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The completion of the space $\mathcal{D}(\mathbb{R})$ in constructive analysis

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1 Introduction

$\mathcal{D}(\mathbb{R})$ is the space of *test functions* -infinitely differentiable functions on $\mathbb{R}$ with compact support- taken together the locally convex structure defined by the seminorms

$$p_{\alpha,\beta}(\phi) := \sup_{n} \max_{l\leq\beta(n)} \sup_{|x|\geq n} 2^{\alpha(n)}|\phi^{(l)}(x)| \quad (\alpha, \beta \in \mathbb{N} \to \mathbb{N}),$$

and is an important example of a locally convex space which is not metrizable and normable; see [1, Appendix A] and [2, Chapter 7, Notes] for more details. Also, $\mathcal{D}(\mathbb{R})$ is the domain of *generalized functions* (or *distributions*), and therefore the topological study of this space is necessary for the theory of generalized functions. In this paper, we survey the recent constructive study of completeness of $\mathcal{D}(\mathbb{R})$ and its dual space. Then we see that the space $\widetilde{\mathcal{D}}(\mathbb{R})$ has an important role in this study. By the way, the arguments in this paper are based on the framework of E.Bishop's constructive mathematics, which is formalized in intuitionistic finite-type arithmetics $\mathbf{HA}^{\omega}$ and has axiom of countable choice; for more details, see [5], [11, Chapter 1] and [12, Chapter 9].

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2 Preliminary

In this section, we define the continuity and differentiability in constructive calculus and locally convex spaces, in oder to introduce $\mathcal{D}(\mathbb{R})$. Also, we give a notion weaker than boundedness for a set of natural numbers and an independent principle of $\mathbf{HA}^\omega$. They will be important for a certain constructive look at completeness of $\mathcal{D}(\mathbb{R})$.

We assume familiarity with constructive calculus, as found in [1, Chapter 2], [3, Appendix], [2, Chapter 2] or [12, Chapter 6]. A function f from a subset X of $\mathbb{R}$ to $\mathbb{R}$ is said to be *uniformly continuous* if for each k, there exists N such that for all $x, y \in X$,

$$|x-y| < 2^{-N} \implies |f(x)-f(y)| < 2^{-k}.$$

We say that a function from $\mathbb{R}$ to $\mathbb{R}$ is *continuous* if it is uniformly continuous on each compact interval. A function f from a subset X of $\mathbb{R}$ into $\mathbb{R}$ is *uniformly differentiable* on X, with a derivative f', if for each k, there exists N such that for all $x, y \in X$,

$$|x-y| < 2^{-N} \implies |f'(x)(x-y) - (f(x)-f(y))| < 2^{-k}.$$

Note that the derivative of a uniformly differentiable function on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$, by a straightforward modification of [3, Appendix A]. A function $f : X \to \mathbb{R}$ is *differentiable* if it is uniformly differentiable on each compact subset of X. We use the familiar notation for iterated derivatives of differentiability: $f^{(0)} := f$, $f^{(l+1)} := (f^{(l)})'$.

Let X be a vector space over the set $\mathbb{R}$ of all real numbers. A function $p : X \to \mathbb{R}^{0+}$ is said to be a *seminorm* on X if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, we have (1) $p(x+y) \leq p(x) + p(y)$ and (2) $p(\lambda x) = |\lambda| p(x)$. Let I be a set, and $\{p_i\}_{i \in I}$ seminorms on X. A pair $(X, \{p_i\}_{i\in I})$ is said to be a *locally convex space* over $\mathbb{R}$ if for each $x \in X$, whenever $p_i(x) = 0$ for all $i \in I$, then $x = 0$.

A sequence $\{x_n\}$ in a locally convex $(X, \{p_i\})$ is said to *converges to x* in X, if for each natural number k and indexes $i_1, \dots, i_l$, there exists N such that $\max_{1 \leq t \leq l} p_{i_t}(x_n - x) < 2^{-k}$ for all $n \geq N$. $\{x_n\}$ in a locally convex $(X, \{p_i\})$ is said to be a *Cauchy sequence in* X, if for each natural number k and indexes $i_1, \dots, i_l$, there exists N such that $\max_{1 \leq t \leq l} p_{i_t}(x_m - x_n) < 2^{-k}$ for all $m, n \geq N$. A locally convex space $(X, \{p_i\})$ is said to be *sequentially complete* if every Cauchy sequence converges in X.

A linear functional on a locally convex space X is said to be *sequentially continuous* if for each sequence $\{x_n\}$ in X, if $\{x_n\}$ converges to 0 in X, then $u(x_n)$ converges to 0 in $\mathbb{R}$. A linear functional on X is said to be *bounded* if there exist a positive real number $C > 0$ and indexes $i_1, \ldots, i_l$ such that $|u(x)| \leq C \max_{1\leq t\leq l} p_{i_t}(x)$ for all x in X. Moreover, let X^* be the set of sequentially continuous functionals on a locally convex space X, and set

$$\|u\|_x := |u(x)| \quad (u \in X^*, x \in X).$$

Then we can define the locally convex space $(X^*, \{\|\cdot\|_x\})$, which is called the *dual space of X with weak topology.*

Let S be a inhabited [2] set. A class $\mathcal{F}$ of inhabited subsets of S is called a *filter* if it satisfies the following conditions:

- If A, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and
- If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.

We say that a filter $\mathcal{F}$ *converges to x* in a locally convex space $(X, \{p_i\})$, if for each k in $\mathbb{N}$ and finitely many $p_{i_1}, \ldots, p_{i_l}$, there exists $U \in \mathcal{F}$ such that for all y in X,

$$y \in U \implies \max_{1\leq t\leq l} p_{i_t}(x-y) < 2^{-k}.$$

$\mathcal{F}$ is a *Cauchy filter* if for each k in $\mathbb{N}$ and finitely many $p_{i_1}, \ldots, p_{i_l}$, there exists $U \in \mathcal{F}$ such that for all x, y in X, if $x, y \in U$ then $\max_{1\leq t\leq l} p_{i_t}(x-y) < 2^{-k}$. A locally convex space is said to be *complete*, if every Cauchy filter converges. *Weak completeness* is completeness of the dual space of a locally convex space, with weak topology.

A subset A of $\mathbb{N}$ is said to be *pseudobounded* if for each sequence $\{a_n\}$ in A, there exists N such that $a_n < n$ for all $n \geq N$. A bounded subset of $\mathbb{N}$ is pseudobounded. On the other hand, a natural recursivisation of the following principle is independent of $\mathbf{HA}^\omega$ (see [4]).

BD-N: Every countable pseudobounded subset of $\mathbb{N}$ is bounded.

Thus the converse for countable sets does not hold in Bishop's constructive mathematics. However, BD-N holds in classical mathematics, L.E.J. Brouwer's intuitionistic mathematics and constructive recursive mathematics of A.A. Markov's school; see [4] and [6] for more details.

Now the main subject starts from the next section.

[2] "A set S is inhabited" means "S has an element".

3 The space $\mathcal{D}(\mathbb{R})$

$\mathcal{D}(\mathbb{R})$ is classically complete relatively to filters, but it had not been known whether the constructive completion of the space $\mathcal{D}(\mathbb{R})$, whose explicit description was given in [1, Appendx 1] and [2, Chapter 7], coincides with the original space or not. This led us to a difficulty in developing the constructive theory of generalized functions. At first, in this section, we see how constructively difficult completeness of $\mathcal{D}(\mathbb{R})$ holds.

Let $\mathcal{K}(\mathbb{R})$ denote the space of continuous functions with compact support, taken together with the locally convex structure defined by the norms

$$q_\alpha(\phi) := \sup_n \sup_{|x|\geq n} 2^{\alpha(n)}|\phi(x)| \quad (\phi \in \mathcal{K}(\mathbb{R}), \alpha \in \mathbb{N} \to \mathbb{N}).$$

Theorem 3.1. *[7, Theorem 4] The following are equivalent.*

1. $\mathcal{K}(\mathbb{R})$ *is sequentially complete.*

2. $\mathcal{D}(\mathbb{R})$ *is sequentially complete.*

3. BD$-\mathbb{N}$.

It follows from Theorem 3.1 that the sequential completeness holds in classical mathematics, intuitionistic mathematics and constructive recursive mathematics but does not in Bishop's constructive mathematics. Moreover, the completeness implies BD-$\mathbb{N}$, since completeness implies sequential completeness, and therefore the completeness does not. Then we have a question what is the sufficient condition to establish the completeness. The following is the answer.

Theorem 3.2. *[10, Corollary 4.5] The following are equivalent.*

1. $\mathcal{K}(\mathbb{R})$ *is complete.*

2. $\mathcal{D}(\mathbb{R})$ *is complete.*

3. BD-$\mathbb{N}$.

That is, completeness of $\mathcal{D}(\mathbb{R})$, its sequential completeness and principle BD-$\mathbb{N}$ are equivalent in Bishop's framework. This fact follows immediately from Theorem 4.4. We will see it through the consideration of the completion $\widetilde{\mathcal{D}}(\mathbb{R})$ in the next section.

4 The space $\widetilde{\mathcal{D}}(\mathbb{R})$

Bishop gave the completion of the space $\mathcal{D}(\mathbb{R})$ in [1, Appendix A] and [2, Chapter 7, Notes], in view of difficulty of the completeness. It consists of infinitely differentiable functions ϕ such that $p_{\alpha,\beta}(\phi)$ exists for all α, β in $\mathbb{N} \to \mathbb{N}$, but Bishop mentioned that it was artificial. Also, details for the completion were not showed fully in [1, Appendix A] and [2, Chapter 7, Notes], and it is not useful as the completion of $\mathcal{D}(\mathbb{R})$. So it is necessary to give another form of the completion.

For a function $f : \mathbb{R} \to \mathbb{R}$, we write

1. $\mathrm{supp} f := \overline{\{x \in \mathbb{R} : |f(x)| > 0\}}$; that is, $\mathrm{supp} f$ is the closure of the set $\{x \in \mathbb{R} : |f(x)| > 0\}$ in euclidean space $\mathbb{R}$.

2. $\mathrm{supp}_{\mathbb{N}} f := \{0\} \cup \{n \in \mathbb{N} : \exists q \in \mathbb{Q}(|q| \geq n \wedge |f(q)| > 0)\}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to *has compact support* if the set $\mathrm{supp} f$ is bounded. It is easy to show that a uniformly continuous function f on $\mathbb{R}$ has compact support if and only if the $\mathrm{supp}_{\mathbb{N}} f$ is bounded. A function $f : \mathbb{R} \to \mathbb{R}$ is said to *has pseudobounded support* if the set $\mathrm{supp}_{\mathbb{N}} f$ is pseudobounded. We let $\widetilde{\mathcal{D}}(\mathbb{R})$ denote the set of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ with pseudobounded support. Note that the seminorms of the space $\mathcal{D}(\mathbb{R})$ can be defined in $\widetilde{\mathcal{D}}(\mathbb{R})$ (see [8, Proposition 2.6]). Also, $\widetilde{\mathcal{K}}(\mathcal{D})$ denotes the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with pseudobounded support. Similarly, the seminorms of the space $\mathcal{K}(\mathbb{R})$ can be defined in $\widetilde{\mathcal{K}}(\mathbb{R})$. Here we have the following theorems.

Theorem 4.1. *[8, Theorem 2.8] A sequence $\{\phi_i\}$ is a Cauchy sequence in $\widetilde{\mathcal{D}}(\mathbb{R})$ if and only if the following hold:*

- *for each l and k, there exists I such that*

$$\sup_{x \in \mathbb{R}} |\phi_i^{(l)}(x) - \phi_j^{(l)}(x)| < 2^{-k} \quad (i, j \geq I), \textit{and}$$

- $\cup_i \mathrm{supp}_{\mathbb{N}} \phi_i$ *is pseudobounded.*

Theorem 4.2. *[8, Theorem 2.9] A sequence $\{\phi_i\}$ converges in $\widetilde{\mathcal{D}}(\mathbb{R})$ if and only if the following hold:*

- *for each l, the sequence $\{\phi_i^{(l)}\}_i$ of l-th derivatives converges uniformly on $\mathbb{R}$, and*

- *$\bigcup_i \mathrm{supp}_{\mathbb{N}}\phi_i$ is pseudobounded.*

Classically, if $\{\phi_i\}$ is a Cauchy sequence in $\widetilde{\mathcal{D}}(\mathbb{R})$, then $\bigcup_i \mathrm{supp}_{\mathbb{N}}\phi_i$ is bounded. So we see that a certain classical definition of Cauchyness is equivalent to one relatively to the seminorms, if and only if BD-N.

The following is showed by Theorems 4.1 and 4.2.

Theorem 4.3. *[8, Theorem 2.12] $\widetilde{\mathcal{D}}(\mathbb{R})$ is sequentially complete.*

Thus it follows from Theorem 4.3 that $\widetilde{\mathcal{D}}(\mathbb{R})$ is identical to $\mathcal{D}(\mathbb{R})$ if and only if BD-N. It is similar to the case of $\mathcal{K}(\mathbb{R})$ and $\widetilde{\mathcal{K}}(\mathbb{R})$. Moreover we can show another completeness of these spaces.

Theorem 4.4. *[10, Theorem 4.3] $\widetilde{\mathcal{D}}(\mathbb{R})$ is complete.*

Therefore $\widetilde{\mathcal{D}}(\mathbb{R})$ is the completion of $\mathcal{D}(\mathbb{R})$, and is a natural extensions from it, and $\widetilde{\mathcal{K}}(\mathbb{R})$ is from $\mathcal{K}(\mathbb{R})$ similarly. It is also clear that this implies Theorem 3.2.

Theorem 4.5. *[10, Corollary 4.6] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Then ϕ has pseudobounded support if and only if $p_{\alpha,\beta}(\phi)$ exists for all $\alpha, \beta \in \mathbb{N} \to \mathbb{N}$.*

$\widetilde{\mathcal{D}}(\mathbb{R})$ is identical to the completion given in [1, Appendix A] and [2, Chapter 7, Notes], from this theorem. It is showed similarly that $\widetilde{\mathcal{K}}(\mathbb{R})$ is complete.

5 The dual space of $\mathcal{D}(\mathbb{R})$

It is expected in [1, Appendix A] and [2, Chapter 7, Notes] that the dual space of $\mathcal{D}(\mathbb{R})$ is not weakly complete in Bishop's framework, where the dual space means the space of bounded linear functionals on $\mathcal{D}(\mathbb{R})$, in them. However we have not known its solution yet. On the other hand, we have some results for the space of sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$, in [8] and [9]. So, in the rest of this paper, we call the space of sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$ the dual space of $\mathcal{D}(\mathbb{R})$, which is considered mainly. Also, a sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$ is said here to be a *distribution*, and a distribution on $\widetilde{\mathcal{D}}(\mathbb{R})$ means a sequentially continuous linear functional on $\widetilde{\mathcal{D}}(\mathbb{R})$.

Theorem 5.1. *[8, Theorem 4.9] Every distribution has an unique extension to* $\widetilde{\mathcal{D}}(\mathbb{R})$.

Theorem 5.1 means that every distribution can be identified with a distribution on $\widetilde{\mathcal{D}}(\mathbb{R})$. the following is called the Banach-Steinhaus theorem for the space $\mathcal{D}(\mathbb{R})$.

Theorem 5.2. *[8, Theorem 4.10] Let* $\{u_k\}$ *be a sequence of distributions on* $\mathcal{D}(\mathbb{R})$ *such that* $\langle u, \phi\rangle := \lim_k \langle u_k, \phi\rangle$ *exists for all* ϕ *in* $\mathcal{D}(\mathbb{R})$. *Then* u *is a distribution on* $\mathcal{D}(\mathbb{R})$.

Many classical proofs of this version of the Banach-Steinhaus theorem require the sequential completeness of $\mathcal{D}(\mathbb{R})$. However, we conclude from Theorem 5.2 that the completeness of $\mathcal{D}(\mathbb{R})$ is unnecessary for proving the Banach-Steinhaus theorem not only in classical mathematics but also in intuitionistic mathematics and constructive recursive mathematics. Moreover our version of the Banach-Steinhaus theorem implies the weak sequential completeness theorem for the dual space $\mathcal{D}^*(\mathbb{R})$ of $\mathcal{D}(\mathbb{R})$. Thus we can verify that at least the sequential completion of $\mathcal{D}^*(\mathbb{R})$ is identified with $\mathcal{D}^*(\mathbb{R})$, although it is mentioned in [1, Appendix] and [2, Chapter 7, Notes] that the completion of $\mathcal{D}^*(\mathbb{R})$ relatively to filters cannot be.

In [8], the consideration of distributions on $\widetilde{\mathcal{D}}(\mathbb{R})$ has important for Theorem 5.2. Actually, Theorem 5.2 is proved in [8], by using Theorem 5.1 and the version for $\widetilde{\mathcal{D}}(\mathbb{R})$.

Now we show an application of Theorem 5.2. $\mathcal{D}_n(\mathbb{R})$ denotes the space of all test functions with compact support contained in $[-n, n]$, taken together with seminorms

$$\|\phi\|_{n,l} := \max_{m\leq l} \sup_{|x|\leq n} |\phi^{(m)}(x)| \quad (\phi \in \mathcal{D}_n(\mathbb{R})).$$

Then every $\mathcal{D}_n(\mathbb{R})$ is a $(\mathcal{F})$-space; that is, metrizable and sequentially complete, by a straightforward modification of the proof of [2, Theorem 2.4.11]. It is clear that $\mathcal{D}(\mathbb{R}) = \cup_n \mathcal{D}_n(\mathbb{R})$. Here, in classical mathematics, the locally convex structure of $\mathcal{D}(\mathbb{R})$ defined by the seminorms $\{p_{\alpha,\beta}\}$ can be also given by the sequence $\{\mathcal{D}_n(\mathbb{R})\}$ of $(\mathcal{F})$-spaces. This is a property of a $(\mathcal{L}F)$-space in the general theory of topological vector spaces. On the other hand, we had not known whether it holds in Bishop's constructive mathematics or not, since we have not had a sufficiently general constructive theory of topological vector spaces in Bishop's framework. However we obtain the following result in the theory of a $(\mathcal{L}F)$-space.

Theorem 5.3. *Let u be a linear functional on $\mathcal{D}(\mathbb{R})$. Then u is sequentially continuous if and only if for each n, u is sequentially continuous on $\mathcal{D}_n(\mathbb{R})$.*

Proof. The part "only if" is trivial. We show the part "if". Let u be sequentially continuous linear functional on each $\mathcal{D}_n(\mathbb{R})$, and fix any ϕin $\mathcal{D}(\mathbb{R})$. Then we have

$$\langle u, \phi\rangle = \sum_{i=0}^{\infty}\langle u, \rho_i\phi\rangle,$$

where a sequence $\{\rho_i\}$ in $\mathcal{D}(\mathbb{R})$ satisfies that

- for each n, $0 \leq \rho_i(x) \leq 1$ if $x \in \mathbb{R}$,
- for each n, $\sup_{n-1\leq|x|\leq n+1} \rho_i(x) > 0$,
- for each n, $\rho_i(x) = 0$ if $x \leq n-1$ or $n+1 \leq x$, and
- $\sum_{n=0}^{\infty} \rho_i(x) = 1$ for all $x \in \mathbb{R}$;

from [8, Lemma 4.6]. For each i, the mapping $\phi \mapsto \rho_i\phi$ is sequentially continuous from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}_{i+1}(\mathbb{R})$, and so the mapping $\phi \mapsto \langle u, \rho_i\phi\rangle$ is a sequentially continuous linear functional on $\mathcal{D}(\mathbb{R})$, since it is a sequentially continuous linear functional on $\mathcal{D}_{i+1}(\mathbb{R})$. Thus the mapping $\phi \mapsto \sum_{i=0}^{\infty}\langle u, \rho_i\phi\rangle$ is a sequentially continuous linear functionals, from Theorem 5.2. □

The sufficient condition of Theorem 5.3 is known as a classical definition of distribution. It is concluded by Theorem 5.2 that our definition is also constructively equivalent to it.

Now some problems have ever remained in Bishop's constructive mathematics. We have not known whether the dual space of $\mathcal{D}(\mathbb{R})$ is weakly complete relatively to filters holds or not. Also, it is necessary to show whether the Banach-Steinhaus theorem relatively to bounded linear functionals on $\mathcal{D}(\mathbb{R})$ holds or not. Moreover, what we want to do is to prove that a distributions is not bounded on $\mathcal{D}(\mathbb{R})$ constructively. This is expected from the work of [6].

6 Concluding remarks

There are some open problems for the completeness of the dual space of $\mathcal{D}(\mathbb{R})$, as described in Section 5. Moreover, in order to develop constructive theory of generalized functions, we have more problems.

We had had little development of constructive generalized function theory, since it was not proved that sequential completeness of $\mathcal{D}(\mathbb{R})$ does not hold constructively. But, after we obtained Theorem 3.1, we have a foresight of the constructive theory. For example, we can define support of a distribution and its compactness, similarly to ones of a function on $\mathbb{R}$, and then a distribution with compact support is classically identified with a continuous linear functional on $\mathcal{D}(\mathbb{R})$. On the other hand, this fact does not hold constructively, but a distribution with pseudobounded support is constructively with one; see [9, Chapter 3] for more details. So pseudoboundedness also has a important role in this scene.

However we had some hard problems. In fact, for open subset Ω of euclidean space $\mathbb{R}^n$, we can define classically the space $\mathcal{D}(\Omega)$ of infinitely differentiable functions on Ω with compact support, and obtain the classical theory for $\mathcal{D}(\Omega)$. On the other hand, we have not established constructive calculus for functions with many variables, and it is not easy to treat vague open subsets constructively. So it is necessary for us to study more constructive foundations of constructive calculus. It is similar to the constructive theory of topological vector spaces.

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