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Fine Continuous Functions and Computable Analysis

京都産業大学 理学部　森 隆一

Takakazu Mori
Faculty of Science, Kyoto-Sangyo University

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1 Introduction

Computable Analysis is regarded widely as a reconstruction of analysis "effectively" by defining every notions effectively such as computable numbers, computable functions and effective integrability. Many approaches of this attempt have taken place. An example is Type II computability with representation, developed by Weihrauch ([13]). This approach is based on the coding theory. Another example is computable metric spaces with a computability structure, developed by Tsujii, Yasugi and Mori ([8], [15]) inheriting the preceding work of Pour-El and Richards ([9]). Roughly speaking, the latter approach simulates the usual analysis using effective convergence instead of usual convergence. Effective way means a method based on the recursion theory. So, we assume the basic knowledge of recursion theory such as recursive functions.

We start with the calculation of $\sqrt{2}$ for the sake of the illustration of computable numbers.

			1	4	1
1			2	00	
1			1		
2	4		1	00	
	4			96	
2	8	1		4	00
		1		2	81
2	8	2		1	19

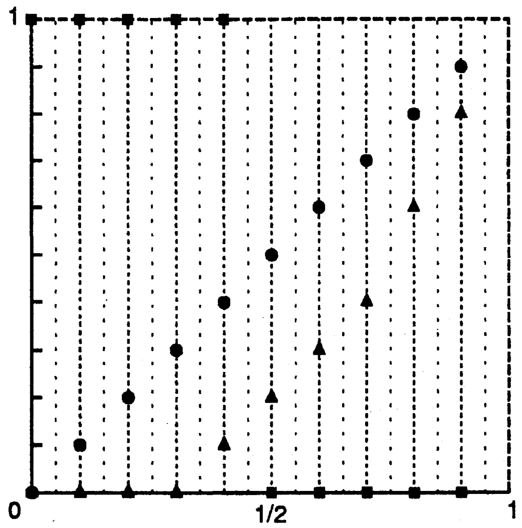
Calculation of $\sqrt{2}$:

To obtain an approximation of order 10^{-k}, it is sufficient to repeat $k+1$ steps. Each step of calculation consists of oprations $+, -, \times$ and finding the maximal integer which is less than some integer and satisfies some conditions.

We interpret the above situation as follows: There exists a sequence of rational numbers $r_1 = 1, r_2 = 1.4, r_3 = 1.41, \ldots$, which can be calculated inductively and satisfies $|r_n - \sqrt{2}| < 10^{-k}$ for $n \geqq k+1$. This property is generalized to the existence of recursive functions α, β, γ and δ such that $r_n = (-1)^{\alpha}\frac{\gamma(n)}{\beta(n)}$ satisfies $|r_n - x| < 10^{-k}$ for $n \geqq \delta(k)$.

We can also write such a program that calculate $\sqrt{x}$ for each input x through the decimal expansion. But the decimal expansion has the difficulty due to the existence of two expansions for rational x.

As a definition of computable functions, it is natural to suppose that a computable function is determined by its values for computable numbers. Continuity is a sufficient condition for this, since every binary rational number is computable. Herling ([5]) proved that computability involves some kind of continuity.

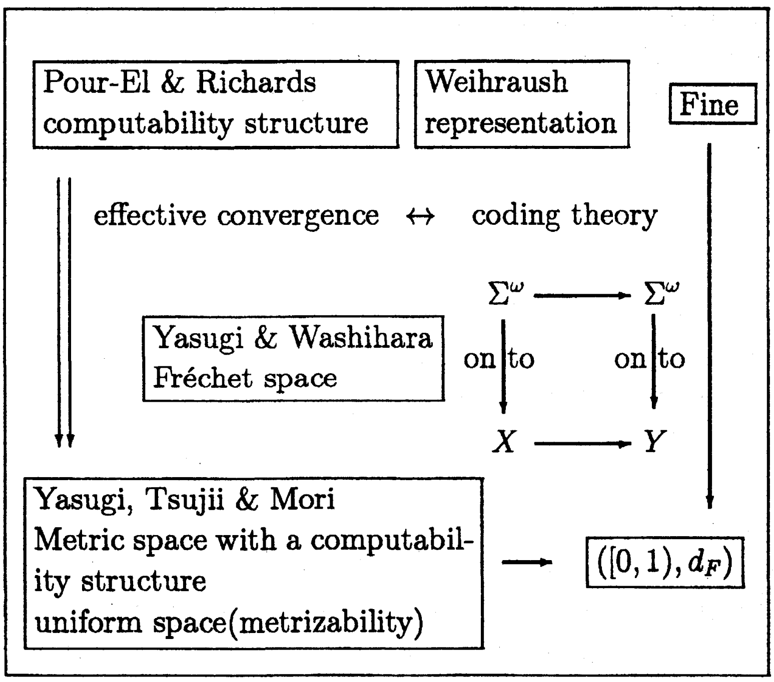


Graphs of simple functions on $[0,1)$:

Consider the graph of
$f(x) = x$ (circular dots),
$g(x) = x^2$ (triangular dots) and

$$h(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ 0 & \text{if } x \geqq \frac{1}{2} \end{cases} \quad \text{(square dots).}$$

Can we determine the value at $\frac{1}{2}$ from values of the function of $(\frac{1}{2}-\frac{1}{n}, \frac{1}{2}+\frac{1}{n})\setminus\{\frac{1}{2}\}$? We can do this for f and g. Moreover, the variations of them become arbitrarily small when we take n sufficiently large.
The above decision cannot be possible for h. But it goes well if we take $[\frac{1}{2}, \frac{1}{2}+\frac{1}{n})$ as a neighborhood.



For the values of functions on the set of non-computable numbers, two way of treatment can be considered: (A) They are out of consideration, (B) they are approximated effectively. Historically, (A) may be major. We employ the stand point (B).
The left graph illustrates the flow chart of Computable Analysis near the scope of this article.
Fine's work is not included in the field of Computable Analysis.

Practically, step functions are important and seems to be natural in some case. It may be usual that cost for some service, such as traffic cost, is determined by a piecewise constant right continuous increasing function. In the digital processing, the Walsh functions and the Haar wavelet become important tools. These functions are continuous w.r.t. the Fine metric d_F, although they are discontinuous w.r.t. the Euclidean metric d_E.

In this article, we summarize the computable analysis on $([0,1), d_F)$. Although, most definitions and fundamental properties can be carried over to separable metric spaces.

2 Fine metric

The Fine metric on $[0,1)$ is defined as follows ([3], [11]): Let $\Omega := \{0,1\}^\omega$ and Ω_0 be the set of all elements in Ω with infinitely many zeros. For $\sigma = (\sigma_0, \sigma_1, \dots) \in \Omega$, we define the function λ by

$$\lambda(\sigma) := \sum_{k=0}^{\infty} \sigma_k 2^{-(k+1)}. \tag{1}$$

For $\sigma, \tau \in \Omega$, the dyadic addition $\sigma \oplus \tau$ is defined by $(\sigma \oplus \tau)_k := |\sigma_k - \tau_k| = \sigma_k \,\mathsf{XOR}\, \tau_k$. We denote the restriction of λ to Ω_0 by ρ_F. It holds that $\lambda(\Omega) = [0,1]$, $\rho_F(\Omega_0) = [0,1)$ and ρ_F is one-to-one. So we can define the inverse of ρ_F and we denote it by $\mu(x)$. This $\mu(x)$ is merely the binary expansion of $x \in [0,1)$ under the restriction that it has infinitely many zeros. ρ_F turns out to be an admissible representation of $([0,1), d_F)$ ([1]). We also define $x \oplus y := \lambda(\mu(x) \oplus \mu(y))$.

Let d_C be the Cantor metric on Ω, i.e.

$$d_C(\sigma, \tau) := \sum_{k=0}^{\infty} |\sigma_k - \tau_k| 2^{-(k+1)} = \lambda(\sigma \oplus \tau), \tag{2}$$

it satisfies following properties.

Lemma 2.1 (i) *If $d_C(\sigma,\tau) < 2^{-k}$, then $\sigma(\ell) = \tau(\ell)$ for $\ell < k$.*
(ii) *If $\sigma(\ell) = \tau(\ell)$ for $\ell < k$, then $d_C(\sigma,\tau) \leqq 2^{-k}$.*

$(\Omega, d_C, \oplus)$ is a compact abelian group. It is proved by Fine that Walsh functions are characters of this group ([3]).

The Fine-metric on $[0,1)$ is defined to be the induced metric from d_C by μ, namely

$$d_F(x,y) := d_C(\mu(x), \mu(y)) = \lambda(\mu(x) \oplus \mu(y)). \tag{3}$$

$d_F(x,y)$ can be thought as the sum of weighted difference between the corresponding bits of $\mu(x)$ and $\mu(y)$. If r is a binary rational, then the inequality $d_F(r,x) < 2^{-(n+1)}$ is equivalent to the coincidence of the first n bits of $\mu(r)$ and $\mu(x)$.

To specify the topological properties w.r.t. the Fine-metric, we attach Fine- at the front of these terminologies. For example, convergence w.r.t. the Fine-metric is called Fine-convergence.

It holds that $d_E(x,y) \leqq d_F(x,y)$, where d_E is the usual Euclidean metric. Therefore, d_F is stronger than d_E. The metric space $([0,1), d_F)$ is totally bounded. But it is not complete, since for a positive binary rational r, the sequence $\{r-2^{-n}\}$ is a Fine-Cauchy sequence but does not Fine-converge. While, it converges to r w.r.t. d_E. In this case, if $\mu(r) = (*,\dots,*,1,0,0,\dots)$ then $\mu(r-2^{-n}) = ((*,\dots,*,0,1,\dots,1,0,0,\dots)$ and it converges to $(*,\dots,*,0,1,1,\dots)$ in (Ω, d_C). This example also shows the existence of a sequence which converges w.r.t. d_E but does not Fine-converge. (Ω, d_C) can be regarded as a completion of $([0,1), d_F)$. The distinct property of the Fine-metric is that a left-closed right-open interval $[r,s)$ is closed and open if r and s are binary rationals. $[r, r+2^{-n})$ is equal to the open ball $\{x \mid d_F(r,x) < 2^{-n})\}$ if r is a binary rational and n is sufficiently large. This property corresponds to prohibition of left convergence to binary rationals and make some discontinuous functions Fine-continuous. It also gives rise to the existence of an open disjoint covering of $[0,1)$.

Definition 2.1 *A sequence of binary rationals in $[0,1)$ is said to be a recursive sequence of binary rationals if there exist recursive functions $\alpha(n)$ and $\beta(n)$, which satisfy*

$$r_n = \frac{\beta(n)}{2^{\alpha(n)}} \ (0 \leqq \beta(n) < 2^{\alpha(n)}).$$

Double sequences $\{x_{m,n}\}$ and $\{f_{m,n}\}$ mean that there exist $\{y_n\}$ and $\{g_n\}$ and a pairing function $\langle m,n\rangle$, which satisfy $x_{m,n} = y_{\langle m,n\rangle}$ and $f_{m,n} = g_{\langle m,n\rangle}$ respectively. We mainly take that $\langle m,n\rangle := \frac{1}{2}(m+n)(m+n+1)+n$. For a triple sequence, we take $\langle m,n,k\rangle = \langle\langle m,n\rangle, k\rangle$.

Definition 2.2 *$\{x_{m,n}\}$ is said to Fine-converge effectively to $\{x_m\}$ if there exists a recursive function $\alpha(m,k)$ such that $n \geqq \alpha(m,k)$ implies $d_F(x_{m,n}, x_m) < 2^{-k}$.*

Definition 2.3 *A sequence $\{x_n\}$ is said to be Fine-computable if there exists a recursive sequence of binary rationals $\{r_{n,k}\}$, which Fine-converges effectively to $\{x_n\}$.*

Remark 2.1 (i) x is called Fine-computable if $\{x, x, \dots\}$ is Fine-computable.

(ii) Every binary rational number is Fine-computable.

(iii) The set of Fine-computable numbers is countable.

(iv) A recursive sequence of binary rationals is computable. But the converse does not hold.

The usual computability of real numbers is defined similarly w.r.t. d_E instead of d_F.

Proposition 2.1 (Brattka [1])

(i) *A real number is computable if and only if it is Fine-computable.*

(ii) *A Fine-computable sequence of reals is a computable sequence.*

(iii) *There exists a computable sequence of reals which is not Fine-computable.*

The set of all Fine-computable sequences $\mathcal{S}_F$ satisfies the following axioms of a computability structure.

Definition 2.4 (Metric Space with a Computability Structure) *$(X, d, \mathcal{S})$ is said to be a metric space with a computability structure if the following three axioms are satisfied.*

Axiom M1 (Metrics) *If $\{x_m\}, \{y_n\} \in \mathcal{S}$, then $\{d(x_m, y_n)\}$ forms a double sequence of computable reals.*

Axiom M2 (Re-Enumerations) *If $\{x_n\} \in \mathcal{S}$, then $\{x_{\alpha(n)}\} \in \mathcal{S}$ for any recursive function α.*

Axiom M3 (Limits) *If $\{x_{m,n}\} \in \mathcal{S}$, $\{x_m\} \subseteq X$ and $\{x_{m,n}\}$ converges effectively to $\{x_m\}$, then $\{x_m\} \in \mathcal{S}$.*

Definition 2.5 *If there exists a computable sequence $\{e_n\}$ which is dense in X, then we say that $(X, d, \mathcal{S})$ is effectively separable and we call $\{e_n\}$ an effective separating set.*

As the definition of computability of a sequence in Ω, we have the following.

Definition 2.6 *If there exists a recursive function $\alpha(n, \ell)$ which satisfies $\sigma_n(\ell) = \alpha(n, \ell)$, then we call $\{\sigma_n\}$ a computable sequence.*

Let $\mathcal{S}_\Omega$ be the set of all computable $\{\sigma_n\}$, then $\mathcal{S}_\Omega$ is a computability structure on Ω w.r.t. d_C.

3 Fine-continuous functions

In this section, we overview the classical results of Fine-continuous functions. We write the set of all binary rationals in $[0, 1)$ as $\mathbb{Q}_2$. The effective enumeration $\{e_i\}$ of $\mathbb{Q}_2$ is an effective separating set of $[0, 1)$. In the rest of this article, we take this $\{e_i\}$ as an effective separating set.

Definition 3.1 *A function on Ω is called a cylinder function if it depends only finite bits, i.e. there exists an integer n such that $\sigma(\ell) = \tau(\ell)$ for $\ell < n$ imply $f(\sigma) = f(\tau)$.*

Since (Ω, d_C) is compact, every continuous function is uniformly continuous. If we denote the set of all continuous functions on Ω as $\mathcal{C}_\Omega$, then $\mathcal{C}_\Omega$ is a Banach space with the maximum norm. We can also prove the following proposition.

Proposition 3.1 *A function on (Ω, d_C) is continuous if and only if it is approximated uniformly by a sequence of cylinder functions.*

Definition 3.2 *A left closed right open interval $[a, b)$ is called a dyadic interval if $a, b \in \mathbb{Q}_2$.*

We write the open ball $\{x \mid d_F(x,a) < r\}$ as $\mathbb{B}_F(a,r)$. In general, it is not an interval. For example, $\mathbb{B}_F(\frac{1}{2},\frac{3}{4}) = [0,\frac{1}{4}) \cup [\frac{1}{2},1)$. In order that $\mathbb{B}_F(a,r)$ becomes an interval, we adopt the following constraint.

Constraint: *If a is binary rational, say $j2^{-k}$, where j is odd or $j = k = 0$, then $r = 2^{-\ell}$ for some $\ell \geqq k$.*

Let $r \in \mathbb{Q}_2$ and $\mu(r) = \sigma$, then there exists an integer k such that $\sigma(k) = 1$ and $\sigma(\ell) = 0$ for $\ell \geqq k$. We denote this k as $L(r)$. In this case, if $\ell \geqq L(r)$ then $\mathbb{B}_F(r, 2^{-\ell}) = [r, r + 2^{-\ell}) = [r, r \oplus 2^{-\ell})$.

Proposition 3.2 *A function f on $[0,1)$ is Fine-continuous if and imply if it satisfies*
(i) *continuous at $x \in Q_2^C$, and*
(ii) *right continuous at $x \in Q_2$.*

We denote $\mathcal{D}$ the set of uniformly Fine-continuous functions.

Proposition 3.3 *The following three are equivalent.*
(i) $f \in \mathcal{D}$.
(ii) *f is Fine-continuous and has a left limit at $x \in Q_2$.*
(iii) *There exists a continuous function on Ω such that $f(x) = g(\mu(x))$ for $x \in [0,1)$.*

In this case, $\sup_{x\in[0,1)} |f(x)| = \sup_{\sigma\in\Omega} |g(\sigma)|$ and $\mathcal{D}$ is a Banach space with sup-norm. We denote the set of all Fine-continuous functions on $[0,1)$ as $\mathcal{C}$. For the convergence of a sequence $\{f_n\}$ to f in $\mathcal{C}$, we have obtained pointwise convergence and uniform convergence. These convergences depend scarcely on the topological structure on X.

We can define another convergence and continuity.

Definition 3.3 *f is said to locally uniformly Fine-continuous if there exists a positive integer valued function $\gamma(i)$ such that f is uniformly continuous on $\mathbb{B}_F(e_i, 2^{-\gamma(i)})$ and $\bigcup_{i=0}^{\infty} \mathbb{B}_F(e_i, 2^{-\gamma(i)}) = [0,1)$.*

Definition 3.4 *$\{f_n\}$ is said to Fine-converge locally uniformly to f if there exists a positive integer valued function $\gamma(i)$ such that $\{f_n\}$ converges uniformly to f on $\mathbb{B}_F(e_i, 2^{-\gamma(i)})$ and $\bigcup_{i=0}^{\infty} \mathbb{B}_F(e_i, 2^{-\gamma(i)}) = [0,1)$.*

Schröder ([10]) considers the following continuous convergence.

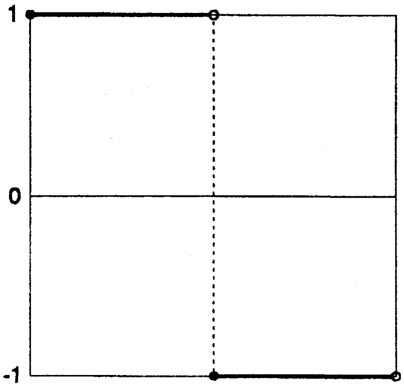
Definition 3.5 *We say that $\{f_n\}$ continuously Fine-converges to f if $\{f_n(x_n)\}$ converges to $f(x)$ for any sequence $\{x_n\}$ which Fine-converges to x.*

The following convergence is stronger than continuous Fine-convergence: For each e_i and $\epsilon > 0$, we can take $N = N(i,\epsilon)$ and $a = a(i,\epsilon)$ such that $n \geqq N$ and $x \in \mathbb{B}_F(e_i, a)$ imply $|f_n(x) - f(x)| < \epsilon$.

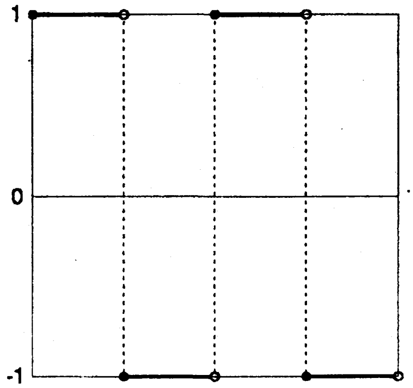
Example 3.1 Let $\mu(x) = (\sigma_0, \sigma_1, \dots)$ and the binary expansion of an integer n be $n_0 + 2n_1 + \cdots + 2^k n_k$ $(n_k \neq 0)$. Then, Walsh functions are defined by $w_n(x) = (-1)^{\sum_{i=0}^{k} \sigma_i n_i}$.

It is easy to see that they are obviously uniformly Fine-continuous but not continuous w.r.t. d_E. It also holds that $w_3(x) = w_1(x)w_2(x)$ and $w_5(x) = w_1(x)w_4(x)$.

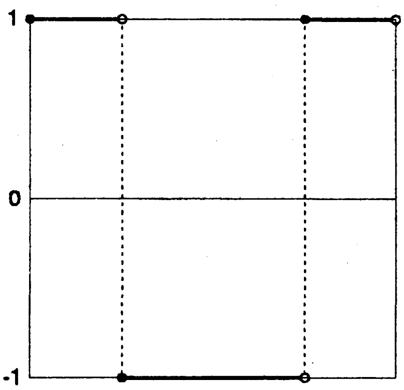
The graphs of them are followings.



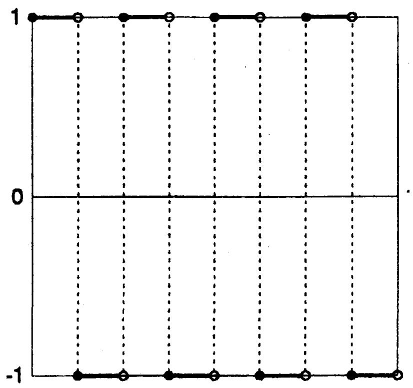
$w_1(x)$



$w_2(x)$



$w_3(x)$



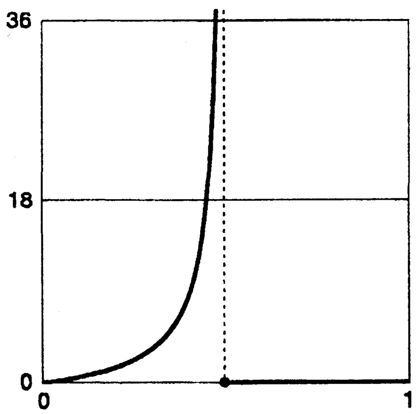
$w_4(x)$

Example 3.2

f defined by

$$f(x) = \begin{cases} \frac{1}{1-2x} - 1 & \text{if } x < \frac{1}{2} \\ 0 & \text{if } x \geqq \frac{1}{2} \end{cases}$$

is locally uniformly Fine-continuous but not uniformly Fine-continuous.



4 Fine-computable functions

In the formulation of Pour-El and Richards, computability of a sequence of functions $\{f_n\}$ is defined by the two properties; (i) Sequential Computability, which requires that $\{f_n(x_m)\}$ is computable for any computable $\{x_m\}$, and (ii) Effective Uniform Continuity. A function f is said to be uniformly computable if $\{f, f, \dots\}$ is computable.

For a function on $([0,1), d_F)$, we have defined three kinds of continuity. So, we can obtain three kinds of computability and corresponding effective convergence.

4.1 Computable functions on Ω

Definition 4.1 *A sequence of functions $\{f_n\}$ on Ω is said to be computable if it satisfies*
(i) *Sequential Computability,*
(ii) *Effective Uniform Continuity, that is, there exist a recursive functions $\alpha(n,k)$ such that $d_C(\sigma,\tau) < 2^{-\alpha(n,k)}$ implies $|f_n(\sigma) - f_n(\tau)| < 2^{-k}$*

We call this α as the modulus of continuity of $\{f_n\}$.

Definition 4.2 *We say that $\{f_n\}$ converges effectively uniformly to f if there exists a recursive function $\beta(k)$ such that $n \geqq \beta(k)$ implies $|f_n(\sigma) - f(\sigma)| < 2^{-k}$ for $\forall\sigma$*

Theorem 4.1 *An effective uniform limit of a computable sequence is computable.*

We also need effective uniform convergence of $\{f_{mn}\}$ to $\{f_m\}$. This is obtained by changing $\beta(k)$ to $\beta(m,k)$.

Proposition 4.1 *A cylinder function which takes only computable values is uniformly computable.*

For $0 \leqq j = j_{n-1}2^{n-1} + \cdots + 2j_1 + j_0 < 2^n$, let $\Gamma_{n,j} = \{\sigma \mid \sigma(\ell) = j_{n-\ell}, 0 \leqq \ell \leqq n-1\}$.

Definition 4.3 *A sequence $\{f_n\}$ is called a computable sequence of cylinder functions if there exist recursive functions $\alpha(n)$ and $\{c_{n,j}\}$ such that $f_n(\sigma) = c_{n,j}$ for $\sigma \in \Gamma_{\alpha(n),j}$.*

Theorem 4.2 *f is computable if and only if there exists a computable sequence of cylinder functions which converges effectively uniformly to f.*

We can take the following approximating sequence of cylinder functions.

$$f_n(\sigma) = f((\sigma(0), \sigma(1), \dots, \sigma(n-1), 0, 0, \dots))$$

4.2 Uniformly Fine-computable functions on $[0,1)$

Definition 4.4 *A sequence of functions $\{f_n\}$ is said to be uniformly Fine-computable if it satisfies*

(i) *Sequential Computability, that is, if $\{x_m\}$ is a Fine-computable sequence of reals, then $\{f_n(x_m)\}$ is a computable sequence of reals.*

(ii) *Effectively Uniformly Fine-Continuity, that is, there exist a recursive functions $\alpha(n,k)$ such that $d_F(x,y) < 2^{-\alpha(n,k)}$ implies $|f_n(x) - f_n(y)| < 2^{-k}$*

Definition 4.5 *We say that $\{f_n\}$ Fine-converges effectively uniformly to f if there exists a recursive function $\beta(k)$ such that $n \geqq \beta(k)$ implies $|f_n(x) - f(x)| < 2^{-k}$ for $\forall x$*

Theorem 4.3 *An effective uniform Fine-limit of an uniformly Fine-computable sequence is uniformly Fine-computable.*

Definition 4.6 *$\{f_n\}$ is called a computable sequence of binary step functions if there exists a recursive function $\alpha(n)$ and a computable sequence of reals $\{c_{n,j}\}$ such that*

$$f_n(x) = c_{n,j} \quad \textit{if} \quad j2^{-n} \leqq x < (j+1)2^{-n}.$$

Theorem 4.4 *The followings are equivalent.*

(i) *f is uniformly Fine-computable.*

(ii) *$f(x) = g(\mu(x))$ for some computable function g on Ω.*

(iii) *There exists a computable sequence of binary step functions which Fine-converges effectively uniformly to f.*

We can apply the results by Yasugi, Tsujii and Mori.

Corollary 4.1 *If $\{f_n\}$ is uniformly Fine-computable, then $\sup_{x\in[0,1)} |f_n(x)|$ is a computable sequence of reals.*

Example 4.1 *We can take the following approximating sequence.*

$$f_n(x) = f(i2^{-n}) \quad \textit{if} \quad i2^{-n} \leqq x < (i+1)2^{-n} \tag{4}$$

4.3 Locally uniformly Fine-computable functions on $[0,1)$

Definition 4.7 *$\{f_n\}$ is said to be locally uniformly Fine-computable if it satisfies*

(i) *sequentially computable*

(ii) *effectively locally uniformly Fine-computable, that is, there exist recursive functions $\gamma(n,i)$ and $\alpha(n,i,k)$, such that*

(ii-a) *$x, y \in \mathbb{B}(e_i, 2^{-\gamma(n,i)})$ and $d_F(x,y) < 2^{-\alpha(n,i,k)}$ imply $|f_n(x) - f_n(y)| < 2^{-k}$,*

(ii-b) $\bigcup_{i=1}^{\infty} \mathbb{B}(e_i, 2^{-\gamma(n,i)}) = [0,1)$.

Definition 4.8 *$\{f_n\}$ Fine-converges effectively locally uniformly to f if there exist recursive functions $\gamma(i)$ and $\beta(i,k)$ such that*

(a) *$x \in \mathbb{B}(e_i, 2^{-\gamma(i)})$ and $n \geqq \beta(i,k)$ imply $|f_n(x) - f(x)| < 2^{-k}$,*

(b) $\bigcup_{i=1}^{\infty} \mathbb{B}(e_i, 2^{-\gamma(i)}) = [0,1)$.

The function of Example (3.2) is locally uniformly Fine-computable but not uniformly Fine-computable.

Definition 4.9 *An effectively locally uniformly Fine-continuous sequence of functions w.r.t. $\gamma(n,i) = \gamma(i)$ is said to be effectively locally uniformly qui-Fine-continuous.*

Definition 4.10 *$\{f_n\}$ is said to be effectively locally uniformly asymptotically qui-Fine-continuous if there exists recursive functions $\gamma(i)$, $\delta(i,k)$ and $\alpha(n,i,k)$ such that*

(a) *$x, y \in \mathbb{B}(e_i, 2^{-\gamma(i)})$, $n \geqq \delta(i,k)$, $d_F(x,y) < 2^{-\alpha(n,i,k)}$ imply $|f_n(x) - f_n(y)| < 2^{-k}$,*

(b) $\bigcup_{i=1}^{\infty} \mathbb{B}(e_i, 2^{-\gamma(i)}) = [0,1)$.

Evidently, an effectively locally uniformly qui-Fine-continuous sequence is effectively locally uniformly Fine-continuous. The sequence of functions $\{f_n\}$ defined by the Equation (4) is effectively locally uniformly qui-Fine-continuous and Fine-converges effectively locally uniformly to f if f is locally uniformly Fine-computable.

Proposition 4.2 *Let $\{f_n\}$ Fine-converges effectively locally uniformly to f w.r.t. recursive functions $\gamma(i)$and $\beta(i,k)$, the the followings are equivalent.*

(i) *f is effectively locally uniformly continuous w.r.t. $\gamma(i)$ and a recursive function $\alpha(i,k)$.*

(ii) *$\{f_n\}$ is effectively locally uniformly asymptotically qui-Fine-continuous w.r.t $\gamma(i)$, recursive functions $\delta(i,k)$ and $\tilde{\alpha}(i,k)$.*

Theorem 4.5 *If a locally uniformly Fine-computable sequence Fine-converges effectively locally uniformly, then the limit function is locally uniformly Fine-computable.*

Theorem 4.6 *f is locally uniformly Fine-computable if and only if there exists a computable sequence of binary step functions which Fine-converges effectively locally uniformly to f.*

4.4 Fine-computable functions on $[0,1)$

Definition 4.11 *$\{f_n\}$ is said to be Fine-computable if satisfies*

(i) *sequentially computable,*

(ii) *there exists a recursive function $\gamma(n,i,k)$ such that*

(ii-a) *$x \in \mathbb{B}(e_i, 2^{-\gamma(n,i,k)})$ implies $|f_n(e_i) - f_n(x)| < 2^{-k}$,* (ii-b) $\bigcup_{i=1}^{\infty} \mathbb{B}(e_i, 2^{-\gamma(n,i,k)}) = [0,1)$ *for* $\forall k$.

(ii-a) can be replaced by $x, y \in \mathbb{B}(e_i, 2^{-\gamma(n,i,k)})$ implies $|f_n(x) - f_n(y)| < 2^{-k}$.

Definition 4.12 *$\{f_n\}$ Fine-converges effectively to f if there exist recursive functions $\gamma(i,k)$ and $\beta(i,k)$ such that*

(a) $x \in \mathbb{B}(e_i, 2^{-\gamma(i,k)})$ *and* $n \geqq \beta(i,k)$ *imply* $|f_n(x) - f(x)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} \mathbb{B}(e_i, 2^{-\gamma(i,k)}) = [0,1)$ *for* $\forall k$.

Proposition 4.3 (Brattka) *There exists a Fine-computable function, which is not locally uniformly Fine-computable.*

Definition 4.13 *An effectively Fine-continuous sequence w.r.t $\gamma(n,i,k) = \gamma(i,k)$ is called effectively equi-Fine-continuous.*

Definition 4.14 *$\{f_n\}$ is said to be effectively asymptotically equi-Fine-continuous if there exist recursive functions $\gamma(i,k)$ and $\delta(i,k)$ such that*

(a) $x, y \in \mathbb{B}(e_i, 2^{-\gamma(i,k)})$ *and* $n \geqq \delta(i,k)$ *imply* $|f_n(x) - f_n(y)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} \mathbb{B}(e_i, 2^{-\gamma(i,k)}) = [0,1)$.

The sequence of functions $\{f_n\}$ defined by Equation (4) is effectively equi-Fine-continuous and Fine-converges effectively to f, if f is Fine-computable.

Proposition 4.4 *Let $\{f_n\}$ Fine-converges effectively to f w.r.t. recursive functions $\gamma(i,k)$ and $\beta(i,k)$, then the followings hold.*

(i) *If f is effectively Fine-continuous w.r.t. $\gamma(i,k)$, then $\{f_n\}$ is effectively asymptotically equi-Fine-continuous.*

(ii) *if $\{f_n\}$ is effectively asymptotically equi-Fine-continuous w.r.t $\gamma(i,k)$ and $\delta(,k)$, then f is effectively Fine-continuous.*

4.5 Examples and Remarks

Definition 4.15 *A subset of $\mathbb{N}$ is called recursively enumerable if it is the range of some recursive function.*

Many examples make use of the following Proposition.

Proposition 4.5 *There exists a recursive enumerable set, which is not recursive.*

If the set is infinite, then we can take a 1:1 recursive function.

Let $\alpha(n)$ be a 1:1 recursive function. Suppose further that $\alpha(\mathbb{N})$ is not recursive.

Define $c_k = \sum_{\ell=0}^{k-1} e^{-\alpha(\ell)}$ and $c = \sum_{\ell=0}^{\infty} e^{-\alpha(\ell)}$. Then, c is not computable and $\{c_n\}$ converges to c, but the convergence is not effective.

Example 4.2 $f(x) = c_k \quad 1 - 2^{-k} \leqq x < 1 - 2^{-(k+1)}$
is locally uniformly Fine-computable but $\sup |f| = c$ is not computable.

Example 4.3 The following $\{\varphi_n\}$ is a computable sequence of binary step functions.

$$\begin{aligned}
\psi_k(x) &= \begin{cases} 1 & \text{if } 1 - 2^{-k} \leqq x < 1 - 2^{-(k+1)} \\ 0 & \text{otherwise} \end{cases}, \qquad (5) \\
\varphi_n(x) &= \sum_{k=0}^{n-1} 2^{-k} \psi_{\alpha(k)}(x), \\
\varphi(x) &= \begin{cases} 2^{-k} & \text{if } \exists k \in A \text{ s.t. } \frac{1}{2} - 2^{-k} \leqq x < \frac{1}{2} - 2^{-(k+1)} \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

$\{\varphi_n\}$ Fine-converges effectively to φ, φ is effectively fine-continuous, but not sequentially computable.

5 Effective Integrability

A Fine-continuous function is continuous at every binary irrational point, so the Lebesgue measure of the set of all discontinuous points is zero. Therefore, a bounded Fine-continuous function is Riemann integrable and also Lebesgue integrable.

For a binary step function φ of the form $\varphi(x) = \sum_{i=0}^{2^k-1} c_i \chi_{[i2^{-k},(i+1)2^{-k})}(x)$, its integral $\int_0^1 \varphi(x)dx$ is equal to $2^{-k} \sum_{i=0}^{2^k-1} c_i$. Hence, $\{\int_0^1 \varphi_n(x)dx\}$ is a computable sequence of reals, if $\{\varphi_n\}$ is a computable sequence of binary step functions.

Let f be an uniformly Fine-computable function and $\{\varphi_n\}$ be an approximating computable sequence of binary step functions defined by Equation (4). Then f is bounded from Corollary 4.1, so f is integrable. In addition, $\int_0^1 \varphi_n dx$ converges effectively to $\int_0^1 f dx$, since

$$\left| \int_0^1 \varphi_n(x)dx - \int_0^1 f(x)dx \right| \leqq \int_0^1 |\varphi_n(x) - f(x)|dx \leqq \sup_{x \in [0,1)} |\varphi_n(x) - f(x)|.$$

Therefore, $\int_0^1 f dx$ is computable. This property holds for an uniformly Fine-computable sequence.

Theorem 5.1 *If $\{f_n\}$ is uniformly Fine-computable, then $\{\int_0^1 f_n dx\}$ is a computable sequence of reals. Moreover, if $\{f_n\}$ Fine-converges effectively uniformly to f, then $\{\int_0^1 f_n dx\}$ converges effectively to $\int_0^1 f(x)dx$.*

The first part of this theorem can be proved in the same way as the proof of Theorem 0.5 in [9], since the supremum of an uniformly Fine-computable function is computable.

Definition 5.1 *A bounded Fine-computable sequence of functions $\{f_n\}$ is called computable if $\{\int_0^1 f_n dx\}$ is a computable sequence of reals.*

Let $\{f_n\}$ be a bounded sequence of Fine-computable function, i.e. $\{f_n\}$ is an Fine-computable sequence and there exists a real number M such that $|f_n(x)| \leqq M$ for all n and x. Assume furthermore that $\{f_n\}$ Fine-converges to f, then $|f(x)| \leqq M$ for all x. It holds that f is integrable and $\int_0^1 f_n(x)dx$ converges to $\int_0^1 f(x)dx$ by virtute of the bounded convergence theorem.

Proposition 5.1 *Let Fine-computable sequences $\{f_m\}$ and $\{g_{m,n}\}$ satisfy the following conditions.*

(i) *There exists a computable sequence of reals $\{M_m\}$ such that $|g_{m,n}(x)| \leqq M_m$ for all n and x,*

(ii) *$\{\int_0^1 g_{m,n}(x)dx\}$ is a computable sequence of reals.*

(iii) (Effective Convergence) *There exist recursive functions $\alpha(m,i,k)$ and $\gamma(m,i,k)$ which satisfy*

(iii-a) *$x \in \mathbb{B}_F(e_i, 2^{-\gamma(m,i,k)})$ and $n \geqq \alpha(m,i,k)$ imply $|g_{m,n}(x) - f_m(x)| < 2^{-k}$,*

(iii-b) *$\bigcup_{i=1}^{\infty} \mathbb{B}_F(e_i, 2^{-\gamma(m,i,k)}) = [0,1)$ for each m, k.*

Then, $\{\int_0^1 g_{m,n}(x)dx\}$ converges effectively to $\{\int_0^1 f_m(x)dx\}$ and consequently $\{f_m\}$ is effectively integrable.

This proposition can be considered as the effective version of Bounded Convergence Theorem. If f is a bounded Fine-computable function, then the sequence of binary step functions defined by Equation (??) is also bounded. So, we obtain the following theorem.

Theorem 5.2 *If $\{f_n\}$ is Fine-computable and there exists a computable sequence of reals such that $|f_n(x)| \leqq M_n$ for all x, then $\{\int_0^1 f_n(x)dx\}$ is a computable sequence of reals.*

For a Fine-computable function, we have the following example:

Example 5.1 (Brattka [1]) We take the same a and $\{c_n\}$ in Example 4.2, then $\varphi(x) = \sum_{k=0}^{\infty} 2^{k+1}2^{-a(k)}\psi_k(x)$, is locally uniformly computable but $\int_0^1 \varphi(x)dx = c$ is not computable. ($\psi_k(x)$ is defined by Equation (5).) In this case, φ is not bounded. If otherwise, there exists an integer N such that $2^{n+1}2^{-a(n)} \leqq 2^N$. This implies that $2^{-a(n)} \leqq 2^{N-1-n}$ and $\{c_n\}$ converges effectively. Moreover, $\varphi_n(x) = \sum_{k=0}^{n-1} 2^{k+1}c_k\psi_k(x)$ Fine-converges to φ but the convergence is not effective.

Let $f \geqq 0$ be a Fine-computable function. Then $\{f_n\} = \{f \wedge 2^n\}$ is a computable sequence and each f_n is bounded by 2^n. So, $\{\int_0^1 f_n dx\}$ is a computable sequence of reals from Theorem 5.2 and monotonically increasing. If we assume that f is integrable, then the computability of $\int_0^1 f(x)dx$ is equivalent to the effective convergence of $\{\int_0^1 f_n dx\}$ from Lemma ??. So we obtain the following definition of effective integrability.

Definition 5.2 *An integrable Fine-computable functions f is said to be effectively integrable if the integrals of f^+ and f^- are computable reals.*

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