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On the Abhyankar's question for affine plane curves with one place at infinity

MITSUSHI FUJIMOTO
Department of Information Education
Fukuoka University of Education
fujimoto@fukuoka-edu.ac.jp

MASAKAZU SUZUKI and KAZUHIRO YOKOYAMA
Faculty of Mathematics
Kyushu University
{suzuki,yokoyama}@math.kyushu-u.ac.jp

1 Introduction

Let C be an irreducible algebraic curve in complex affine plane $\mathbb{C}^2$. We say that C has *one place at infinity*, if the closure of C intersects with the ∞-line in $\mathbb{P}^2$ at only one point P and C is locally irreducible at that point P.

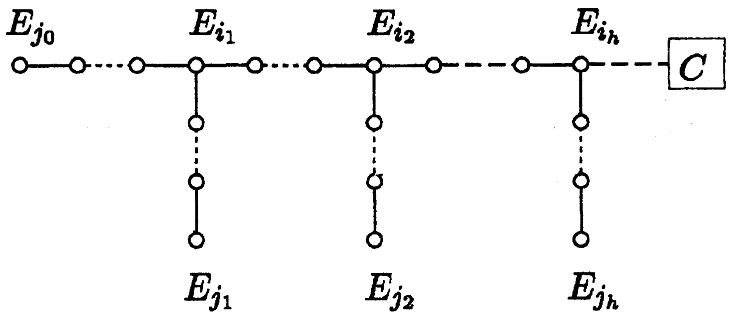
The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of $\mathbb{C}^2$ has been studied by many mathematicians since Suzuki [10] and Abhyankar–Moh [2] proved independently that the canonical model of C is a line when C is non-singular and simply connected.

Sathaye [8] introduce the Abhyankar's question for curves with one place at infinity and Sathaye–Stenerson [9] suggested a candidate of counter example for this question. However, they could not give the answer to the question since the root computation for a huge polynomial system was required.

We found a counter example for the Abhyankar's question using computer algebra system. In this report, we give the details.

2 Preliminaries

Let C be a curve with one place at infinity defined by a polynomial equation $f(x,y) = 0$ in the complex affine plane $\mathbb{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m,n)$. The dual graph corresponding to the minimal resolution of the singularity of C at infinity is the following [11]:

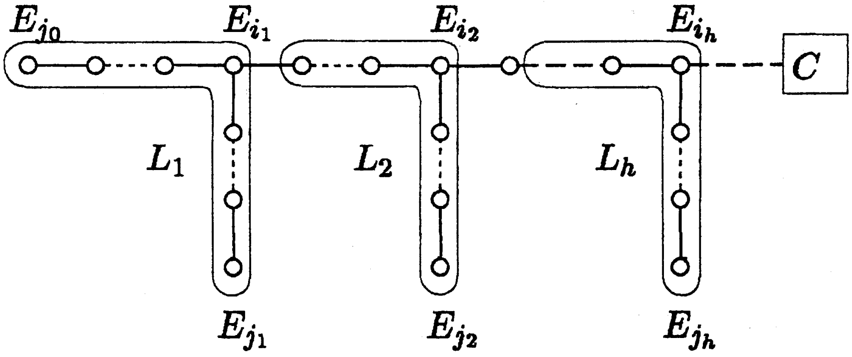


Definition 1 (δ-sequence) *Let f be the defining polynomial of a curve C with one place at infinity. Let δ_k $(0 \leq k \leq h)$ be the order of the pole of f on E_{j_k} in the above dual graph. We shall call the sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$ the δ-sequence of C (or of f).*

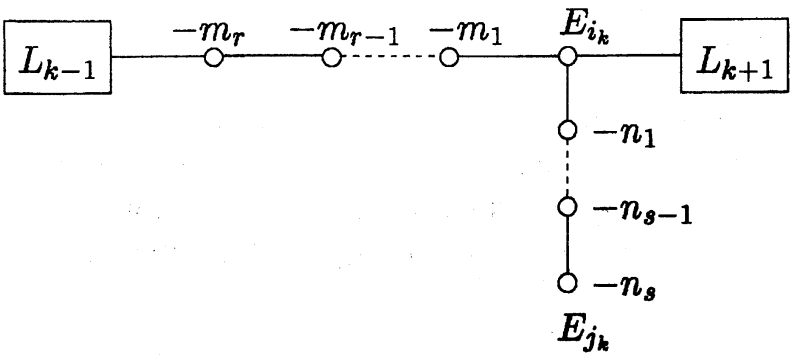
We have the following fact since $\deg_x f = m$ and $\deg_y f = n$.

Fact 1 $\delta_0 = n$, $\delta_1 = m$

We set L_k for each k $(1 \leq k \leq h)$ like the following figure:



Definition 2 ((p,q)-sequence) *Now, we assume that the weights of L_k is of the following form:*



We define the natural numbers p_k, a_k, q_k, b_k *satisfying*

$$(p_k, a_k) = 1,\ (q_k, b_k) = 1,\ 0 < a_k < p_k,\ 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots \; - \cfrac{1}{m_r}}} \quad \textit{and} \quad \frac{q_k}{b_k} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots \; - \cfrac{1}{n_s}}}.$$

We shall call the sequence $\{(p_1, q_1), (p_2, q_2), \ldots, (p_h, q_h)\}$ *the* (p, q)*-sequence of* C *(or of* f*).*

There are the following Abhyankar–Moh's semigroup theorem and its converse theorem by Sathaye–Stenerson as results for δ-sequence. We set $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Theorem 1 (Abhyankar-Moh [1, 3, 4]) *Let* C *be an affine plane curve with one place at infinity. Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *be the* δ*-sequence of* C *and* $\{(p_1, q_1), \ldots, (p_h, q_h)\}$ *be the* (p, q)*-sequence of* C. *We set* $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \leq k \leq h+1)$. *We have then,*

(i) $q_k = d_k / d_{k+1},\ d_{h+1} = 1\,(1 \leq k \leq h)$,

(ii) $d_{k+1} p_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \leq k \leq h) \end{cases}$,

(iii) $q_k \delta_k \in \mathbb{N}\delta_0 + \mathbb{N}\delta_1 + \cdots + \mathbb{N}\delta_{k-1}\ (1 \leq k \leq h)$.

Theorem 2 (Sathaye–Stenerson [9]) *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}\,(h \geq 1)$ *be the sequence of* $h{+}1$ *natural numbers. We set* $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \leq k \leq h+1)$ *and* $q_k = d_k / d_{k+1}\,(1 \leq k \leq h)$. *Furthermore, suppose that the following conditions are satisfied :*

(1) $\delta_0 < \delta_1$,
(2) $q_k \geq 2\,(1 \leq k \leq h)$,
(3) $d_{h+1} = 1$,
(4) $\delta_k < q_{k-1}\delta_{k-1}\,(2 \leq k \leq h)$,
(5) $q_k\delta_k \in \mathbb{N}\delta_0 + \mathbb{N}\delta_1 + \cdots + \mathbb{N}\delta_{k-1}\ (1 \leq k \leq h)$.

Then, there exists a curve with one place at infinity of the δ-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$.

Suzuki [11] gave an algebrico-geometric proof of the above two theorem by the consideration of the resolution graph at infinity. Further, Suzuki gave an algorithm for mutual conversion of a dual graph and a δ-sequence.

3 Construction of defining polynomials of curves

We shall assume that $f(x, y)$ is monic in y. We define approximate roots by Abhyankar's definition.

DEFINITION 3 (approximate roots) *Let $f(x,y)$ be the defining polynomial, monic in y, of a curve with one place at infinity. Let $\{\delta_0, \delta_1, \ldots, \delta_h\}$ be the δ-sequence of f. We set $n = \deg_y f$, $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ and $n_k = n/d_k\,(1 \leq k \leq h+1)$. Then, for each $k\,(1 \leq k \leq h+1)$, a pair of polynomials $(g_k(x,y), \psi_k(x,y))$ satisfying the following conditions is uniquely determined:*
(i) *g_k is monic in y and $\deg_y g_k = n_k$,*
(ii) *$\deg_y \psi_k < n - n_k$,*
(iii) *$f = g_k^{d_k} + \psi_k$.*
We call this g_k the k-th approximate root of f.

We can easily get the following fact from the definition of approximate roots.

Fact 2 *We have*

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_k x^k, \qquad g_{h+1} = f$$

where $c_k \in \mathbb{C}$, $p = \deg_x f/d$, $q = \deg_y f/d$, $d = \gcd\{\deg_x f, \deg_y f\}$ and $\lfloor p/q \rfloor$ is the maximal integer ℓ such that $\ell \leq p/q$.

DEFINITION 4 (Abhyankar-Moh's condition) *We shall call the conditions* (1) - (5) *concerning* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *in* Theorem 2 *Abhyankar-Moh's condition.*

The following theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

Theorem 3 ([5]) *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}\,(h \geq 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition (see* DEFINITION 4*). Set* $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}\,(1 \leq k \leq h+1)$ *and* $q_k = d_k/d_{k+1}\,(1 \leq k \leq h)$.
(1) *We define* $g_k\,(0 \leq k \leq h+1)$ *as follows:*

$$\left\{\begin{array}{rcl} g_0 & = & x, \\ g_1 & = & y + \displaystyle\sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbb{C},\ p = \delta_1/d_2,\ q = \delta_0/d_2, \\ g_{i+1} & = & g_i^{q_i} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{i-1}^{\bar{\alpha}_{i-1}} \\ & & + \displaystyle\sum_{(\alpha_0,\alpha_1,\cdots,\alpha_i)\in\Lambda_i} c_{\alpha_0\alpha_1\cdots\alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_i^{\alpha_i}, \\ & & a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} \in \mathbb{C}^*,\ c_{\alpha_0\alpha_1\cdots\alpha_i} \in \mathbb{C} \quad (1 \leq i \leq h), \end{array}\right.$$

where $(\bar{\alpha}_0, \bar{\alpha}_1, \cdots, \bar{\alpha}_{i-1})$ *is the sequence of* i *non-negative integers satisfying*

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i,\ \bar{\alpha}_j < q_j\,(0 < j < i)$$

and

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \cdots, \alpha_i) \in \mathbb{N}^{i+1} \;\middle|\; \alpha_j < q_j\,(0 < j < i), \alpha_i < q_i - 1, \sum_{j=0}^{i} \alpha_j \delta_j < q_i \delta_i \right\}.$$

Then, $g_0, g_1, \ldots, g_h$ *are approximate roots of* $f(= g_{h+1})$*, and* f *is the defining polynomial, monic in* y*, of a curve with one place at infinity of the* δ*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$.

(2) *The defining polynomial* f*, monic in* y*, of a curve with one place at infinity of the* δ*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *is obtained by the procedure of* (1)*, and the values of parameters* $\{a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}}\}_{1\leq i\leq h}$ *and* $\{c_{\alpha_0\alpha_1\cdots\alpha_i}\}_{0\leq i\leq h}$ *are uniquely determined for* f.

4 Abhyankar's Question

Definition 5 (planar semigroup) *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ $(h \geq 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *is said to be a planar semigroup.*

Definition 6 (polynomial curve) *Let* C *be an algebraic curve defined by* $f(x,y)=0$, *where* $f(x,y)$ *is an irreducible polynomial in* $\mathbb{C}[x,y]$. *We call* C *a polynomial curve, if* C *has a parametrisation* $x = x(t)$, $y = y(t)$, *where* $x(t)$ *and* $y(t)$ *are polynomials in* $\mathbb{C}[t]$.

Abhyankar's Question: Let Ω be a planar semigroup. Is there a polynomial curve with δ-sequence generating Ω ?

Moh [6] showed that there is no polynomial curve with δ-sequence $\{6,8,3\}$. But there is a polynomial curve $(x,y) = (t^3, t^8)$ with δ-sequence $\{3,8\}$ which generates the same semigroup as above. Sathaye–Stenerson [9] proved that the semigroup generated by $\{6,22,17\}$ has no other δ-sequence generating the same semigroup, and proposed the following conjecture for this question.

Sathaye–Stenerson's Conjecture: There is no polynomial curve having the δ-sequence $\{6,22,17\}$.

By Theorem 3, the defining polynomial of the curve with one place at infinity of the δ-sequence $\{6,22,17\}$ as follows:

$$\begin{aligned} f &= (g_2^2 + a_{2,1}x^2 g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\ &\quad + c_{1,1,0}x g_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0} \end{aligned}$$

where

$$\begin{aligned} g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 &= (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ &\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ &\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

Since C has one place at infinity and genus zero if and only if C has polynomial parametrization (Abhyankar), $\{6,22,17\}$ is a counter example if it can be shown that the above type curve does not include a polynomial curve.

5 Approach by using a computer algebra system

We assume that C is a polynomial curve and has the δ-sequence $\{6,22,17\}$. Therefore C has the following polynomial parametrization:

$$\begin{cases} x &= t^6+a_1t^5+a_2t^4+a_3t^3+a_4t^2+a_5t+a_6 \\ y &= t^{22}+b_1t^{21}+b_2t^{20}+b_3t^{19}+\cdots+b_{21}t+b_{22} \end{cases}$$

It follows that $\deg_t g_2(x(t),y(t)) = 17$ from the form of f and g_2 in the previous section. We can get the polynomial system I with 11 variables and 17 polynomials after eliminating variables from the coefficients of all terms of t-degree more than 18 in $g_2(x(t),y(t))$.

$\{6,22,17\}$ is a counter example of Abhyankar's question if I does not have a root. For such a huge polynomial system it is suitable to compute the Gröbner basis of the ideal. However, it was impossible to compute the Gröbner basis of I even if using a computer with 8GB memory.

We classified δ-sequences with genus ≤ 50 into groups which generate the same semigroup. Furthermore, we listed δ-sequences with the following three properties: (i) There is no other δ-sequence which generates the same semigroup. (ii) The number of generators is 3. (iii) k-number ≥ -1. Then, we obtained $\{6,15,4\},\{4,14,9\},\{6,15,7\},\{6,21,4\},\cdots$. The Gröbner basis computations for the polynomial systems corresponding to these δ-sequences showed that $\{6,21,4\}$ was a counter example of Abhyankar's question.

The defining polynomial of the curve with one place at infinity of the δ-sequence $\{6,21,4\}$ as follows:

$$f = g_2^3+a_{2,0}x^2+c_{1,0,1}xg_2+c_{1,0,0}x+c_{0,0,1}g_2+c_{0,0,0}$$

where

$$\begin{aligned} g_2 &= g_1^2+a_7x^7+c_{6,0}x^6+c_{5,0}x^5+c_{4,0}x^4+c_{3,0}x^3 \\ &\quad +c_{2,0}x^2+c_{1,0}x+c_{0,0} \\ g_1 &= y+c_3x^3+c_2x^2+c_1x+c_0 \end{aligned}$$

Let the following be the polynomial parametrization of the polynomial curve with δ-sequence $\{6,21,4\}$:

$$\begin{cases} x &= t^6+a_1t^5+a_2t^4+a_3t^3+a_4t^2+a_5t+a_6 \\ y &= t^{21}+b_1t^{20}+b_2t^{19}+b_3t^{18}+\cdots+b_{20}t+b_{21} \end{cases}$$

By the same operation as the case of $\{6,22,17\}$ we can get the polynomial system J with 7 variables $\{a_2,a_3,a_4,a_5,a_6,b_{12},b_{18}\}$ and 13 polynomials from $\deg_t g_2(x(t),y(t))=4$.

We used the total degree reverse lexicographic ordering (DRL) with $a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_6 \succ b_{12} \succ b_{18}$ to the Gröbner basis computation. CPU time for the computation is 3 hours 40 minutes and the required memory is 850MB. The computer is a PC AthlonMP 2200+ with 4GB memory. The computer algebra system is Risa/Asir [7] on FreeBSD 4.7.

The obtained Gröbner basis G of J was not $\{1\}$. However, the normal form of the coefficient p of the term with t-degree $=4$ in $g_2(x(t),y(t))$ with respect to G is 0. This shows that $p \in J$. Thus, we get $\deg_t g_2(x(t),y(t)) < 4$. Since this is contradictory for $\deg_t g_2(x(t),y(t))=4$, there is no polynomial curve with δ-sequence $\{6,21,4\}$.

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