mathcal{H}^{\rho}_{\mathcal{A}})$. It is called a Hilbert *C^*-symbolic bimodule over $\mathcal{A}$*, and written as $(\phi_\rho, \mathcal{H}^{\rho}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$.

We will consider C^*-algebras constructed from the Hilbert C^*-symbolic bimodules $(\phi_\rho, \mathcal{H}^{\rho}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$. A general construction of C^*-algebras from Hilbert C^*-bimodules has been established by M. Pimsner [Pim] (see [Ka] for the case of von Neumann algebras). The C^*-algebras are called Cuntz-Pimsner algebras. Its ideal structure and simplicity conditions have been studied by Kajiwara-Pinzari-Watatani [KPW] and Muhly-Solel [MS] (see also [KW], [Sch]). The constructed C^*-algebra from the Hilbert C^*-symbolic bimodule $(\phi_\rho, \mathcal{H}^{\rho}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ is denoted by $\mathcal{A} \rtimes_\rho \Lambda$, where Λ is the subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ associated with the C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$. We call the algebra $\mathcal{A} \rtimes_\rho \Lambda$ *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift* Λ. As in [Pim] (cf. [KPW]), the gauge action, denoted by $\hat{\rho}$, on the algebra $\mathcal{A} \rtimes_\rho \Lambda$ of the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is defined as a generalization of that of the Cuntz-Krieger algebras. We remark that Pimsner showed the following fact [Pim]: For every Hilbert C^*-bimodule E over a C^*-algebra $\mathcal{A}$, if $\mathcal{A}$ is commutative and finite dimensional, and if E is projective and finitely generated, the associated C^*-algebra is a Cuntz-Krieger algebra. We present the following theorem

Theorem A (Theorem 5.2). *Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system and Λ be the associated subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$. Assume that $\mathcal{A}$ is commutative.*

(i) *If $\mathcal{A} = \mathbb{C}$, the subshift Λ is the full shift $\Sigma^{\mathbb{Z}}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$.*

(ii) *If $\mathcal{A}$ is finite dimensional, the subshift Λ is a sofic shift $\Lambda_{\mathcal{G}}$ presented by a left-resolving labeled graph $\mathcal{G}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is a Cuntz-Krieger algbera $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph. Conversely, for any sofic shift, that is presented by a left-resolving labeled graph $\mathcal{G}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the sofic shift, the algebra $\mathcal{A}$ is finite dimensional, and the algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz-Krieger algbera $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph.*

(iii) *If $\mathcal{A}$ is an AF-algebra, there uniquely exists a λ-graph system $\mathfrak{L}$ up to equivalence such that the subshift Λ is presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$. Conversely, for any subshift, that is presented by a left-resolving λ-graph system $\mathfrak{L}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated* *subshift is the subshift presented by $\mathfrak{L}$, the algebra $\mathcal{A}$ is a commutative AF-algebra, and the algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.*

We will introduce notions of strong shift equivalence and shift equivalence of C^*-symbolic dynamical systems, that are generalizations of those of square nonnegative matrices defined by Williams [Wi], of finite symbolic square matrices defined by Nasu [N] and Boyle-Krieger [BK] and of symbolic matrix systems defined by [Ma]. They are generalizations of conjugacy of single automorphisms of C^*-algebras. Strong shift equivalence and shift equivalence of Hilbert C^*-symbolic bimodules are introduced. We know that two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent (resp. shift equivalent) if and only if their associated Hilbert C^*-symbolic bimodules $(\phi_\rho, \mathcal{H}^\rho_\mathcal{A}, \{u_\alpha\}_{\alpha\in\Sigma})$ and $(\phi_{\rho'}, \mathcal{H}^{\rho'}_{\mathcal{A}'}, \{u'_\alpha\}_{\alpha\in\Sigma'})$ are strong shift equivalent (resp. shift equivalent). A notion of strong shift equivalence of C^*-symbolic crossed products with gauge actions is introduced. We finally obtain the following theorem.

Theorem B(Theorem 7.5). *Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two C^*-symbolic dynamical systems. Let Λ and Λ' be their associated subshifts $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ and $\Lambda_{(\mathcal{A}',\rho',\Sigma')}$ respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, then*

(i) *the subshifts Λ and Λ' are topologically conjugate,*
(ii) *the C^*-symbolic crossed products $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ with gauge actions are strong shift equivalent, and*
(iii) *the stabilized gauge actions $(\mathcal{A} \rtimes_\rho \Lambda \otimes \mathcal{K}, \hat{\rho} \otimes \mathrm{id}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda' \otimes \mathcal{K}, \hat{\rho}' \otimes \mathrm{id}, \mathbb{T})$ are cocycle conjugate, where $\mathcal{K}$ denotes the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space.*

The result (iii) is a generalization of the main result of [Ma4] (cf.[CK:3.8. Theorem]).

We define the K-groups $K_*(\mathcal{A}, \rho, \Sigma)$, the Bowen-Franks groups $BF^*(\mathcal{A}, \rho, \Sigma)$ and the dimension groups $D_*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting for $* = 0, 1$

$$K_*(\mathcal{A}, \rho, \Sigma) = K_*(\mathcal{A} \rtimes_\rho \Lambda), \qquad BF^*(\mathcal{A}, \rho, \Sigma) = \mathrm{Ext}_*(\mathcal{A} \rtimes_\rho \Lambda),$$
$$D_*(\mathcal{A}, \rho, \Sigma) = (K_*((\mathcal{A} \rtimes_\rho \Lambda) \rtimes_{\hat{\rho}} \mathbb{T}), \hat{\hat{\rho}}_*)$$

where $\hat{\hat{\rho}}_*$ is the automorphism of $K_*((\mathcal{A} \rtimes_\rho \Lambda) \rtimes_{\hat{\rho}} \mathbb{T})$ induced from the dual action $\hat{\hat{\rho}}$ of the gauge action $\hat{\rho}$. The dimension groups and the Bowen-Franks groups are generalizations of those groups for a finite square nonnegative matrix, that is regarded as a finite labeled graph for which labels are edges itself (cf.[BF], [Kr], [LM]). Then Theorem B implies that all the abelian groups $K_*(\mathcal{A}, \rho, \Sigma), BF^*(\mathcal{A}, \rho, \Sigma)$ and $D_*(\mathcal{A}, \rho, \Sigma)$ are invariant under strong shift equivalence of C^*-symbolic dynamical systems (Proposition 7.6).

2. λ-GRAPH SYSTEMS AND ITS C^*-ALGEBRAS

Let Σ be a finite set with its discrete topology. We call it an alphabet. Each element of Σ is called a symbol or a label. Let $\Sigma^{\mathbb{Z}}$ be the infinite product spaces

$\prod_{i\in\mathbb{Z}}\Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $(\sigma(x_i))_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We write the subshift as Λ for brevity. A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a word of length $|\mu| = k$. For a subshift Λ, we denote by Λ^l the set of all admissible words of length l of Λ. By a symbolic matrix $\mathcal{B}$ over Σ we mean a finite matrix with entries in finite formal sums of elements of Σ. A square symbolic matrix $\mathcal{B}$ naturally gives rise to a finite labeled directed graph which we denote by $\mathcal{G}_\mathcal{B}$. The labeled directed graph defines a subshift over Σ which consists of all infinite labeled sequences following the labeled directed edges in $\mathcal{G}_\mathcal{B}$. Such a subshift is called a sofic shift presented by $\mathcal{G}_\mathcal{B}$ and written as $\Lambda_{\mathcal{G}_\mathcal{B}}$ ([Fi],[Kr2],[Kr3],[We], cf. [Kit],[LM]). Throughout this paper, a labeled graph means a labeled directed graph with finite vertices and finite directed edges such as every vertex has at least one in-coming edge and at least one out-going edge.

Let $\mathcal{B}$ and $\mathcal{B}'$ be symbolic matrices over Σ and Σ' respectively. Let ϕ be a bijection from a subset of Σ onto a subset of Σ', that is called a specification. Following M. Nasu in [N],[N2], we say that $\mathcal{B}$ and $\mathcal{B}'$ are specified equivalent under specification ϕ if $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by replacing every symbol α appearing in $\mathcal{B}$ by $\phi(\alpha)$. We write it as $\mathcal{B} \overset{\phi}{\simeq} \mathcal{B}'$. Let $\mathbb{Z}_+$ be the set of all nonnegative integers.

Recall that a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is a directed Bratteli diagram with a vertex set $V = \cup_{l\in\mathbb{Z}_+} V_l$, an edge set $E = \cup_{l\in\mathbb{Z}_+} E_{l,l+1}$, and a map $\lambda : E \to \Sigma$, and that is supplied with a sequence of surjective maps $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets $V_l, l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l, its terminal vertex $t(e)$ in V_{l+1} and its label $\lambda(e)$ in Σ. Every vertex in V has successors and every vertex in V, except V_0, has predecessors. It is then required that for $u \in V_{l-1}$ and $v \in V_{l+1}$, there exists a bijective correspondence between the edge set $\{e \in E_{l,l+1} | t(e) = v, \iota(s(e)) = u\}$ and the edge set $\{e \in E_{l-1,l} | s(e) = u, t(e) = \iota(v)\}$ that preserves labels. The required property is called the local property.

Two λ-graph systems $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ and $\mathfrak{L}' = (V', E', \lambda', \iota')$ over Σ' are said to be isomorphic if there exist bijections $\Phi_V : V_l \to V_l'$, $\Phi_E : E_{l,l+1} \to E'_{l,l+1}$ and a specification $\phi : \Sigma \to \Sigma'$ such that $\Phi_V(s(e)) = s(\Phi_E(e)), \Phi_V(t(e)) = t(\Phi_E(e))$ and $\lambda'(\Phi_E(e)) = \phi(\lambda(e))$ for $e \in E$, and $\iota'(\Phi_V(v)) = \Phi_V(\iota(v))$ for $v \in V$.

A symbolic matrix system over Σ consists of a sequence of pairs of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have their entries in formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0, 1\}$. The matrices $\mathcal{M}_{l,l+1}$ and $I_{l,l+1}$ have the same size for each $l \in \mathbb{Z}_+$ and satisfy the following relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{2.1}$$

The matrices $I_{l,l+1}, l \in \mathbb{Z}_+$ have one 1 in each column and at least one 1 in each row. We denote it by $(\mathcal{M}, I)$. A λ-graph system naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The edges from a vertex $v_i^l \in V_l$ to a vertex v_j^{l+1} are given by the (i, j)-component $\mathcal{M}_{l,l+1}(i, j)$ of the matrix $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection $\iota_{l,l+1}$ from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$.

Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if there exists a specification ϕ from Σ to Σ' and an $m(l) \times m(l)$-square permutation matrix P_l for each $l \in \mathbb{N}$ such that

$$P_l \mathcal{M}_{l,l+1} \overset{\phi}{\simeq} \mathcal{M}'_{l,l+1} P_{l+1}, \qquad P_l I_{l,l+1} = I'_{l,l+1} P_{l+1} \qquad \text{for} \quad l \in \mathbb{Z}_+.$$

There exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of λ-graph systems.

Let $\mathcal{G} = (G, \lambda)$ be a labeled graph with finite directed graph G and labeling λ. Let $\{v_1, \dots, v_n\}$ be the vertex set of G. Put $V_l = \{v_1, \dots, v_n\}$ for all $l \in \mathbb{Z}_+$. We regard the sets $V_l, l \in \mathbb{Z}_+$ as disjoint sets. Define $\iota : V_{l+1} \to V_l$ by $\iota(v_i) = v_i$ for $i = 1, \dots, n$. Write labeled edges from V_l to V_{l+1} for $l \in \mathbb{N}$ following the directed graph G with labeling λ. The resulting labeled Bratteli diagram with ι becomes a λ-graph system. A labeled graph and also a λ-graph system are said to be left-resolving if different edges with the same label have different terminals. Hence a lebeled graph defines a λ-graph system such that if the lebeled graph is left-resolving, so is the λ-graph system. We call the resulting λ-graph system the associated λ-graph system with the labeled graph graph. We note that any sofic shift may be presented by left-resolving labeled graph ([Kr2],[Kr3],[We]).

A λ-graph system $\mathfrak{L}$ gives rise to a subshift $\Lambda_{\mathfrak{L}}$ on the sequence space of labels appearing in the labeled Bratteli diagram. We say that $\mathfrak{L}$ presents the subshift $\Lambda_{\mathfrak{L}}$. A canonical method to construct a λ-graph system from an arbitrary subshift Λ has been introduced in [Ma]. The λ-graph system and its symbolic matrix system are said to be canonical for the subshift and written as $\mathfrak{L}^{\Lambda}$ and $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ respectively.

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. For a vertex $v \in V_l$, we denote by $\Gamma^-_{\mathfrak{L},l}(v)$ the set of all label sequences of length l in $\mathfrak{L}$ that start at vertices of V_0 and terminate at v. We say that $\mathfrak{L}$ is predecessor-separated if for $u, v \in V_l$ the condition $\Gamma^-_{\mathfrak{L},l}(u) = \Gamma^-_{\mathfrak{L},l}(v)$ implies $u = v$. The canonical λ-graph systems are left-resolving and predecessor-separated.

We will introduce an equivalence relation of predecessor-separated λ-graph systems. Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be the symbolic matrix systems over Σ and Σ' respectively. We denote by $m(l)$ the row size of the matrix $\mathcal{M}_{l,l+1}$ and by $m'(l)$ that of $\mathcal{M}'_{l,l+1}$ respectively. We say that $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are *equivalent* if there exist $N \in \mathbb{Z}_+$ and a bijection $\pi : \Sigma \to \Sigma'$ such that for each $l \in \mathbb{Z}_+$, there exist an $m(l) \times m'(N+l)$ matrix H_l over $\{0,1\}$ and an $m'(l) \times m(N+l)$ matrix K_l over $\{0,1\}$ satisfying the following equations:

$$\mathcal{M}_{l,l+1} H_{l+1} \overset{\pi}{\simeq} H_l \mathcal{M}'_{l+N,l+N+1}, \qquad \mathcal{M}'_{l,l+1} K_{l+1} \overset{\pi^{-1}}{\simeq} K_l \mathcal{M}_{l+N,l+N+1},$$

$$I_{l,l+1} H_{l+1} = H_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1} K_{l+1} = K_l I_{l+N,l+N+1}$$

and

$$H_l K_{N+l} = I_{l,2N+l}, \qquad K_l H_{N+l} = I'_{l,2N+l}.$$

We write this equivalence relation as $(\mathcal{M}, I) \cong (\mathcal{M}', I')$. Two λ-graph systems are called *equivalent* if their respect symbolic matrix systems are equivalent.

In the rest of this section, we briefly review the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$. The C^*-algebras have been originally constructed in [Ma3] as groupoid C^*-algberas of certain r-discrete groupoids constructed from continuous graphs in the sense of Deaconu ([De],[De2],[De3],cf.[Re]) obtained by the λ-graph systems. They are realized as universal unique C^*-algebras as in the following way. For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, let $\{v_1^l, \dots, v_{m(l)}^l\}$ be the vertex set V_l. We put

(2.2)

$$A_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

(2.3)

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1,2,\dots,m(l),\ j = 1,2,\dots,m(l+1),\ \alpha \in \Sigma$.

Lemma 2.1([Ma3; Theorem A]). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is the universal concrete C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1,2,\dots,m(l),\ l \in \mathbb{Z}_+$ satisfying the following relations called $(\mathfrak{L})$:*

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \tag{2.4}$$

$$\sum_{k=1}^{m(l)} E_k^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}, \tag{2.5}$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{2.6}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) E_j^{l+1}, \tag{2.7}$$

for $i = 1,2,\dots,m(l), l \in \mathbb{Z}_+, \alpha \in \Sigma$.

If $\mathfrak{L}$ satisfies condition (I), a generalized conditon of condition (I) for a finite square matrix with entries in $\{0,1\}$ defined in [CK], the algebra $\mathcal{O}_{\mathfrak{L}}$ is the unique C^*-algebra subject to the above relations $(\mathfrak{L})$. Furthermore, if $\mathfrak{L}$ is irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([Ma3],[Ma5]). The gauge action $\alpha^{\mathfrak{L}}$ on $\mathcal{O}_{\mathfrak{L}}$ is defined by an action of $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ such that ${\alpha^{\mathfrak{L}}}_z(S_\alpha) = zS_\alpha$, ${\alpha^{\mathfrak{L}}}_z(E_i^l) = E_i^l$ for $\alpha \in \Sigma, i = 1,2,\dots,m(l), l \in \mathbb{Z}_+$.

3. C^*-SYMBOLIC DYNAMICAL SYSTEMS

Let $\mathcal{A}$ be a unital C^*-algebra. Throughout this paper, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_{\mathcal{A}}$ of $\mathcal{A}$. The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. We denote by $\mathrm{End}(\mathcal{A})$ the set of all endomorphisms of $\mathcal{A}$. Let Σ be a finite set. A finite family of endomorphisms $\rho_\alpha \in \mathrm{End}(\mathcal{A}), \alpha \in \Sigma$ is said to be *essential* if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\Sigma_{\alpha\in\Sigma}\rho_\alpha(1) \geq 1$.

It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.

Definition. A *C^*-symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and a finite family of endomorphisms ρ_α of $\mathcal{A}$ indexed by $\alpha \in \Sigma$, that is essential and faithful.

Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ and a bijection $\pi : \Sigma \to \Sigma'$ such that $\Phi \circ \rho_\alpha = \rho'_{\pi(\alpha)} \circ \Phi$ for all $\alpha \in \Sigma$.

Proposition 3.1. *For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, there uniquely exists a subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for the subshift if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$.*

Suppose that $\mathcal{A}$ is a commutative C^*-algebra $C(\Omega)$ of all continuous functions on a compact Hausdorff space Ω. An endomorphism of $\mathcal{A}$ bijectively corresponds to a continuous map from a clopen set of Ω to Ω. Hence a C^*-symbolic dynamical system $(C(\Omega), \rho, \Sigma)$ bijectively corresponds to a family $\{f_\alpha, E_\alpha\}_{\alpha \in \Sigma}$ of clopen sets $E_\alpha \subset \Omega$ and continuous maps $f_\alpha : E_\alpha \to \Omega$, $\alpha \in \Sigma$ such that

$$\bigcup_{\alpha\in\Sigma} E_\alpha = \Omega \quad \text{and} \quad \bigcup_{\alpha\in\Sigma} f_\alpha(E_\alpha) = \Omega.$$

We will study this situation in more graphical examples for a while.

For a left-resolving labeled graph $\mathcal{G} = (G, \lambda)$, let $v_1, \dots, v_n$ be its vertex set. Consider the n-dimensional commutative C^*-algbera $\mathcal{A}_\mathcal{G} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, n$. Then we may define an $n \times n$-matrix for $\alpha \in \Sigma$ with entries in $\{0,1\}$ by

$$A^\mathcal{G}(i, \alpha, j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for $i, j = 1, \dots, n$. We set

$$\rho^\mathcal{G}_\alpha(E_i) = \sum_{j=1}^n A^\mathcal{G}(i, \alpha, j) E_j, \qquad i = 1, \dots, n, \alpha \in \Sigma.$$

Then $\rho^\mathcal{G}_\alpha, \alpha \in \Sigma$ define endomorphisms of $\mathcal{A}_\mathcal{G}$ such that $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ is a C^*-symbolic dynamical system.

Conversely, let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system such that $\mathcal{A}$ is n-dimensional and commutative. Take $E_1, \dots, E_n$ the orthogonal minimal projections of $\mathcal{A}$ such that $\mathcal{A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$. Define an $n \times n$ matrix $[A(i, \alpha, j)]_{i,j=1,\dots,n}$ for $\alpha \in \Sigma$ by setting

$$A(i, \alpha, j) = \begin{cases} 1 & \text{if } \rho_\alpha(E_i) \geq E_j, \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

so that one has

$$\rho_\alpha(E_i) = \sum_{j=1}^n A(i, \alpha, j) E_j, \qquad i = 1, \dots, n, \alpha \in \Sigma.$$

Let $v_1, \dots, v_n$ be the vertex set corresponding to the projections $E_1, \dots, E_n$. Define a directed labeled edge e such as the source vertex $s(e) = v_i$, the terminal vertex $t(e) = v_j$ and the label $\lambda(e) = \alpha$ if $A(i, \alpha, j) = 1$. Then we have a left-resolving labeled graph $\mathcal{G}$ which presents the subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$. Hence we have

Proposition 3.2. *For a left-resolving labeled graph $\mathcal{G}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ such that the algebra $\mathcal{A}_{\mathcal{G}}$ is commutative and finite dimensional, and the associated subshift $\Lambda_{(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if $\mathcal{A}$ is commutative and finite dimensional, there exists a left-resolving labeled graph $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ and the associated subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$.*

Let us apply the above discussions to general subshifts and λ-graph systems. For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, let $(\mathcal{M}, I)$ be its corresponding symbolic matrix system. Let $A_{l,l+1}$ be the matrices defined by (2.2). We equip V_l with discrete topology. We denote by $\Omega_{\mathfrak{L}}$ the topological space of the projective limit

$$V_0 \overset{\iota}{\leftarrow} V_1 \overset{\iota}{\leftarrow} V_2 \overset{\iota}{\leftarrow} \cdots,$$

that is a compact, totally disconnected, second countable topological space. We regard the algebra of all continuous functions on V_l as the direct sum

$$C(V_l) = \mathbb{C}E_1^l \oplus \mathbb{C}E_2^l \oplus \cdots \oplus \mathbb{C}E_{m(l)}^l,$$

where the vertices $v_i^l \in V_l, i = 1, \dots, m(l)$ correspond to the minimal projections $E_i^l \in V_l, i = 1, \dots, m(l)$. We denote $C(V_l)$ by $\mathcal{A}_{\mathfrak{L},l}$. Let $\mathcal{A}_{\mathfrak{L}}$ be the commutative C^*-algebra of all continuous functions on $\Omega_{\mathfrak{L}}$, that is the inductive limit algebra

$$\mathcal{A}_{\mathfrak{L},0} \overset{I^t_{0,1}}{\to} \mathcal{A}_{\mathfrak{L},1} \overset{I^t_{1,2}}{\to} \mathcal{A}_{\mathfrak{L},2} \overset{I^t_{2,3}}{\to} \mathcal{A}_{\mathfrak{L},3} \overset{I^t_{3,4}}{\to} \cdots.$$

Hence $\mathcal{A}_{\mathfrak{L}}$ is a unital commutative AF-algebra. For a symbol $\alpha \in \Sigma$ we set

$$\rho_\alpha^{\mathfrak{L}}(E_i^l) = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j)E_j^{l+1} \qquad \text{for } i = 1, 2, \dots, m(l).$$

By the commutation relation (2.1), $\rho_\alpha^{\mathfrak{L}}$ defines an endomorphism of $\mathcal{A}_{\mathfrak{L}}$. Since each vertex $v_i^l \in V_l$ except $l = 0$ has an in-coming edge, the family $\{\rho_\alpha^{\mathfrak{L}}\}_{\alpha\in\Sigma}$ is essential. It is also faithful because each vertex $v_i^l \in V_l$ has an out-going edge. Thus we have

Proposition 3.3. *For a λ-graph system $\mathfrak{L}$ over Σ, there exists a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that the C^*-algebra $\mathcal{A}_{\mathfrak{L}}$ is commutative and AF, and the associated subshift $\Lambda_{(\mathcal{A}^{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)}$ coincides with the subshift $\Lambda_{\mathfrak{L}}$ presented by $\mathfrak{L}$.*

Conversely

Theorem 3.4. *Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system. If the algebra $\mathcal{A}$ is commutative and AF, there exists a λ-graph system $\mathfrak{L}$ over Σ such that the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$.*

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *predecessor-separated* if the projections $\{(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \mid \alpha_1, \dots, \alpha_k \in \Sigma, k \in \mathbb{N}\}$ generate the C^*-algebra $\mathcal{A}$.

Proposition 3.5.

(i) *If a λ-graph system $\mathfrak{L}$ is predecessor-separated, the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is predecessor-separated.*

(ii) *Suppose that an algebra $\mathcal{A}$ is unital, commutative and AF. If a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is predecessor-separated, there exists a predecessor-separated λ-graph system $\mathfrak{L}$ over Σ such that the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$.*

Proposition 3.6. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be predecessor-separated λ-graph systems over Σ and Σ respectively. Then $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}_{\mathfrak{L}'}, \rho^{\mathfrak{L}'}, \Sigma')$ if and only if $\mathfrak{L}$ and $\mathfrak{L}'$ are equivalent. In this case, the presented subshifts $\Lambda_{\mathfrak{L}}$ and $\Lambda_{\mathfrak{L}'}$ are identified through a symbolic conjugacy.*

Therefore we have

Corollary 3.7. *The equivalence classes of the predecessor-separated λ-graph systems are identified with the isomorphism classes of the predecessor-separated C^*-symbolic dynamical systems of the commutative AF-algebras.*

We formulate here an action of a subshift to a C^*-algebra. We say that *a subshift Λ acts on a C^*-algebra $\mathcal{A}$* if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ coincides with Λ.

4. Hilbert C^*-symbolic bimodules

In this section we will construct a Hilbert C^*-bimodule from a C^*-symbolic dynamical system. Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system. We put the projections $P_\alpha = \rho_\alpha(1)$ in $\mathcal{A}$ for $\alpha \in \Sigma$. Let $\{e_\alpha\}_{\alpha\in\Sigma}$ denote the standard basis of the $|\Sigma|$-dimensional vector space $\mathbb{C}^{|\Sigma|}$, where $|\Sigma|$ denotes the cardinal number of the set Σ. Set

$$\mathcal{H}^\rho_{\mathcal{A}} := \sum_{\alpha\in\Sigma} \mathbb{C}e_\alpha \otimes P_\alpha \mathcal{A}.$$

Define a right $\mathcal{A}$-action and an $\mathcal{A}$-valued inner product on $\mathcal{H}^\rho_{\mathcal{A}}$ by setting

$$(e_\alpha \otimes P_\alpha x)y := e_\alpha \otimes P_\alpha xy,$$
$$\langle e_\alpha \otimes P_\alpha x \mid e_\beta \otimes P_\beta y\rangle := \begin{cases} x^* P_\alpha y & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha, \beta \in \Sigma$ and $x, y \in \mathcal{A}$. Then $\mathcal{H}^\rho_{\mathcal{A}}$ forms a Hilbert C^*-right $\mathcal{A}$-module. We put

$$u_\alpha := e_\alpha \otimes P_\alpha, \qquad \alpha \in \Sigma.$$

Lemma 4.1. *The family $u_\alpha, \alpha \in \Sigma$ forms an orthogonal finite basis of $\mathcal{H}^\rho_{\mathcal{A}}$ in the sense of [KPW] such that*

$$\sum_{\alpha\in\Sigma} \langle u_\alpha \mid u_\alpha\rangle \geq 1. \tag{4.1}$$

We say that a finite basis of a Hilbert C^*-module is *essential* if the basis satisfies the condition (4.1). We will next define a diagonal left action ϕ_ρ of $\mathcal{A}$ to the set of all adjointable bounded $\mathcal{A}$-module maps $L(\mathcal{H}^\rho_\mathcal{A})$ on $\mathcal{H}^\rho_\mathcal{A}$ as follows:

$$\phi_\rho(a)u_\alpha x := u_\alpha \rho_\alpha(a)x, \qquad a, x \in \mathcal{A}, \alpha \in \Sigma.$$

The above definition is well-defined. If $u_\alpha x = u_\alpha y$, then $P_\alpha x = P_\alpha y$ so that $\rho_\alpha(a1)x = \rho_\alpha(a1)y$ for $a \in \mathcal{A}$. Hence one has that $u_\alpha\rho_\alpha(a)x = u_\alpha\rho_\alpha(a)y$. Since the family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ is faithful, the left action ϕ_ρ of $\mathcal{A}$ on $\mathcal{H}^\rho_\mathcal{A}$ is faithful, that is, the element $\phi_\rho(x)$ is nonzero for any nonzero $x \in \mathcal{A}$. Therefore we have

Proposition 4.2. *For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, there exists a Hilbert C^*-right $\mathcal{A}$-module $\mathcal{H}^\rho_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha\in\Sigma}$ and a unital faithful diagonal left action $\phi_\rho : \mathcal{A} \to L(\mathcal{H}^\rho_\mathcal{A})$ such that*

$$\begin{aligned} &(4.2) & \phi_\rho(a)u_\alpha &= u_\alpha\rho_\alpha(a), & & \\ &(4.3) & \langle u_\alpha \mid u_\alpha\rangle &= \rho_\alpha(1), & & a \in \mathcal{A}, \alpha \in \Sigma. \end{aligned}$$

We note that the above two conditions imply

$$(4.4) \qquad \langle u_\alpha \mid \phi_\rho(a)u_\alpha\rangle = \rho_\alpha(a), \qquad a \in \mathcal{A}, \alpha \in \Sigma.$$

Conversely

Proposition 4.3. *For a Hilbert C^*-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha\in\Sigma}$ and a unital faithful diagonal left action $\phi : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$, define ρ_α for $\alpha \in \Sigma$ by setting*

$$\rho_\alpha(a) = \langle u_\alpha \mid \phi(a)u_\alpha\rangle, \qquad a \in \mathcal{A}.$$

Then ρ_α gives rise to an endomorphism of $\mathcal{A}$ such that $(\mathcal{A}, \rho, \Sigma)$ yields a C^*-symbolic dynamical system.

A Hilbert C^*-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with a left action $\phi : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$ is called a Hilbert C^*-bimodule over $\mathcal{A}$ ([Pim], cf.[KW], [KPW], [MS]). Two Hilbert C^*-bimodules $(\phi, \mathcal{H}_\mathcal{A})$ and $(\phi', \mathcal{H}'_\mathcal{A})$ over $\mathcal{A}$ are said to be unitary equivalent if there exists a bimodule isomorphism $\Phi : \mathcal{H}_\mathcal{A} \to \mathcal{H}'_\mathcal{A}$ such that Φ is unitary with respect to their respect inner products.

Definition. A Hilbert C^*-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha\in\Sigma}$ and a unital faithful diagonal left action $\phi : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$ is called a *Hilbert C^*-symbolic bimodule over $\mathcal{A}$*. It is written as $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha\in\Sigma})$.

A Hilbert C^*-symbolic bimodule $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha\in\Sigma})$ over $\mathcal{A}$ bijectively corresponds to a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ by the above discussions. Two Hilbert C^*-symbolic bimodules $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha\in\Sigma})$ and $(\phi', \mathcal{H}'_\mathcal{A}, \{u'_{\alpha'}\}_{\alpha'\in\Sigma'})$ over $\mathcal{A}$ are said to be unitary equivalent if there exists a bimodule isomorphism $\Phi : \mathcal{H}_\mathcal{A} \to \mathcal{H}'_\mathcal{A}$ and a bijection $\pi : \Sigma \to \Sigma'$ such that Φ is unitary with respect to their respect inner products and satisfies $\Phi(u_\alpha) = u'_{\pi(\alpha)}, \alpha \in \Sigma$. Let $\rho_\alpha, \alpha \in \Sigma$ and $\rho'_{\alpha'}, \alpha' \in \Sigma'$ be their respect endomorphisms of $\mathcal{A}$. In this case, we have

$\rho_\alpha(a) = \rho'_{\pi(\alpha)}(a), a \in \mathcal{A}$ because the equality (4.2) implies $\phi'(a)\Phi(u_\alpha) = \Phi(u_\alpha)\rho_\alpha(a)$ and hence $\phi'(a)u'_{\pi(\alpha)} = u'_{\pi(\alpha)}\rho_\alpha(a)$. This means that $\rho_\alpha(a) = \rho'_{\pi(\alpha)}(a), a \in \mathcal{A}$.

Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are said to be *inner conjugate* if there exists an element $U_{\alpha,\beta} \in \mathcal{A}$ for $\alpha \in \Sigma, \beta \in \Sigma'$ such that

(i) $\rho_\alpha(a)U_{\alpha,\beta} = U_{\alpha,\beta}\rho'_\beta(a)$,

(ii) $\sum_{\epsilon\in\Sigma'} U_{\alpha,\epsilon}U^*_{\gamma,\epsilon} = \delta_{\alpha,\gamma}\rho_\alpha(1)$, $\quad \sum_{\gamma\in\Sigma} U^*_{\gamma,\beta}U_{\gamma,\epsilon} = \delta_{\beta,\epsilon}\rho'_\beta(1)$ $\quad$ and

(iii) $\rho_\alpha(1)U_{\alpha,\beta} = U_{\alpha,\beta} = U_{\alpha,\beta}\rho'_\beta(1)$

for $\alpha, \gamma \in \Sigma$, $\beta, \epsilon \in \Sigma'$ and $a \in \mathcal{A}$. The family $\{U_{\alpha,\beta}\}_{\alpha\in\Sigma,\beta\in\Sigma'}$ is called an intertwiner between $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$.

Proposition 4.4. *Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are inner conjugate if and only if their associated Hilbert C^*-bimodules $(\phi_\rho, \mathcal{H}^\rho_\mathcal{A})$ and $(\phi_{\rho'}, \mathcal{H}^{\rho'}_\mathcal{A})$ are unitary equivalent as a Hilbert C^*-bimodule.*

We note that if $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are inner conjugate with intertwiner $\{U_{\alpha,\beta}\}_{\alpha\in\Sigma,\beta\in\Sigma'}$, then the equalities for $\alpha \in \Sigma, \beta \in \Sigma'$ and $a \in \mathcal{A}$

$$\rho_\alpha(a) = \sum_{\epsilon\in\Sigma'} U_{\alpha,\epsilon}\rho'_\epsilon(a)U^*_{\alpha,\epsilon}, \qquad \rho'_\beta(a) = \sum_{\gamma\in\Sigma} U^*_{\gamma,\beta}\rho_\gamma(a)U_{\gamma,\beta},$$

hold. For $(\mathcal{A}, \rho, \Sigma)$, let $D_\rho(a)$ for $a \in \mathcal{A}$ be the $|\Sigma| \times |\Sigma|$-diagonal matrix $D_\rho(a)$ with diagonal entries $[\rho_\alpha(a)]_{\alpha\in\Sigma}$. One knows $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are inner conjugate if and only if there exists an $|\Sigma| \times |\Sigma'|$-matrix U over $\mathcal{A}$ such that

$$D_\rho(a) = UD_{\rho'}(a)U^* \qquad \text{for } a \in \mathcal{A}, \text{ and} \tag{4.5}$$

$$UU^* = D_\rho(1), \quad U^*U = D_{\rho'}(1). \tag{4.6}$$

Let $\mathcal{A}$ be an n-dimensional commutative C^*-algebra. By Proposition 3.2, a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ defines a left-resolving labeled graph $\mathcal{G}^\rho = (G^\rho, \lambda^\rho)$ over Σ with underlying finite directed graph G^ρ. Let $v_1, \dots, v_n$ denote the vertex set of G^ρ. We denote by $A^\rho(i,j)$ the cardinal number of the edges $E^\rho(i,j)$ whose source vertex is v_i and terminal vertex is v_j. In this case, inner conjugacy is completely characterized as in the following way.

Proposition 4.5. *Let $\mathcal{A}$ be the n-dimensional commutative C^*-algebra. Then C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \eta, \Sigma)$ are inner conjugate if and only if $A^\rho(i,j) = A^\eta(i,j)$ for all $i, j = 1, 2, \dots, n$. That is, the directed graphs G^ρ and G^η are isomorphic.*

5. Crossed products by symbolic dynamical systems

We will study C^*-algebras constructed from Hilbert C^*-symbolic bimodules. A general construction of C^*-algebras from Hilbert C^*-bimodules has been established by Pimsner [Pim] (cf. [Ka]). The C^*-algebras are called Cuntz-Pimsner algebras. Its ideal structure and simplicity conditions have been studied by Kajiwara-Pinzari-Watatani [KPW] and Muhly-Solel [MS], see also [KW], [PWY], [Sch]. For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, we have a C^*-algebra from the Hilbert C^*-symbolic bimodule $(\phi_\rho, \mathcal{H}^\rho_\mathcal{A}, \{u_\alpha\}_{\alpha\in\Sigma})$ by using Pimsner's general construction of C^*-algebras from Hilbert C^*-bimodules. We denote the C^*-algebra by $\mathcal{A} \rtimes_\rho \Lambda$, where Λ is the subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$ associated with $(\mathcal{A}, \rho, \Sigma)$. We call the algebra $\mathcal{A} \rtimes_\rho \Lambda$ *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift Λ.*

Proposition 5.1. *The C^*-symbolic crossed product $\mathcal{A} \rtimes_\rho \Lambda$ is the universal unital C^*-algebra $C^*(\mathcal{A}, S_\alpha, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following operator relations:*

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x \tag{5.1}$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. Furthermore for $\alpha_1, \dots, \alpha_k \in \Sigma$, a word $(\alpha_1, \dots, \alpha_k)$ is admissible for the subshift $\Lambda = \Lambda_{(\mathcal{A},\rho,\Sigma)}$ if and only if $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$.

As in [Pim] (cf. [KPW]), the gauge action, denoted by $\hat\rho$, on the algebra $\mathcal{A} \rtimes_\rho \Lambda$ of the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is defined by

$$\hat\rho_z(x) = x, \qquad \hat\rho_z(S_\alpha) = zS_\alpha, \qquad x \in \mathcal{A}, \alpha \in \Sigma, z \in \mathbb{T}.$$

We have the following theorem.

Theorem 5.2. *Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system and Λ be the associated subshift $\Lambda_{(\mathcal{A},\rho,\Sigma)}$. Assume that $\mathcal{A}$ is commutative.*

(i) *If $\mathcal{A} = \mathbb{C}$, the subshift Λ is the full shift $\Sigma^{\mathbb{Z}}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$.*

(ii) *If $\mathcal{A}$ is finite dimensional, the subshift Λ is a sofic shift $\Lambda_{\mathcal{G}}$ presented by a left-resolving labeled graph $\mathcal{G}$, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is a Cuntz-Krieger algbera $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph. Conversely, for any sofic shift $\Lambda_{\mathcal{G}}$, that is presented by a left-resolving labeled graph $\mathcal{G}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the sofic shift $\Lambda_{\mathcal{G}}$, the algebra $\mathcal{A}$ is finite dimensional, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the Cuntz-Krieger algbera $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph.*

(iii) *If $\mathcal{A}$ is an AF-algebra, there uniquely exists a λ-graph system $\mathfrak{L}$ up to equivalence such that the subshift Λ is presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$. Conversely, for any subshift $\Lambda_{\mathfrak{L}}$, that is presented by a left-resolving λ-graph system $\mathfrak{L}$, there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the subshift $\Lambda_{\mathfrak{L}}$, the algebra $\mathcal{A}$ is a commutative AF-algebra, and the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.*

We remark that Pimsner showed the following fact [Pim]: For every Hilbert C^*-bimodule E over a C^*-algebra $\mathcal{A}$, if $\mathcal{A}$ is commutative and finite dimensional, and if E is projective and finitely generated, the associated C^*-algebra is a Cuntz-Krieger algebra.

We will give some examples

(i) Let $\alpha_1, \dots, \alpha_m \in \mathrm{Aut}(\mathcal{B})$ be automorphisms of a unital C^*-algebra $\mathcal{B}$. Let $\mathcal{G} = (G, \lambda)$ be a left-resolving labeled graph with symbols $\Sigma = \{\alpha_1, \dots, \alpha_m\}$. Let $V = \{v_1, \dots, v_n\}$ be the vertex set. Let $[A^{\mathcal{G}}(i, \alpha_k, j)]_{i,j=1,\dots,n}$ be the $n \times n$-matrix for $\alpha_k \in \Sigma$ with entries in $\{0, 1\}$ defined by (3.1). We put $\mathcal{A} = \mathcal{B} \oplus \cdots \oplus \mathcal{B}$ the direct sum of the n-copies of $\mathcal{B}$. For $\alpha_k \in \Sigma$, define $\rho^{\mathcal{G}}_{\alpha_k} \in \mathrm{End}(\mathcal{A})$ by setting

$$\begin{aligned}&\rho^{\mathcal{G}}_{\alpha_k}(b_1,\dots,b_n)\\ =&(\sum_{i=1}^n A^{\mathcal{G}}(i,\alpha_k,1)\alpha_k(b_i),\dots,\sum_{i=1}^n A^{\mathcal{G}}(i,\alpha_k,n)\alpha_k(b_i)), \qquad (b_1,\dots,b_n)\in\mathcal{A}.\end{aligned}$$

Since we assume that every vertex of G has an in-coming edge, one has $\sum_{k=1}^n \rho^{\mathcal{G}}_{\alpha_k}(1) \geq 1$. Since we also assume that every vertex of G has an out-going edge, the family $\{\rho^{\mathcal{G}}_{\alpha_k}\}_{k=1}^n$ is faithful. Hence we have a C^*-symbolic dynamical system $(\mathcal{A},\rho^{\mathcal{G}},\Sigma)$. The associated subshift $\Lambda_{(\mathcal{A},\rho^{\mathcal{G}},\Sigma)}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by the labeled graph $\mathcal{G}$. If the underlying directed graph G is irreducible with condition (I) in the sense of [CK] and each automorphism α_k has no nontrivial invariant ideal of $\mathcal{B}$, the associated crossed product $\mathcal{A}\rtimes_\rho \Lambda_{\mathcal{G}}$ is simple and purely infinite.

The following example is a special case of this example.

(ii) Let $A = C(\mathbb{T})$ and $\Sigma = \{1,2,\dots,n\}, n > 1$. Take irrational numbers $\theta_1,\dots,\theta_n \in \mathbb{R}\backslash\mathbb{Q}$. Define $\rho_i(f)(z) = f(e^{2\pi\sqrt{-1}\theta_i}z)$ for $f \in C(\mathbb{T}), z \in \mathbb{T}$. We have a C^*-symbolic dynamical system $(C(\mathbb{T}),\rho,\Sigma)$. Since the endomorphisms $\rho_i, i = 1,\dots,n$ are automorphisms and hence the associated subshift is the full shift $\Sigma^{\mathbb{Z}}$. We denote by $\mathcal{O}_{\theta_1,\dots,\theta_n}$ the C^*-symbolic crossed product $C(\mathbb{T})\rtimes_{\theta_1,\dots,\theta_n}\Sigma^{\mathbb{Z}}$. As the algebra $\mathcal{O}_{\theta_1,\dots,\theta_n}$ is the universal unital C^*-algebra generated by n isometries and one unitary U satisfying the following relations:

$$\sum_{j=1} S_j S_j^* = 1, \qquad S_i^* S_i = 1, \qquad U S_i = e^{2\pi\sqrt{-1}\theta_i} S_i U, \qquad i = 1,\dots,n.$$

Hence $\mathcal{O}_{\theta_1,\dots,\theta_n}$ is realized as the ordinary crossd product $\mathcal{O}_n \rtimes_{\alpha_{\theta_1,\dots,\theta_n}} \mathbb{Z}$ of the Cuntz algebra $\mathcal{O}_n$ by the automorphism $\alpha_{\theta_1,\dots,\theta_n}$ defined by $\alpha_{\theta_1,\dots,\theta_n}(S_i) = e^{2\pi\sqrt{-1}\theta_i}S_i$. It is simple and purely infinite whose K-groups are

$$K_0(\mathcal{O}_{\theta_1,\dots,\theta_n}) = K_1(\mathcal{O}_{\theta_1,\dots,\theta_n}) \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

(iii) Let $A = [A(i,j)]_{i,j=1,\dots,n}$ be an $n\times n$ matrix with entries in $\{0,1\}$. We denote by Λ_A^+ the compact Hausdorff space

$$\Lambda_A^+ = \{(x_i)_{i\in\mathbb{N}} \in \{1,\dots,n\}^{\mathbb{N}} \mid A(x_i,x_{i+1}) = 1 \text{ for all } i\in\mathbb{N}\}$$

of the right one-sided topological Markov shift associated with the matrix A. Let $S_i, i = 1,\dots,,n$ be the generating partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_A$ such that $\sum_{j=1}^n S_jS_j^* = 1, S_i^*S_i = \sum_{j=1}^n A(i,j)S_jS_j^*$. The algebra $\mathcal{A}_A = C(\Lambda_A^+)$ of all continuous functions on Λ_A^+ is identified with the subalgebra of $\mathcal{O}_A$ generated by the projections $S_\mu S_\mu^*$ for $\mu = \mu_1\cdots\mu_k$, where $S_\mu = S_{\mu_1}\cdots S_{\mu_k}$ for $\mu_1,\dots,\mu_k \in \{1,\dots,n\}$. Let $\Sigma = \{(_1,(_2,\dots,(_n,)_1,)_2,\dots,)_n\}$ be $2n$-brackets. We define $2n$-endomorphisms of $\mathcal{A}_A$ by setting

$$\rho^A_{(_i}(a) = S_i^* a S_i, \qquad \rho^A_{)_i}(a) = S_i a S_i^*, \qquad i = 1,\dots,n,\ a\in\mathcal{A}_A.$$

We have a C^*-symbolic dynamical system $(\mathcal{A}_A, \rho^A, \Sigma)$. If in particular all entries $A(i,j), i,j = 1,\dots,n$ of A are 1, then Λ_A^+ is the right one-sided full shift $\{1,\dots,n\}^{\mathbb{N}}$ and the associated subshift is the Dyck shift D_n of the $2n$-brackets. Let $\mathfrak{L}^{Ch(D_n)}$ be the corresponding λ-graph system for $(\mathcal{A}_A, \rho^A, \Sigma)$. It is called the Cantor horizon λ-graph system of the Dyck shift D_n, that has been studied in [KM]. The C^*-symbolic crossed product $C(\{1,\dots,n\}^{\mathbb{N}}) \rtimes_{\rho^A} D_n$ is a simple purely infinite C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_n)}}$ that is the C^*-algebra associated with $\mathfrak{L}^{Ch(D_n)}$. Its K-groups have been computed so that

$$K_0(C(\{1,\dots,n\}^{\mathbb{N}}) \rtimes_{\rho^A} D_n) = \mathbb{Z}/n\mathbb{Z} \oplus C(\mathcal{C},\mathbb{Z}),$$
$$K_1(C(\{1,\dots,n\}^{\mathbb{N}}) \rtimes_{\rho^A} D_n) = 0$$

where $C(\mathcal{C},\mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$-valued continuous functions on the Cantor set $\mathcal{C}$ ([KM]).

For a general matrix A with entries in $\{0,1\}$, let $\mathfrak{L}^{Ch(D_A)}$ be the corresponding λ-graph system to $(\mathcal{A}_A, \rho^A, \Sigma)$. It is easy to see that the associated subshift is a subshift of Dyck shift D_n that has some forbidden words coming from the forbidden words of the topological Markov shift Λ_A. The subshift is a version of topological Markov shift of the Dyck shifts, and appear in [HIK], [KM2]. We call it the topological Markov Dyck shift associated with the matrix A and write it as D_A. We then see that the C^*-symbolic crossed product $C(\Lambda_A^+) \rtimes_{\rho^A} D_A$ is a simple purely infinite C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_A)}}$ if the matrix A is irreducible.

6. Strong shift equivalence of C^*-symbolic dynamical systems and Hilbert C^*-bimodules

As in the preceding section, we may regard a λ-graph system as a C^*-symbolic dynamical system. The matrix interpretation of a λ-graph system is called a symbolic matrix system. In [Ma], we have formulated strong shift equivalence of symbolic matrix systems, as a generalization of nonnegative square matrices ([Wi]) and symbolic square matrices ([N]). Strong shift equivalence of symbolic matrix systems is a basic equivalence relation related to topological conjugacy of subshifts. It has been proved that two subshifts Λ and Λ' are topologically conjugate if and only if their canonical symbolic matrix systems $(\mathcal{M}^\Lambda, I^\Lambda)$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are strong shift equivalent ([Ma]).

In this section, we will formulate strong shift equivalences and shift equivalences of C^*-symbolic dynamical systems and of Hilbert C^*-symbolic bimodules as generalizations of those of λ-graph systems.

Definition. Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be *strong shift equivalent in* 1-*step* if there exist finite sets C and D, two families of homomorphisms $\eta_c : \mathcal{A} \to \mathcal{A}', c \in C$ and $\zeta_d : \mathcal{A}' \to \mathcal{A}, d \in D$ and two into bijections $\kappa : \Sigma \to CD$ and $\kappa' : \Sigma' \to DC$ such that

$$\rho_\alpha = \zeta_{d_\alpha} \circ \eta_{c_\alpha} \text{ if } \kappa(\alpha) = c_\alpha d_\alpha, \quad \text{and} \quad \rho'_{\alpha'} = \eta_{c_{\alpha'}} \circ \zeta_{d_{\alpha'}} \text{ if } \kappa'(\alpha') = d_{\alpha'} c_{\alpha'}$$

and

$$\zeta_d \circ \eta_c = 0 \text{ if } cd \notin \kappa(\Sigma), \quad \text{and} \quad \eta_c \circ \zeta_d = 0 \text{ if } dc \notin \kappa'(\Sigma').$$

We write this situation as $(\mathcal{A},\rho,\Sigma) \underset{1}{\approx} (\mathcal{A}',\rho',\Sigma')$.

We set $\widetilde{\mathcal{A}} = \mathcal{A}\oplus\mathcal{A}'$ and $\widetilde{\Sigma} = C\sqcup D$ disjoint union of C and D. Define $\tilde{\rho}_{\tilde{\alpha}} \in \mathrm{End}(\widetilde{\mathcal{A}})$ for $\tilde{\alpha} \in \widetilde{\Sigma}$ by setting

$$\tilde{\rho}_{\tilde{\alpha}}(x,y) = \begin{cases} (0,\eta_c(x)) & \text{if } \tilde{\alpha} = c \in C, \\ (\zeta_d(y),0) & \text{if } \tilde{\alpha} = d \in D \end{cases}$$

for $(x,y) \in \mathcal{A}\oplus\mathcal{A}'$. Then we have

Lemma 6.1. *$(\widetilde{\mathcal{A}},\tilde{\rho},\widetilde{\Sigma})$ is a C^*-symbolic dynamical system.*

We call $(\widetilde{\mathcal{A}},\tilde{\rho},\widetilde{\Sigma})$ the *bipartite* C^*-symbolic dynamical system related to $(\mathcal{A},\rho,\Sigma)$ and $(\mathcal{A}',\rho',\Sigma')$. If there exists an N-chain of strong shift equivalences in 1-step between $(\mathcal{A},\rho,\Sigma)$ and $(\mathcal{A}',\rho',\Sigma')$, they are said to be *strong shift equivalent in N-step* and written as $(\mathcal{A},\rho,\Sigma) \underset{N}{\approx} (\mathcal{A}',\rho',\Sigma')$. They are simply said to be strong shift equivalent.

Recall that two C^*-symbolic dynamical systems $(\mathcal{A},\rho,\Sigma)$ and $(\mathcal{A}',\rho',\Sigma')$ are said to be isomorphic if there exists an isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ of C^*-algebras and a bijection $\pi : \Sigma \to \Sigma'$ such that $\rho_\alpha = \phi^{-1}\circ {\rho'}_{\pi(\alpha)}\circ\phi$ for all $\alpha \in \Sigma$.

Lemma 6.2.

(i) *If $(\mathcal{A},\rho,\Sigma)$ and $(\mathcal{A}',\rho',\Sigma')$ are isomorphic, they are strong shift equivalent in 1-step.*

(ii) *Suppose that both sets Σ and Σ' are one points $\{\alpha\}$ and $\{\alpha'\}$ respectively and both ρ_α and $\rho'_{\alpha'}$ are automorphisms. Then $(\mathcal{A},\rho,\Sigma)$ and $(\mathcal{A}',\rho',\Sigma')$ are isomorphic if and only if they are strong shift equivalent in 1-step.*

We next formulate shift equivalence of C^*-symbolic dynamical systems

Definition. C^*-symbolic dynamical systems $(\mathcal{A},\rho,\Sigma)$ and $(\mathcal{A}',\rho',\Sigma')$ are said to be *shift equivalent of lag N* if there exist two finite sets C and D, two families $\eta_c : \mathcal{A}\to\mathcal{A}', c \in C$ and $\zeta_d : \mathcal{A}'\to\mathcal{A}, d\in D$ of homomorphisms and four specifications $\kappa_C : \Sigma C \to C\Sigma'$, $\kappa_D : \Sigma' D \to D\Sigma$, $\kappa_\Sigma : \Sigma^N \to CD$ and $\kappa_{\Sigma'} : {\Sigma'}^N \to DC$ such that

$$\begin{aligned} \eta_c\circ\rho_\alpha &= \rho'_{\alpha'}\circ\eta_{c'} \qquad \text{if} \quad \kappa_C(\alpha c) = c'\alpha', \\ \zeta_{d'}\circ\rho'_{\alpha'} &= \rho_\alpha\circ\zeta_d \qquad \text{if} \quad \kappa_D(\alpha' d') = d\alpha, \end{aligned}$$

and

$$\begin{aligned} \rho_{\alpha_N}\circ\cdots\circ\rho_{\alpha_2}\circ\rho_{\alpha_1} &= \zeta_d\circ\eta_c \qquad \text{if} \quad \kappa_\Sigma(\alpha_1\alpha_2\cdots\alpha_N) = cd, \\ \rho'_{\alpha'_N}\circ\cdots\circ\rho'_{\alpha'_2}\circ\rho'_{\alpha'_1} &= \eta_{c'}\circ\zeta_{d'} \qquad \text{if} \quad \kappa_{\Sigma'}(\alpha'_1\alpha'_2\cdots\alpha'_N) = d'c'. \end{aligned}$$

We write this situation as $(\mathcal{A},\rho,\Sigma) \underset{N}{\sim} (\mathcal{A}',\rho',\Sigma')$.

The following proposition is proved by similar ideas to the case of matrices ([Wi], cf.[LM]).

Proposition 6.3. *Let $(\mathcal{A},\rho,\Sigma)$, $(\mathcal{A}',\rho',\Sigma')$ and $(\mathcal{A}'',\rho'',\Sigma'')$ be C^*-symbolic dynamical systems.*

(i) $(\mathcal{A},\rho,\Sigma) \underset{N}{\approx} (\mathcal{A}',\rho',\Sigma')$ *implies* $(\mathcal{A},\rho,\Sigma) \underset{N}{\sim} (\mathcal{A}',\rho',\Sigma')$.

(ii) $(\mathcal{A},\rho,\Sigma) \underset{N}{\sim} (\mathcal{A}',\rho',\Sigma')$ *implies* $(\mathcal{A},\rho,\Sigma) \underset{N'}{\sim} (\mathcal{A}',\rho',\Sigma')$ *for all* $N' \geq N$.

(iii) $(\mathcal{A},\rho,\Sigma) \underset{N}{\sim} (\mathcal{A}',\rho',\Sigma')$ *and* $(\mathcal{A}',\rho',\Sigma') \underset{L}{\sim} (\mathcal{A}'',\rho'',\Sigma'')$ *imply* $(\mathcal{A},\rho,\Sigma) \underset{N+L}{\sim} (\mathcal{A}'',\rho'',\Sigma'')$.

Thus shift equivalence of C^*-symbolic dynamical systems is an equivalence relation.

We will next formulate strong shift equivalence and shift equivalence of Hilbert C^*-bimodules. Let $\mathcal{A}$ and $\mathcal{A}'$ be C^*-algebras. We define a ***Hilbert*** C^****-symbolic right*** $\mathcal{A}'$*-module* $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_\alpha\}_{\alpha\in\Sigma})$ *over* Σ *with left* $\mathcal{A}$*-action* by a Hilbert C^*-right $\mathcal{A}'$-module with orthogonal essential finite basis $\{w_\alpha\}_{\alpha\in\Sigma}$ and a unital faithful diagonal left action φ of $\mathcal{A}$ on ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}$. Let $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_\alpha\}_{\alpha\in\Sigma})$ be a Hilbert C^*-symbolic right $\mathcal{A}'$-module over Σ with left $\mathcal{A}$-action and $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w'_{\alpha'}\}_{\alpha'\in\Sigma'})$ a Hilbert C^*-symbolic right $\mathcal{A}''$-module over Σ' with left $\mathcal{A}'$-action. Define the relative tensor product

$$
\begin{aligned}
&(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_\alpha\}_{\alpha\in\Sigma}) \otimes_{\mathcal{A}'} (\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w'_{\alpha'}\}_{\alpha'\in\Sigma'})\\
:=&(\varphi\otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w_\alpha \otimes_{\mathcal{A}'} w'_{\alpha'}\}_{(\alpha,\alpha')\in\Sigma\otimes_{\mathcal{A}'}\Sigma'})
\end{aligned}
$$

where ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}$ is the tensor product Hilbert C^*-right $\mathcal{A}''$-module relative to $\mathcal{A}'$, and $\varphi\otimes 1$ is the natural left $\mathcal{A}$-action on it. The finite set $\Sigma\otimes_{\mathcal{A}'}\Sigma'$ is defined as follows: As both the left action φ and ψ are diagonal with respect to the bases $\{w_\alpha\}_{\alpha\in\Sigma}$ and $\{w'_{\alpha'}\}_{\alpha'\in\Sigma'}$ respectively, there exist $\eta_\alpha(a)\in\mathcal{A}'$ for $a\in\mathcal{A}$ and $\zeta_{\alpha'}(b)\in\mathcal{A}''$ for $b\in\mathcal{A}'$ such that

$$
\varphi(a)w_\alpha = w_\alpha\eta_\alpha(a), \qquad \psi(b)w'_{\alpha'} = w'_{\alpha'}\zeta_{\alpha'}(b).
$$

The finite set $\Sigma\otimes_{\mathcal{A}'}\Sigma'$ is defined by

$$
\Sigma\otimes_{\mathcal{A}'}\Sigma' = \{(\alpha,\alpha')\in\Sigma\times\Sigma' \mid \zeta_{\alpha'}(\eta_\alpha(1_{\mathcal{A}}))\neq 0\}.
$$

It is easy to check that

$$
(\varphi\otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w_\alpha \otimes_{\mathcal{A}'} w'_{\alpha'}\}_{(\alpha,\alpha')\in\Sigma\otimes_{\mathcal{A}'}\Sigma'})
$$

is a Hilbert C^*-symbolic right $\mathcal{A}''$-module over $\Sigma\otimes_{\mathcal{A}'}\Sigma'$ with left $\mathcal{A}$-action.

Definition. Let $(\phi,\mathcal{H}_{\mathcal{A}})$ be a Hilbert C^*-bimodule over $\mathcal{A}$ and $(\phi',\mathcal{H}_{\mathcal{A}'})$ a Hilbert C^*-bimodule over $\mathcal{A}'$. They are said to be ***strong shift equivalent in*** 1***-step*** and written as $(\phi,\mathcal{H}_{\mathcal{A}}) \underset{1}{\approx} (\phi',\mathcal{H}_{\mathcal{A}'})$ if there exist a Hilbert C^*-right $\mathcal{A}'$-module $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'})$ with left $\mathcal{A}$-action and a Hilbert C^*-right $\mathcal{A}$-module $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$ with left $\mathcal{A}'$-action such that

(6.1)
$$
\begin{cases}
(\varphi\otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}) = & (\phi,\mathcal{H}_{\mathcal{A}}) \text{ as a Hilbert } C^*\text{-bimodule over } \mathcal{A},\\
(\psi\otimes 1, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}) = & (\phi',\mathcal{H}_{\mathcal{A}'}) \text{ as a Hilbert } C^*\text{-bimodule over } \mathcal{A}'.
\end{cases}
$$

The above all equalities of Hilbert C^*-bimodules mean unitary equivalences as Hilbert C^*-bimodules. In this situation, we say that $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'})$ and $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$ satisfy the strong shift equivalence relation between $(\phi, \mathcal{H}_{\mathcal{A}})$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$. Consider the direct sum

$$(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}) \oplus (\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}) := (\varphi \oplus \psi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \oplus {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$$

that is a Hilbert C^*-right $\mathcal{A}' \oplus \mathcal{A}$-module with left $\mathcal{A} \oplus \mathcal{A}'$-action. It is denoted by $(\xi, \mathcal{H}_X)$ and satisfies

$${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} = \xi(\mathcal{A})\mathcal{H}_X = \mathcal{H}_X\mathcal{A}', \qquad {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} = \xi(\mathcal{A}')\mathcal{H}_X = \mathcal{H}_X\mathcal{A}.$$

As $\mathcal{H}_X$ is regarded as a Hilbert C^*-right $\mathcal{A}\oplus\mathcal{A}'$-module, $(\xi, \mathcal{H}_X)$ is considered to be a Hilbert C^*-bimodule over $\mathcal{A} \oplus \mathcal{A}'$, that is called a *bipartite* Hilbert C^*-bimodule related to $(\phi, \mathcal{H}_{\mathcal{A}})$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$. We note that the condition (6.1) is equivalent to the condition:

$(\xi\otimes 1, \mathcal{H}_X \otimes_{\mathcal{A}\oplus\mathcal{A}'} \mathcal{H}_X) = (\phi, \mathcal{H}_{\mathcal{A}})\oplus(\phi', \mathcal{H}_{\mathcal{A}'})$ as a Hilbert C^*-bimodule over $\mathcal{A}\oplus\mathcal{A}'$.

If there exists an N-chain of strong shift equivalences in 1-step between $(\phi, \mathcal{H}_{\mathcal{A}})$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$, they are said to be *strong shift equivalent in N-step* and we write it as $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N}{\approx} (\phi', \mathcal{H}_{\mathcal{A}'})$. They are simply said to be strong shift equivalent.

In particular, Hilbert C^*-symbolic bimodules $(\phi, \mathcal{H}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ and $(\phi', \mathcal{H}_{\mathcal{A}'}, \{u'_{\alpha'}\}_{\alpha'\in\Sigma'})$ are said to be strong shift equivalent in 1-step if there exist a Hilbert C^*-symbolic right $\mathcal{A}'$-module $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_c\}_{c\in C})$ with left $\mathcal{A}$-action and a Hilbert C^*-right $\mathcal{A}$-module $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}, \{w'_d\}_{d\in D})$ with left $\mathcal{A}'$-action such that the qualities (6.1) are taken to be unitary equivalent as Hilbert C^*-symbolic bimodules.

Definition. Let $(\phi, \mathcal{H}_{\mathcal{A}})$ be a Hilbert C^*-bimodule over $\mathcal{A}$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$ a Hilbert C^*-bimodule over $\mathcal{A}'$. They are said to be *shift equivalent of lag N* if there exist a Hilbert C^*-right $\mathcal{A}'$-module $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'})$ with left $\mathcal{A}$-action and a Hilbert C^*-right $\mathcal{A}$-module $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$ with left $\mathcal{A}'$-action such that

$$(\phi, \underbrace{\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{H}_{\mathcal{A}}}_{N}) = (\varphi \otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}),$$

$$(\phi', \underbrace{\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} \cdots \otimes_{\mathcal{A}'} \mathcal{H}_{\mathcal{A}'}}_{N}) = (\psi \otimes 1, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}),$$

and

$$(\varphi\otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}\otimes_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}'}) = (\phi, \mathcal{H}_{\mathcal{A}}\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}), \qquad (\psi\otimes 1, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathcal{H}_{\mathcal{A}}) = (\phi', \mathcal{H}_{\mathcal{A}'}\otimes_{\mathcal{A}'}{}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$$

We write this situation as $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N}{\sim} (\phi', \mathcal{H}_{\mathcal{A}'})$.

We similarly define a shift equivalence between Hilbert C^*-symbolic bimodules by equipping with finite bases.

The above formulations of a strong shift equivalence and a shift equivalence of Hilbert C^*-bimodules are generalizations of those of nonnegative square matrices defined by Wiiliams (cf.[N],[Ma6]). The following proposition is parallel to Proposition 6.3. ([Wi], cf.[LM]).

Proposition 6.4. *Let $(\phi, \mathcal{H}_{\mathcal{A}})$, $(\phi', \mathcal{H}_{\mathcal{A}'})$ and $(\phi'', \mathcal{H}_{\mathcal{A}''})$ be Hilbert C^*-bimodules.*

(i) $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N}{\approx} (\phi', \mathcal{H}_{\mathcal{A}'})$ *implies* $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N}{\sim} (\phi', \mathcal{H}_{\mathcal{A}'})$.

(ii) $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N}{\sim} (\phi', \mathcal{H}_{\mathcal{A}'})$ *implies* $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N'}{\sim} (\phi', \mathcal{H}_{\mathcal{A}'})$ *for all* $N' \geq N$.

(iii) $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N}{\sim} (\phi', \mathcal{H}_{\mathcal{A}'})$ *and* $(\phi', \mathcal{H}_{\mathcal{A}'}) \underset{L}{\sim} (\phi'', \mathcal{H}_{\mathcal{A}''})$ *imply* $(\phi, \mathcal{H}_{\mathcal{A}}) \underset{N+L}{\sim} (\phi'', \mathcal{H}_{\mathcal{A}''})$.

The similar statements hold for Hilbert C^*-symbolic bimodules.

Therefore shift equivalence of Hilbert C^*-bimodules and similarly shift equivalence of Hilbert C^*-symbolic bimodules are equivalence relations.

Proposition 6.5. *If C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent in 1-step, their associated Hilbert C^*-symbolic bimodules $(\phi_\rho, \mathcal{H}^{\rho}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ and $(\phi_{\rho'}, \mathcal{H}^{\rho'}_{\mathcal{A}'}, \{u'_{\alpha'}\}_{\alpha'\in\Sigma'})$ are strong shift equivalent in 1-step.*

Its converse implication holds.

Proposition 6.6. *If Hilbert C^*-symbolic bimodules $(\phi, \mathcal{H}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ and $(\phi', \mathcal{H}'_{\mathcal{A}'}, \{u'_{\alpha'}\}_{\alpha'\in\Sigma'})$ are strong shift equivalent in 1-step, their associated C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent in 1-step.*

We may similarly see that two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are shift equivalent of lag N if and only if their associated Hilbert C^*-symbolic bimodules $(\phi_\rho, \mathcal{H}^{\rho}_{\mathcal{A}}, \{u_\alpha\}_{\alpha\in\Sigma})$ and $(\phi_{\rho'}, \mathcal{H}^{\rho'}_{\mathcal{A}'}, \{u'_{\alpha'}\}_{\alpha'\in\Sigma'})$ are shift equivalent of lag N.

7. Strong shift equivalence of gauge actions

In this section we introduce the notion of strong shift equivalence of C^*-symbolic crossed products with gauge actions.

Definition. Two C^*-symbolic crossed products $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho'}, \mathbb{T})$ with gauge actions are said to be strong shift equivalent in 1-step if there exists a C^*-symbolic dynamical system $(\mathcal{A}_0, \rho_0, \Sigma_0)$ and full projections $p, p' \in \mathcal{A}_0 \rtimes_{\rho_0} \Lambda_0$ satisfying $p + p' = 1$ and $\hat{\rho_0}_z(p) = p, \hat{\rho_0}_z(p') = p'$ for $z \in \mathbb{T}$ where Λ_0 is the subshift associated with $(\mathcal{A}_0, \rho_0, \Sigma_0)$, and

$$
\begin{aligned}
(p(\mathcal{A}_0 \rtimes_{\rho_0} \Lambda_0)p, \hat{\rho_0}, \mathbb{T}) &= (\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}^2, \mathbb{T}), \\
(p'(\mathcal{A}_0 \rtimes_{\rho_0} \Lambda_0)p', \hat{\rho_0}, \mathbb{T}) &= (\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho'}^2, \mathbb{T}).
\end{aligned}
$$

We write this situation as $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T}) \underset{1}{\approx} (\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho'}, \mathbb{T})$. If there exists an N-chain of strong shift equivalences in 1-step, they are said to be strong shift equivalent in N-step and written as $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T}) \underset{N}{\approx} (\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho'}, \mathbb{T})$. It is simply said to be strong shift equivalent.

Theorem 7.1. *Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two C^*-symbolic dynamical systems whose associated subshifts are denoted by Λ and Λ' respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, the C^*-symbolic crossed products $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho'}, \mathbb{T})$ with gauge actions are strong shift equivalent.*

This theorem and its proof are generalizations of [Ma4:Theorem 3.15].

Suppose that $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent in 1-step. There exist finite sets C and D, two families of homomorphisms $\eta_c : \mathcal{A} \to \mathcal{A}', c \in C$ and $\zeta_d : \mathcal{A}' \to \mathcal{A}, d \in D$ and two into bijections $\kappa : \Sigma \to CD$ and $\kappa' : \Sigma' \to DC$ that give rise to the strong shift equivalence between $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$. Let $(\widetilde{\mathcal{A}}, \tilde{\rho}, \widetilde{\Sigma})$ be the bipartite C^*-symbolic dynamical system related to $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$. As $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A}'$, we identify $\mathcal{A}$ and $\mathcal{A}'$ with the subalgebras of $\widetilde{\mathcal{A}}$ by regarding $a \in \mathcal{A}$ as $(a, 0) \in \widetilde{\mathcal{A}}$ and $a' \in \mathcal{A}'$ as $(0, a') \in \widetilde{\mathcal{A}}$ respectively. The symbolic crossed product

$$\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda} = C^*(S_{\tilde{\alpha}}, x \mid \tilde{\alpha} \in \widetilde{\Sigma}, x \in \widetilde{\mathcal{A}})$$

of $(\widetilde{\mathcal{A}}, \tilde{\rho}, \widetilde{\Sigma})$ is the universal C^*-algebra generated by partial isometries $S_{\tilde{\alpha}}, \tilde{\alpha} \in \widetilde{\Sigma} = C \sqcup D$ and elements $x \in \widetilde{\mathcal{A}}$ that satisfy the relations (5.1). Let $C^*(S_{CD}, \mathcal{A})$ and $C^*(S_{DC}, \mathcal{A}')$ be the C^*-subalgebras of $\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}$ defined by setting

$$C^*(S_{CD}, \mathcal{A}) = C^*(S_{c_\alpha d_\alpha}, (a, 0) \mid c_\alpha d_\alpha = \kappa(\alpha), \alpha \in \Sigma, a \in \mathcal{A}) \quad \text{and}$$
$$C^*(S_{DC}, \mathcal{A}') = C^*(S_{d_{\alpha'} c_{\alpha'}}, (0, a') \mid d_{\alpha'} c_{\alpha'} = \kappa'(\alpha'), \alpha' \in \Sigma', a' \in \mathcal{A}')$$

respectively, where $S_{c_\alpha d_\alpha} = S_{c_\alpha} S_{d_\alpha}$ and $S_{d_{\alpha'} c_{\alpha'}} = S_{d_{\alpha'}} S_{c_{\alpha'}}$. Put the projections

$$P_C = \sum_{c \in C} S_c S_c^*, \qquad P_D = \sum_{d \in D} S_d S_d^* \qquad \text{in } \widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}.$$

Hence $P_C + P_D = 1$.

We see that the following propositions hold.

Proposition 7.2.

$$C^*(S_{CD}, \mathcal{A}) = P_C(\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}) P_C, \quad C^*(S_{DC}, \mathcal{A}') = P_D(\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}) P_D.$$

Proposition 7.3. *The C^*-symbolic crossed products $\mathcal{A} \rtimes_\rho \Lambda$ and $\mathcal{A}' \rtimes_{\rho'} \Lambda'$ are canonically isomorphic to the algebras $C^*(S_{CD}, \mathcal{A})$ and $C^*(S_{DC}, \mathcal{A}')$ respectively.*

The following lemma shows that the subalgebras $P_C(\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}) P_C$ and $P_D(\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}) P_D$ are complementary full corners in $\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}$.

Lemma 7.4. *The projections P_C, P_D are full in the algebra $\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}$.*

Proof of sketch of Theorem 7.1. By Proposition 7.2 and Proposition 7.3, we may identify the algebras $\mathcal{A} \rtimes_\rho \Lambda$ with $P_C(\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}) P_C$, and $\mathcal{A}' \rtimes_{\rho'} \Lambda'$ with $P_D(\widetilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \widetilde{\Lambda}) P_D$. By these identifications, one has

$$\hat{\rho}_z^2(s_\alpha) = \hat{\tilde{\rho}}_z(S_c S_d), \qquad \hat{\rho'}_z^2(s'_{\alpha'}) = \hat{\tilde{\rho}}_z(S_d S_c)$$

for $\kappa(\alpha) = cd \in CD, \kappa'(\alpha') = dc \in DC$. Thus $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ are strong shift equivalent in 1-step. $\square$

Remark. It is possible to generalize the above discussions such as strong shift equivalent Hilbert C^*-bimodules give rise to strong shift equivalent C^*-algebras of the Hilbert C^*-bimodules. We will discuss this generalization in a forth coming paper [Ma6].

We present the following theorem.

Theorem 7.5. *Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two C^*-symbolic dynamical systems whose associated subshifts are denoted by Λ and Λ' respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, then we have*

(i) *the subshifts Λ and Λ' are topologically conjugate,*
(ii) *the C^*-symbolic crossed products $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho'}, \mathbb{T})$ with gauge actions are strong shift equivalent, and*
(iii) *the stabilized gauge actions $(\mathcal{A} \rtimes_\rho \Lambda \otimes \mathcal{K}, \hat{\rho} \otimes \mathrm{id}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda' \otimes \mathcal{K}, \hat{\rho'} \otimes \mathrm{id}, \mathbb{T})$ are cocycle conjugate, where $\mathcal{K}$ denotes the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space.*

In the rest of this section, we will concern K-theory for the C^*-algebra $\mathcal{A} \rtimes_\rho \Lambda$ constructed from a C^*-dynamical system $(\mathcal{A}, \rho, \Sigma)$. The endomorphisms $\rho_\alpha : \mathcal{A} \to \mathcal{A}$ for $\alpha \in \Sigma$ yield endomorphisms $\rho_{\alpha *} : K_*(\mathcal{A}) \to K_*(\mathcal{A})$ for $\alpha \in \Sigma$ on the K-theory groups of $\mathcal{A}$. Define an endomorphism

$$\rho_* : K_*(\mathcal{A}) \to K_*(\mathcal{A}), \qquad * = 0, 1$$

by setting $\rho_*(g) = \sum_{\alpha \in \Sigma} \rho_{\alpha *}(g), g \in K_*(\mathcal{A})$. By [Pim] (cf. [KPW]), one has the following six term exact sequence of K-theory:

$$\begin{array}{ccccc}
K_0(\mathcal{A}) & \xrightarrow{id-\rho_*} & K_0(\mathcal{A}) & \xrightarrow{\iota_*} & K_0(\mathcal{A} \rtimes_\rho \Lambda) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{A} \rtimes_\rho \Lambda) & \xleftarrow[\iota_*]{} & K_1(\mathcal{A}) & \xleftarrow[id-\rho_*]{} & K_1(\mathcal{A}).
\end{array}$$

Hence if in particular $K_1(\mathcal{A}) = 0$, one has

$$K_0(\mathcal{A} \rtimes_\rho \Lambda) = K_0(\mathcal{A})/(\mathrm{id} - \rho_*)K_0(\mathcal{A}),$$
$$K_1(\mathcal{A} \rtimes_\rho \Lambda) = \mathrm{Ker}(\mathrm{id} - \rho_*) \text{ in } K_0(\mathcal{A}).$$

This formula is a generalization of K-theory formulae proved in [C2] and [Ma3]. As in [Ma3:Lemma 5.2], one sees that the fixed point algebra $\mathcal{F}_{(\mathcal{A},\rho,\Sigma)}$ of $\mathcal{A} \rtimes_\rho \Lambda$ under the gauge action $\hat{\rho}$ is stably isomorphic to $(\mathcal{A} \rtimes_\rho \Lambda) \rtimes_{\hat{\rho}} \mathbb{T}$. We define the K-groups $K_*(\mathcal{A}, \rho, \Sigma)$ and the dimension groups $D_*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting

$$K_*(\mathcal{A}, \rho, \Sigma) = K_*(\mathcal{A} \rtimes_\rho \Lambda)$$
$$D_*(\mathcal{A}, \rho, \Sigma) = (K_*(\mathcal{F}_{(\mathcal{A},\rho,\Sigma)}), \hat{\hat{\rho}}_*) \qquad * = 0, 1$$

where $\hat{\hat{\rho}}_*$ is the automorphism on the abelian group $K_*(\mathcal{F}_{(\mathcal{A},\rho,\Sigma)})$ induced by the dual action $\hat{\hat{\rho}}$ of the gauge action $\hat{\rho}$. We also define the Bowen-Franks groups $BF^*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting

$$BF^*(\mathcal{A}, \rho, \Sigma) = \mathrm{Ext}_*(\mathcal{A} \rtimes_\rho \Lambda), \qquad * = 0, 1$$

Then Theorem 7.5 (iii) implies

Proposition 7.6. *The abelian groups $K_*(\mathcal{A},\rho,\Sigma), BF^*(\mathcal{A},\rho,\Sigma)$ and the abelian group with automorphisms $D_*(\mathcal{A},\rho,\Sigma)$ for $(\mathcal{A},\rho,\Sigma)$ are invariant under strong shift equivalence of C^*-symbolic dynamical systems.*

The above results are generalization of [Ma4] see also [C2], [CK], [Ma2].

In [Ma8], dynamical property of a "subshift"

$$\mathcal{S}_{(A,\rho,\Sigma)} = \{(\rho_{\alpha_i})_{i\in\mathbb{Z}} \mid (\rho_{\alpha_i}\circ\cdots\circ\rho_{\alpha_{i+k}})(1)\neq 0, i\in\mathbb{Z}, k\in\mathbb{Z}_+\}$$

will be studied.

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