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THREE-PLAYER GAME OF 'KEEP-OR-EXCHANGE'

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ABSTRACT. A three-player, sequential-move game with imperfect information is analyzed and the explicit solution is given. This work is the first extension of the present author's recent paper Ref.[8] to the three-player games. The solution derived is surprisingly complicate in comparison with the one for the two-player game. Our intuition, that the last-mover has an advantage over the middle-mover, and the middle-mover, in turn, has an advantage over the first-mover, is proven to be correct. Three-player simultaneous-move game is also solved. A conjecture for the solution to the four-player game is given.

1 Three-Player Games of 'Score Showdown'. Consider the three players I II and III (sometimes they are denoted by 1,2 and 3). Let $X_{ij} (i = 1, 2, 3; j = 1, 2)$ be the random variable (r.v.) observed by player i at the j-th observation. We assume that X_{ij}'s are i.i.d., each with uniform distribution in $[0, 1]$. The game is played in the three stages.

In the first stage, I observes that $X_{11} = x$ and chooses one of the either A_1 (*i.e.*, I accepts x) or R_1 (*i.e.*, I rejects x and resamples a new r.v. X_{12}). The observed value x and I's choice of either A_1 or R_1 are informed to II and III. But X_{12} is a r.v. for the all players (including I himself).

In the second stage, II observes that $X_{21} = y$, and chooses either one of A_2 (*i.e.*, II accepts y) or R_2 (*i.e.*, II rejects y and resamples a new r.v. X_{22}). The observed value y and II's choice of either A_2 or R_2 are informed to III. But X_{22} is a r.v. for III, and II himself.

In the third stage, III observes that $X_{31} = z$ and chooses either one of A_3 (*i.e.*, III accepts z) or R_3 (*i.e.*, III rejects z and resamples a new r.v. X_{32}). X_{32} is a r.v. for III himself, that is, III doesn't know its realized value until the showdown is made.

Let, for $i = 1, 2, 3$,

$$S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1} \\ X_{i2} \end{cases}, \text{ if } X_{i1} \text{ is} \begin{cases} \text{accepted} \\ \text{rejected} \end{cases} \text{by player } i, \tag{1.1}$$

which we call the *score* for player i.

After the third stage is over, the showdown is made, the scores are compared, and the player with the highest score among the players becomes the *winner*. Each player aims to maximize the probability of his (or her) winning. We assume that all players are intelligent, and each player should prepare for that any subsequent player must use their optimal strategies.

The three-player game of 'Keep-or-Exchange' (*i.e.*, the score is defined by (1.1)) is solved in Section 3. The solution is found to be very complicate far more than expected. It is compared with that of the two-player case, given in Section 2. In Ref.[8] the other two-player games of 'Competing Average', where the score is

$$S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1} \\ \frac{1}{2}(X_{i1} + X_{i2}) \end{cases}, \text{ if } \begin{cases} X_{i1} \text{ is accepted} \\ X_{i2} \text{ is resampled,} \end{cases} \tag{1.2}$$

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and 'Showcase Showdown' where the score is

$$(1.3)\quad S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1} \\ (X_{i1}+X_{i2})I(X_{i1}+X_{i2}\le 1) \end{cases}, \text{ if } \begin{cases} X_{i1} \text{ is accepted} \\ X_{i2} \text{ is resampled,} \end{cases}$$

are solved. The three-player game versions of these two-player games remain to be solved as yet. See also Ref.[1~7].

Intuitively it would seem in the three-player games, that the last-mover has an advantage over the middle-mover, and the middle-mover, in turn, has an advantage over the first-mover. Theorem 2 in the present paper shows that this intuition is correct in 'Keep-or-Exchange', where the score is (1.1). It is an interresting work to investigate whether the counter examples do exist or not.

Three-player simultaneous-move game is solved in Section 4. A conjecture for the solution to the four-player game is given. We observe that player behaves more cautious as he has more competitors.

2 Keep-or-Exchange —— Two-Player game. First we solve the two-player game. We will find, in the next section, that the three-player game is surprisingly complicate to solve, compared with in the two-player case.

Let $W_i (i = 1, 2)$ be the event that player i wins. To find the players' optimal strategies we must derive them in reverse order. Define state $\begin{Bmatrix} (y|x, A_1) \\ (y|x, R_1) \end{Bmatrix}$ for II, to mean that I $\begin{Bmatrix} \text{accepted} \\ \text{rejected} \end{Bmatrix} X_{11} = x$ in the first stage and II has just observed $X_{21} = y$ in the second stage. Then we have

$$(2.1)\quad p_{2A}(y|x, A_1) = P\{W_2|\text{II accepts } X_{21} = y \text{ in state } (y|x, A_1)\} = I(y > x),$$

$$(2.2)\quad \begin{aligned} p_{2R}(y|x, A_1) &= P\{W_2|\text{II rejects } X_{21} = y \text{ in state } (y|x, A_1)\} \\ &= P(X_{22} > x) = \bar{x} \equiv 1 - x, \quad \text{indep.of } y, \end{aligned}$$

$$(2.3)\quad \begin{aligned} p_{2A}(y|x, R_1) &= P\{W_2|\text{II accepts } X_{21} = y \text{ in state } (y|x, R_1)\} \\ &= P(X_{12} < y) = y, \quad \text{indep.of } x, \end{aligned}$$

and

$$(2.4)\quad \begin{aligned} p_{2R}(y|x, R_1) &= P\{W_2|\text{II rejects } X_{21} = y \text{ in state } (y|x, R_1)\} \\ &= P(X_{12} < X_{22}) = \frac{1}{2}, \quad \text{indep.of } x \text{ and } y, \end{aligned}$$

Theorem 1 *The solution to the two-player game with the score function* (1.1) *is as follows. The optimal strategy for* I *in the first stage is given by:*

$$(2.5)\quad \textit{Accept (Reject)} X_{11} = x, \ \textit{if } x > (<)\sqrt{3/8} \approx 0.6124.$$

The optimal strategy for II *in the second stage is given by*

$$(2.6)\quad \textit{Accept (Reject) } X_{21} = y, \ \textit{if } y > (<)\begin{Bmatrix} x \\ 1/2 \end{Bmatrix} \textit{ in state } \begin{Bmatrix} (y|x, A_1) \\ (y|x, R_1) \end{Bmatrix}.$$

The optimal values are

$$(2.7)\quad \begin{aligned} P(W_1) &= \frac{1}{3}\left\{1 + 2(3/8)^{3/2}\right\} \approx 0.4864 \\ P(W_2) &= 1 - P(W_1) = \frac{2}{3}\left\{1 - (3/8)^{3/2}\right\} \approx 0.5136. \end{aligned}$$

Proof is given in Ref.[8].

3 Keep-or-Exchange —— Three-Player game. Let W_i be the event that player i wins. To find the players' optimal strategies, we must derive them in reverse order. Define state $\left\{\begin{array}{c}(z|xA_1, yA_2)\\(z|xA_1, yR_2)\end{array}\right\}$ for III, to mean that I accepted $X_{11} = x$ in the first stage, II $\left\{\begin{array}{c}\text{accepted}\\\text{rejected}\end{array}\right\}X_{21} = y$ in the second stage, and III has just observed $X_{31} = z$ in the third stage. Also define the other two states $(z|xR_1, yR_2)$ and $(z|xR_1, yA_2)$ similarly. Then we easily find that

$$\begin{aligned}(3.1)\qquad p_{3A}(z|xA_1, yA_2) &\equiv P\{W_3|\text{III accepts } X_{31} = z \text{ in state } (z|xA_1, yA_2)\}\\ &= I(z > y), \quad \text{(since II behaves optimally)}\end{aligned}$$

$$\begin{aligned}(3.2)\qquad p_{3R}(z|xA_1, yA_2) &\equiv P\{W_3|\text{III rejects } X_{31} = z \text{ in state } (z|xA_1, yA_2)\}\\ &= P(X_{32} > y) = \bar{y},\end{aligned}$$

$$\begin{aligned}(3.3)\qquad p_{3A}(z|xA_1, yR_2) &\equiv P\{W_3|\text{III accepts } X_{31} = z \text{ in state } (z|xA_1, yR_2)\}\\ &= I(z > x)P(z > X_{22}) = zI(z > x),\end{aligned}$$

$$\begin{aligned}(3.4)\qquad p_{3R}(z|xA_1, yR_2) &\equiv P\{W_3|\text{III rejects } X_{31} = z \text{ in state } (z|xA_1, yR_2)\}\\ &= P\{X_{32} > (x \vee X_{22})\} = \frac{1}{2}(1 - x^2),\end{aligned}$$

$$\begin{aligned}(3.5)\qquad p_{3A}(z|xR_1, yR_2) &\equiv P\{W_3|\text{III accepts } X_{31} = z \text{ in state } (z|xR_1, yR_2)\}\\ &= P(z > X_{12} \vee X_{22}) = z^2,\end{aligned}$$

$$\begin{aligned}(3.6)\qquad p_{3R}(z|xR_1, yR_2) &\equiv P\{W_3|\text{III rejects } X_{31} = z \text{ in state } (z|xR_1, yR_2)\}\\ &= 1/3, \quad \forall(x, y, z),\end{aligned}$$

and

$$\begin{aligned}(3.7)\qquad p_{3A}(z|xR_1, yA_2) &\equiv P\{W_3|\text{III accepts } X_{31} = z \text{ in state } (z|xR_1, yA_2)\}\\ &= I(z > y)P(z > X_{12}) = zI(z > y),\end{aligned}$$

$$\begin{aligned}(3.8)\qquad p_{3R}(z|xR_1, yA_2) &\equiv P\{W_3|\text{III rejects } X_{31} = z \text{ in state } (z|xR_1, yA_2)\}\\ &= P\{X_{32} > (y \vee X_{12})\} = \frac{1}{2}(1 - y^2).\end{aligned}$$

Theorem 2 *The solution to the three-player game with the score function* (1.1) *is as follows. The optimal strategy for* I *in the first stage is given by:*

$$(3.9)\qquad \textit{Accept (Reject) } X_{11} = x, \ \textit{if } x > (<) x_0 = c^{1/4} \approx 0.68774,$$

where $c \approx 0.22372$ *is a four-order polynomial of* $k^{1/3} \equiv \left(\frac{9-\sqrt{3}}{27}\right)^{1/3} \approx 0.64568$, *given by* (3.24). *The optimal strategy for* II *in the second stage is:*

$$(3.10)\qquad \textit{Accept (Reject) } X_{21} = y,$$

$$\textit{if } y > (<) \left\{\begin{array}{l} y_0(x). \\ k^{1/3} \approx 0.64568 \end{array}\right., \quad \textit{in state } \left\{\begin{array}{l}(y|xA_1)\\(y|xR_1)\end{array}\right.$$

where $y_0(x) \equiv \sqrt{h^-(x)}I(0 < x \le \sqrt{2}-1) + \sqrt{h^+(x)}I(\sqrt{2}-1 < x \le \xi) + xI(\xi < x \le 1), h^-(x) \equiv \frac{1}{8}(3 - 2x^2 + 3x^4), h^+(x) \equiv \frac{1}{2}(1 - x + x^2 - x^3), k \equiv \frac{9-\sqrt{3}}{27} \approx 0.26918$ *and* $\xi \approx 0.54368$ *is in a unique root in* $(0,1)$ *of the equation* $x^3 + x^2 + x - 1 = 0$ *(See Figure 2). Note that* $y_0(x) \ge x, \forall x \in (0,1)$. *The optimal strategy for* III *in the third stage is given by:*

(3.11) *Accept (Reject)* $X_{31} = z$,

$$\text{if } z > (<) \begin{cases} y \\ \frac{1}{2}(1-x^2) \vee x \\ 1/\sqrt{3} \approx 0.57735 \\ \frac{1}{2}(1-y^2) \vee y \end{cases}, \quad \text{in state} \begin{cases} (z|xA_1, yA_2) \\ (z|xA_1, yR_2) \\ (z|xR_1, yR_2) \\ (z|xR_1, yA_2). \end{cases}$$

The optimal value for the three players are

(3.12) $P(W_1) \approx 0.32309, \quad P(W_2) \approx 0.33270 \quad \text{and} \quad P(W_3) \approx 0.34421.$

Proof. The theorem is proven in the four steps. (略)

Remark 1 We observe, by Theorems 1 and 2, that the difference between the players' winning probabilities is diminished in the three-player case than in the two-player case.

Remark 2 We give a numerical example which shows how Theorems 1 and 2 work.

	Two-player game	Three-player game
1st stage	If I observes $X_{11} = x = 0.482$, then he announces 0.482 & R_1 (since $x < \sqrt{3/8} \approx 0.6124$) and exchange x to X_{12}	If I observes $X_{11} = x = 0.482$, then he announces 0.482 & R_1 (since $x < x_0 = 0.6877$) and exchange x to X_{12}
2nd stage.	If II obs. $X_{21} = y = 0.644$, then he accepts it (since $y > 1/2$).	If II obs. $X_{21} = y = 0.644$, then he announces 0.644 & R_2 (since $y < y_0 = 0.6457$) and exchange it to X_{22}
3rd Stage		If III obs. $X_{31} = z = 0.581$, he accepts it (since $z > 1/\sqrt{3} \approx 0.5774$)
Showdown	I(II) wins if $X_{12} > (<) 0.644$.	Players' scores are X_{12}, X_{22}, and 0.581, resp. Player with the highest score wins.

Remark 3 It seems to us that the sequential game discussed in the present paper doesn't belong to the area of dynamic programming. The result obtained in the two-player game is not applicable to the three-player game.

4 Simultaneous-Move Game. In the simultaneous-move version of the game, the unfair information acquisition by the players disappears. Each player $i, i = 1, 2, 3$, privately observes X_{i1} and chooses either one of A_i or R_i. The observed value and choice by each player are unknown to his (or her) opponents. Suppose that players' strategies have the form of :

$$\text{I accepts (rejects) } X_{11} = x, \text{ if } x > (<) a,$$

$$\text{II accepts (rejects) } X_{21} = y, \text{ if } y > (<) b,$$

$$\text{III accepts (rejects) } X_{31} = z, \text{ if } z > (<) c.$$

Let $M_i(a,b,c) \equiv P\{W_i|\text{I, II, and III choose } a, b \text{ and } c, \text{ respectively}\}, i = 1,2,3$. Evidently $\sum_{i=1}^{3} M_i(a,b,c) = 1, \forall(a,b,c) \in [0,1]^3$, and, by symmetry, $M_i(a,a,a) = 1/3, \forall i, \forall a \in [0,1]$.

Let $p_{AAA}, p_{RRR}, p_{AAR}$, etc., denote the winning probability for I when the players' choice-triple is $A{-}A{-}A$, $R{-}R{-}R$, $A{-}A{-}R$, etc. Also let $q_{AAA}, q_{RRR}, q_{AAR} (r_{AAA}, r_{RRR}, r_{AAR})$ etc, similarly denote the winning probability for II (III). Then we find that

$$M_1(a,b,c) = p_{AAA} + p_{RRR} + (\text{other six probabilities}), \tag{4.1}$$

$$M_2(a,b,c) = q_{AAA} + q_{RRR} + (\text{other six probabilities}), \tag{4.2}$$

$$M_3(a,b,c) = r_{AAA} + r_{RRR} + (\text{other six probabilities}), \tag{4.3}$$

where

$$\begin{aligned}
p_{AAA} &= P\{X_{11} > a, X_{21} > b, X_{31} > c, X_{11} > X_{21} \vee X_{31}\} = \int_{a\vee(b\vee c)}^{1} (t-b)(t-c)dt, \\
p_{RRR} &= P\{X_{11} < a, X_{21} < b, X_{31} < c, X_{12} > X_{22} \vee X_{32}\} = \frac{1}{3}abc, \\
p_{ARR} &= P\{X_{11} > a, X_{21} < b, X_{31} < c, X_{11} > X_{22} \vee X_{32}\} = bc\int_{a}^{1} t^2 dt, \\
p_{AAR} &= P\{X_{11} > a, X_{21} > b, X_{31} < c, X_{11} > X_{21} \vee X_{32}\} = c\int_{a\vee b}^{1} t(t-b)dt, \\
p_{ARA} &= P\{X_{11} > a, X_{21} < b, X_{31} > c, X_{11} > X_{22} \vee X_{31}\} = b\int_{a\vee c}^{1} t(t-c)dt, \\
p_{RAA} &= P\{X_{11} < a, X_{21} > b, X_{31} > c, X_{12} > X_{21} \vee X_{31}\} = a\int_{b\vee c}^{1} (t-b)(t-c)dt, \\
p_{RRA} &= P\{X_{11} < a, X_{21} < b, X_{31} > c, X_{12} > X_{22} \vee X_{31}\} = ab\int_{c}^{1} t(t-c)dt, \\
p_{RAR} &= P\{X_{11} < a, X_{21} > b, X_{31} < c, X_{12} > X_{21} \vee X_{32}\} = ac\int_{b}^{1} t(t-b)dt,
\end{aligned}$$

etc.

First we have to noice that

$$M_i(a,a,a) = \frac{1}{3}, i = 1,2,3, \forall a \in [0,1].$$

We prove this for $i = 1$ only. Proof is the same for $i = 2,3$. From (4.1) we have

$$\begin{aligned}
M_1(a,a,a) &= [p_{AAA} + p_{RRR} + (\text{other six probabilities})]_{a=b=c} \\
&= (1+a)\int_a^1 (t-a)^2 dt + \frac{1}{3}a^3 + a^2\int_a^1 t^2 dt + 2(a+a^2)\int_a^1 t(t-a)dt \\
&= \frac{1}{3}(1+a)(1-a)^3 + \frac{1}{3}a^3 + \frac{1}{3}a^2(1-a^3) + 2(a+a^2)\cdot\frac{1}{6}(2-3a+a^3),
\end{aligned}$$

which is easily shown to be equal to $1/3, \forall a \in [0,1]$.

Theorem 3 *Solution to the simultaneous-move three-player game. The game has a unique equilibrium point (a^*, a^*, a^*), and the common equilibrium value $1/3$, where a^* is a unique root in $[0,1]$ of the equation* ($\fallingdotseq .691$)

$$2a^4 = 1 - a + a^2 - a^3. \tag{4.4}$$

(Proof is omitted)

The two-player game is solved in Ref.[8]. Let

$$M_1(a,b) \equiv P\{W_1 | \text{I (II) chooses } a(b)\} = 1 - M_2(a,b).$$

Then it is shown that

$$M_1(a,b) = \frac{1}{2}\{-a^2b + (a+1)(1-b+b^2)\} - I(a \geq b)\frac{1}{2}(a-b)^2,$$

$$\frac{\partial M_1(a,b)}{\partial a} = -ab + \frac{1}{2}(1-b+b^2) - I(a \geq b)(a-b).$$

And we have the following

Theorem 4 *Solution to the simultaneous-move two-player game. The game has a unique saddle point* (g,g) *and the saddle value* $\frac{1}{2}$*, where* $g = \frac{1}{2}(\sqrt{5}-1) \approx 0.61803$ *(For the proof, see* Ref. [8]).

Remark 4 The optimal threshould number is g (golden bisection number) in the two-player game and it increases to $a^* \approx 0.691$ in the three-player game. Furthermore, by considering Theorems 3 and 4 we have a conjecture that the simultaneous-move four-player game has a unique eq.point (a^*, a^*, a^*, a^*) and the common eq.value 1/4, where $a^* \approx 0.738$ is a unique root in $[0,1]$ of the equation $3a^6 = 1 - a + a^2 - a^3 + a^4 - a^5$. Player behaves more cautious as he (or her) has more competitors.

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