On the use of explicit bounds on residues of Dedekind zeta functions taking into account the behavior of small primes (Abridged version)

Stéphane R. LOUBOUTIN
Institut de Mathématiques de Luminy, UMR 6206
163, avenue de Luminy, Case 907
13288 Marseille Cedex 9, FRANCE
loubouti@iml.univ-mrs.fr

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Abstract

Lately, we have obtained explicit upper bounds on $|L(1,\chi)|$ (for primitive Dirichlet characters χ) taking into account the behaviors of χ on a given finite set of primes. This yields explicit upper bounds on residues of Dedekind zeta functions of abelian number fields taking into account the behavior of small primes, and we explained how such bounds yield improvements on lower bounds of relative class numbers of CM-fields whose maximal totally real subfields are abelian. We present here some other applications of such bounds together with new bounds for non-abelian number fields.

1   Introduction

Theorem 1 *(See [Lou04a] and [Lou0?], and [Ram1] and [Ram2] for a slight improvement, and see [BHM], [Le], [MP], [Mos], [MR], [SSW] and [Ste] for various applications). Let S be a given finite set of pairwise distinct rational primes. Then, for any primitive Dirichlet character χ of conductor $q_\chi > 1$ we have*

$$|L(1,\chi)| \le \frac{1}{2}\left\{\prod_{p\in S}\frac{p-1}{|p-\chi(p)|}\right\}\left(\log q_\chi + \kappa_\chi + \omega\log 4 + 2\sum_{p\in S}\frac{\log p}{p-1}\right) + R_S(q_\chi),$$

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where

$$\kappa_\chi = \begin{cases} \kappa_{even} = 2 + \gamma - \log(4\pi) = 0.046191\cdots & \text{if } \chi(-1) = +1 \\ \kappa_{odd} = 2 + \gamma - \log\pi = 1.432485\cdots & \text{if } \chi(-1) = -1, \end{cases}$$

where $\omega \geq 0$ is the number of primes $p \in S$ which do not divide q_χ, and where $R_S(q_\chi)$ is an explicit error term which tends rapidly to zero when q_χ goes to infinity. Moreover, if $S = \emptyset$ or if $S = \{2\}$, then this error term $R_S(q_\chi)$ is always less than or equal to zero, and if none of the prime in S divides q_χ then this error term $R_S(q_\chi)$ is less than or equal to zero for q_χ large enough.

Lemma 2 *Assume that none of the primes $p \in S$ divide q_χ. If χ is even then $R_S(q_\chi) \leq 0$ for $S = \{2\}$ and $q_\chi \geq 3$, $R_S(q_\chi) \leq 0$ for $S = \{2,3\}$ and $q_\chi \geq 3$, and $R_S(q_\chi) \leq 0$ for $S = \{2,3,5\}$ and $q_\chi \geq 3$. If χ is odd then $R_S(q_\chi) \leq 0$ for $S = \{2,3\}$ and $q_\chi \geq 9$, and $R_S(q_\chi) \leq 0$ for $S = \{2,3,5\}$ and $q_\chi \geq 217$.*

Proof. We assume that $S \neq \emptyset$ and that none of the primes $p \in S$ divide q_χ. Hence, $\omega = \#S \geq 1$ and we set $d = \prod_{p \in S} p > 1$.

1. Assume that χ is even. Set

$$\theta(x) = \sum_{n \geq 1} e^{-\pi n^2 x} \quad (x > 0).$$

According to [Lou04a, Theorem 5] [1], we may take

$$R_S(q_\chi) = \frac{\phi(d)}{2d\sqrt{q_\chi}}\left(\kappa_{even} - 2 - \log(q_\chi/4^\omega) - 2\sum_{p|d}\frac{\log p}{p-1}\right) + \tilde{R}_S(q_\chi)$$

where in setting $\delta_{max} = \max\{\delta \geq 1; \ \delta \mid d \text{ and } \mu(\delta) = -1\}$ we have

$$\tilde{R}_S(q_\chi) = -\sum_{\delta|d}\frac{\mu(\delta)}{\delta}\int_1^\infty \theta(4^\omega q_\chi x/\delta^2)\frac{dx}{x}$$

$$\leq \frac{\pi e^{-4^\omega \pi q_\chi/\delta_{max}^2}}{6 \cdot 4^\omega q_\chi}\sum_{\substack{\delta|d \\ \mu(\delta)=-1}}\delta = \frac{\pi e^{-4^\omega \pi q_\chi/\delta_{max}^2}}{3 \cdot 4^{\omega+1}q_\chi}\left(\prod_{p|d}(p+1) - \prod_{p|d}(1-p)\right),$$

in using

$$\int_1^\infty \theta(Ax)\frac{dx}{x} \leq \int_1^\infty \theta(Ax)dx = \sum_{n \geq 1}\frac{e^{-\pi n^2 A}}{\pi n^2 A} \leq \frac{\pi}{6A}e^{-\pi A}.$$

2. Assume that χ is odd. Then (see [Lou0?]),

$$R_S(q_\chi) = -\frac{\pi}{2\sqrt{q_\chi}}\frac{\phi(d)}{d} + \frac{\pi}{12q_\chi}\left(2^{\omega+1}(\frac{\phi(d)}{d})^2 + \frac{\mu(d)}{4^\omega}\phi(d)\right) + \frac{0.124}{q_\chi^{3/2}}R_d$$

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[1]Note the misprint in [Lou04a, page 128, line 2] where the factor 2^ω should read $1/2^\omega$.

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where

$$R_d = \frac{4^\omega}{d^{3/2}}\left(\prod_{p|d}(p^2(p-2)+1)\right) + \frac{d^{3/2}}{8^\omega}\left(\prod_{p|d}(p^2+1)\right).$$

3. Notice that the error term is much more satisfactory for even characters. •

2   Upper bounds for relative class numbers

Corollary 3 *Let $q \equiv 5$ (mod 8), $q \neq 5$, be a prime, let χ_q denote any one of the two conjugate odd quartic characters of conductor q and let h_q^- denote the relative class number of the imaginary cyclic quartic field N_q of conductor q. Then,*

$$h_q^- = \frac{q}{2\pi^2}|L(1,\chi_q)|^2 \leq \frac{q}{A_\chi\pi^2}(\log q + 2 + \gamma - \log(\pi) + \log B_\chi)^2,$$

which implies $h_q^- < q$ for $q \leq C_\chi$, where A_χ, B_χ and C_χ are as follows:

<p align="center">Values of (A_χ, B_χ, C_χ)</p>

	$\chi_q(3) = +1$	$\chi_q(3) = -1$	$\chi_q(3) = \pm i$
$\chi_q(5) = +1$	$(40, 16, 6350867)$	$(160, 192, 2 \cdot 10^{14})$	$(100, 192, 5 \cdot 10^{10})$
$\chi_q(5) = -1$	$(90, 64\sqrt{5}, 10^{10})$	$(360, 768\sqrt{5}, 10^{22})$	$(225, 768\sqrt{5}, 4 \cdot 10^{16})$
$\chi_q(5) = \pm i$	$(65, 64\sqrt{5}, 10^8)$	$(260, 768\sqrt{5}, 10^{18})$	$(325/2, 768\sqrt{5}, 3 \cdot 10^{13})$

Proof. Since $q \equiv 5$ (mod 8), we have $\chi_q(2)^2 = \left(\frac{2}{q}\right) = -1$ and $\chi_q(2) = \pm i$. Set $S = \{p \in \{2,3,5\};\ \chi(p) \neq +1\}$. Then $2 \in S$ and according to Theorem 1 we may choose

$$A_\chi = 8\prod_{p\in S}\left|\frac{p-\chi(p)}{p-1}\right|^2 = 40\prod_{2\neq p\in S}\left|\frac{p-\chi(p)}{p-1}\right|^2$$

and

$$\log B_\chi = \omega\log 4 + 2\sum_{p\in S}\frac{\log p}{p-1} = (\omega+1)\log 4 + 2\sum_{2\neq p\in S}\frac{\log p}{p-1},$$

for according to Lemma 2 we have $R_S(q_\chi) \leq 0$ for $q_\chi \geq 217$. •

Remarks 4 *Using Corollary 3 to alleviate the amount of required relative class number computation, M. Jacobson and the author are now trying to solve the open problem hinted at in [Lou98]: determine the least (or at least one) prime $q \equiv 5$ (mod 8) for which $h_q^- > q$. Indeed, according to Corollary 3, for finding such a q in the range $q < 5 \cdot 10^{10}$, we may assume that $\chi_q(3) = +1$, which amounts to eliminating three quarters of the primes q in this range. In the same way, in the range $q < 3 \cdot 10^{13}$ we may assume that $\chi_q(3) = +1$ or $\chi_q(5) = +1$, which amounts to eliminating 9/16 of the primes q in this range.*

3   Real cyclotomic fields of large class numbers

3.1   Using simplest cubic fields

In [CW], G. Cornell and L. C. Washington used simplest cubic fields to produce real cyclotomic fields $\mathbf{Q}^+(\zeta_p)$ of class number $h_p^+ > p$, where the **simplest cubic fields** are the real cyclic cubic number fields associated with the $\mathbf{Q}$-irreducible cubic polynomials $P_m(x) = x^3 - mx^2 - (m+3)x - 1$ of discriminants

$$d_m = \Delta_m^2 \quad \text{where } \Delta_m := m^2 + 3m + 9.$$

Since $-x^3 P_m(1/x) = P_{-m-3}(x)$, we may assume that $m \geq -1$. We let

$$\rho_m = \frac{1}{3}\left(2\sqrt{\Delta_m}\cos\left(\frac{1}{3}\arctan(\frac{\sqrt{27}}{2m+3})\right) + m\right) = \sqrt{\Delta_m} - \frac{1}{2} + O(\frac{1}{\sqrt{\Delta_m}}) \quad (1)$$

denote the only positive root of $P_m(\overset{\bullet}{x})$. Moreover, we will assume that the conductor of K_m is equal to Δ_m, which amounts to asking that (i) $m \not\equiv 0$ (mod 3) and Δ_m is squarefree, or (ii) $m \equiv 0, 6$ (mod 9) and $\Delta_m/9$ is squarefree (see [Wa, Prop. 1 and Corollary]). In that situation, $\{-1, \rho_m, -1/(\rho_m + 1)\}$ generate the full group of algebraic units of K_m and the regulator of K_m is

$$\text{Reg}_{K_m} = \log^2 \rho_m - (\log \rho_m)(\log(1 + \rho_m)) + \log^2(1 + \rho_m), \quad (2)$$

which in using (1) yields

$$\text{Reg}_{K_m} = \frac{1}{4}\log^2 \Delta_m - \frac{\log \Delta_m}{\sqrt{\Delta_m}} + O(\frac{\log \Delta_m}{\Delta_m}) \leq \frac{1}{4}\log^2 \Delta_m. \quad (3)$$

Lemma 5 *The polynomial $P_m(x)$ has no root mod 2, has at least one root mod 3 if and only if $m \equiv 0$ (mod 3), and has at least one root mod 5 if and only if $m \equiv 1$ (mod 5). Hence, if Δ_m is square-free, then 2 and 3 are inert in K_m, and if $m \not\equiv 1$ (mod 5) then 5 is also inert in K_m.*

As in [Lou02b, Section 5.1], we let χ_{K_m} be the primitive cubic Dirichlet characters modulo Δ_m associated with K_m satisfying

$$\chi_{K_m}(2) = \begin{cases} \omega^2 & \text{if } m \equiv 0 \pmod 2 \\ \omega & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

Since the regulators of these K_m's are small, they should have large class numbers. In fact, we proved (see [Lou02d, (12)]):

$$h_{K_m} = \frac{\Delta_m}{4\text{Reg}_{K_m}}|L(1, \chi_{K_m})|^2 \geq \frac{\Delta_m}{e \log^3 \Delta_m} \quad (4)$$

Corollary 6 *Assume that $m \geq -1$ is such that Δ_m is squarefree. Then, $\chi_{K_m}(2) \neq +1$, $\chi_{K_m}(3) \neq +1$ and*

$$h_{K_m} \leq \begin{cases} \Delta_m/60 & \text{if } m \geq 6, \\ \Delta_m/100 & \text{if } m \not\equiv 1 \pmod 5 \text{ and } m \geq 9. \end{cases}$$

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Proof. If a prime $l \geq 2$ is inert in K_m then $\chi_{K_m}(l) \in \{\exp(2i\pi/3), \exp(4i\pi/3)\}$. According to the previous Lemma, to Theorem 1 (with $S = \{2,3\}$ and $S = \{2,3,5\}$) and to Lemma 2, we have

$$|L(1, \chi_{K_m})|^2 \leq \begin{cases} (\log \Delta_m + \kappa_{even} + \log(192))^2/91, \\ 16(\log \Delta_m + \kappa_{even} + \log(768\sqrt{5}))^2/2821 & \text{if } m \not\equiv 1 \pmod 5. \end{cases}$$

Now, according to (4) and (3), the desired results follow for $mm \geq 95000$. The numerical computation of the class numbers of the remaining K_m provides us with the desired bounds (see [Lou02b]). •

From now on, we assume (i) that $p = \Delta_m = m^2 + 3m + 9$ is prime and (ii) that $p \equiv 1 \pmod{12}$. In that case, both K_m and $k_m := Q(\sqrt{\Delta_m})$ are subfields of the real cyclotomic field $Q^+(\zeta_p)$ and the product $h_2 h_3$ of the class numbers $h_2 := h_{k_m}$ and $h_3 := h_{K_m}$ of k_m and K_m divides the class number h_p^+ of $Q^+(\zeta_p)$. Since $h_3 \leq \Delta_m/60$, $h_2 h_3 \geq \Delta_m$ implies $h_2 \geq 60$, hence $h_2 \geq 61$ (for h_2 is odd), and Cohen-Lenstra heuristics predict that real quadratic number fields of prime conductors with class numbers greater than or equal to 61 are few and far between. Hence, such simplest cubic fields K_m of prime conductors $\Delta_m = m^2 + 3m + 9 \equiv 1 \pmod 4$ with $h_2 h_3 > \Delta_m$ are few and far between. As we have at hand a very efficient method for computing class numbers of real quadratic fields (see [Lou02c] and [WB]), we used this explicit necessary condition $h_2 \geq 61$ to compute (using [Lou02b]) the class numbers of only few of the simplest cubic fields K_m of prime conductors $\Delta_m \equiv 1 \pmod{12}$ with $-1 \leq m \leq 1066285$ to obtain the following Table. Notice that the authors of [CW] and [SWW] only came up with one such K_m, the one for $m = 106253$.

Least values of $m \geq -1$ for which $\Delta_m = m^2 + 3m + 9$ is prime and $h_2 h_3 \geq \Delta_m$

m	$|\theta(\chi_{K_m})|$	$\arg W(\chi_{K_m})$	h_2	h_3	$h_2 h_3/\Delta_m$
102496	$20.268\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3}$	891	13152913	$1.115\cdots$
106253	$34.364\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	2685	6209212	$1.476\cdots$
319760	$202.162\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1887	57772549	$1.066\cdots$
554869	$88.861\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3}$	7983	93739324	$2.430\cdots$
726845	$20.938\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	13533	176702419	$4.526\cdots$
791021	$129.812\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1737	445142272	$1.235\cdots$
796616	$357.252\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1155	696739264	$1.268\cdots$
839401	$293.373\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \pi$	1575	554491633	$1.239\cdots$
906437	$93.697\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3})$	1955	469911916	$1.118\cdots$
1066285	$140.662\cdots$	$\frac{1}{3}\arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3}$	5389	473034223	$2.242\cdots$

Finally, it is much more efficient to use simplest sextic fields to produce real cyclotomic fields of prime conductors and class numbers greater than their conductors(see [Lou04c]).

3.2 Using simplest quartic fields

In [Laz1] and [Laz2], A. Lazarus dealt with various class number problems for the so called **simplest quartic fields**, the real cyclic quartic number fields associated with the quartic polynomials $P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1$ of discriminants

$$d_m = 4\Delta_m^3 \quad \text{where} \quad \Delta_m := (m^2 + 16)^3.$$

Since $P_m(-x) = P_{-m}(x)$, we may and we will assume that $m \geq 0$. The reader will easily check (i) that $P_m(x)$ has no rational root, (ii) that $P_m(x)$ is $\mathbf{Q}$-irreducible, except for $m = 0$ and $m = 3$, and (iii) that $P_m(x)$ has a only one root $\rho_m > 1$. Set $\beta_m = \rho_m - \rho_m^{-1} > 0$. Then, $\beta_m^2 - m\beta_m - 4 = 0$ and $\beta_m = (m + \sqrt{\Delta_m})/2$. In particular, $k_m = \mathbf{Q}(\sqrt{\Delta_m})$ is the quadratic subfield of the cyclic quartic field K_m. It is known that h_{k_m} divides h_{K_m}, and we set $h_{K_m}^* = h_{K_m}/h_{k_m}$. Since $\rho_m > 1$ and $\rho_m^2 - \beta_m\rho_m - 1 = 0$, we obtain

$$\rho_m = \frac{1}{2}\left(\frac{m + \sqrt{\Delta_m}}{2} + \sqrt{\frac{\Delta_m + m\sqrt{\Delta_m}}{2}}\right) = \sqrt{\Delta_m}\left(1 - \frac{3}{\Delta_m} + O(\frac{1}{\Delta_m^2})\right)$$

(use $m = \sqrt{\Delta_m - 16}$),

$$\rho_m' := \frac{1}{2}\left(\frac{m - \sqrt{\Delta_m}}{2} + \sqrt{\frac{\Delta_m - m\sqrt{\Delta_m}}{2}}\right) = 1 - \frac{2}{\sqrt{\Delta_m}} + O(\frac{1}{\Delta_m}),$$

and

$$\text{Reg}_{\rho_m}^* = \log^2 \rho_m + \log^2 \rho_m' = \frac{1}{4}\log^2 \Delta_m - \frac{3\log \Delta_m}{\Delta_m} + O(\frac{1}{\Delta_m}) \leq \frac{1}{4}\log^2 \Delta_m \quad (5)$$

for $m \geq 1$. We will say that K_m is a **simplest quartic field** if $m \geq 1$ is such that Δ_m is squarefree (which implies m odd and $m \neq 3$).

Proposition 7 *Assume that $m \geq 1$ is odd and that $\Delta_m = m^2 + 16$ is prime. Then, the discriminant of the real quadratic subfield $k_m = \mathbf{Q}(\sqrt{\Delta_m})$ of K_m is equal to Δ_m, the discriminant of K_m is equal to Δ_m^3, its conductor is equal to Δ_m, the class numbers of K_m and k_m are odd, and (see [Lou04b])*

$$h_{K_m}^* = \frac{\Delta_m}{4\text{Reg}_{K_m}^*}|L(1, \chi_{K_m})|^2 \geq \frac{2\Delta_m}{3e(\log \Delta_m + 0.35)^4} \quad (6)$$

where χ_{K_m} is any one of the two conjugate primitive quartic Dirichlet characters modulo Δ_m associated with K_m. Moreover, $\chi_{K_m}(2) = -1$, and $m \geq 5$ implies

$$h_{K_m}^* < \Delta_m/26.$$

Proof. According to the class number formula (6), to Theorem 1 (with $S = \{2\}$) and Lemma 2 which yield

$$|L(1, \chi_{K_m})|^2 \leq (\log \Delta_m + \kappa_{odd} + \log(16))^2/36,$$

and to the asymptotic (5), we have $h^*_{K_m} \leq \Delta_m/(36 + o(1))$. Hence, $h^*_{K_m} < \Delta_m/24$. for $m \geq 148000$. The numerical computation of the class numbers of the remaining K_m provides us with the desired bound (see [Lou02b]). •

Since, $h_{K_m} = h_{k_m} h^*_{K_m} \geq \Delta_m$ and $h^*_{K_m} < \Delta_m/26$ imply $h_{k_m} \geq 27$ (for h_{k_m} is odd), and Cohen-Lenstra heuristics predict that real quadratic number fields of prime discriminants with class numbers greater than or equal to 27 are few and far between. Hence such simplest quartic fields K_m of prime conductors $\Delta_m = m^2 + 16$ with $h_{K_m} > \Delta_m$ are few and far between. As we have at hand a very efficient method for computing rigorously class numbers of real quadratic fields (see [Lou02c] and [WB]), we used this explicit necessary condition $h_{k_m} \geq 27$ to compute only few of the class numbers of the simplest quartic fields K_m of prime conductors $\Delta_m = m^2 + 16 \equiv 1 \pmod 4$ with $1 \leq m \leq 1680401$ to obtain the following Table. Notice that G. Cornell and L. C. Washington did not find any such K_m (see [CW, bottom of page 268]).

Least values of $m \geq 1$ for which $\Delta_m = m^2 + 16$ is prime and $h_{K_m} \geq \Delta_m$

m	Δ_m	h_{k_m}	$h^*_{K_m}$	h_{K_m}/Δ_m
524285	274874761241	1911	181442581	$1.261\cdots$
1680401	2823747520817	1537	1878644993	$1.022\cdots$

4   The imaginary cyclic quartic fields with ideal class groups of exponent ≤ 2

We explain how one could alleviate the determination in [Lou95] of all the non-quadratic imaginary cyclic fields of 2-power degrees $2n = 2^r \geq 4$ with ideal class groups of exponents ≤ 2 (the time consuming part bieng the computation of the relative class numbers of the fields sieved by Proposition 9). To simplify, we will only deal with imaginary cyclic quartic fields of odd conductors, and we will prove Proposition 9 below. Recall the following result whose proof makes use of Theorem 1 (for even characters) with $S = \{2\}$:

Theorem 8 *(See [Lou03, Theorem 22]). Let K be an imaginary cyclic quartic field of conductor f_K, Let k, f_k and χ_K denote the real quadratic subfield of K, the conductor of k, and any one of the two conjugate primitive quartic Dirichlet characters modulo f_K associated with K. Then,*

$$h^-_K \geq \frac{C_K f_K}{e\pi^2(\log f_k + \kappa_k)\log(f_k f_K^2)}, \tag{7}$$

where

$$C_K = \frac{32}{|2 - \chi_K(2)|^2} = \begin{cases} 32 & \text{if } \chi_K(2) = +1, \\ 32/9 & \text{if } \chi_K(2) = -1, \\ 32/5 & \text{if } \chi_K(2) = \pm i, \end{cases}$$

and

$$\kappa_k = \begin{cases} 2 + \gamma - \log(4\pi) = 0.046\cdots & \text{if } f_k \equiv 1 \pmod{8} \\ 2 + \gamma - \log(\pi/4) = 2.818\cdots & \text{if } f_k \equiv 5 \pmod{8}. \end{cases}$$

Proposition 9 *Assume that the exponent of the ideal class group of an imaginary cyclic quartic field K of odd conductor f_K is less than or equal to 2. Then, $f_k \leq 1889$ and $f_K \leq 10^7$ (where k is the real quadratic subfield of K). Moreover, whereas there are 1 377 361 imaginary cyclic fields K of odd conductors $f_K \leq 10^7$ and such that $f_k \leq 1889$, only 400 out of them may have their ideal class groups of exponents ≤ 2, the largest possible conductor being $f_K = 5619$ (for $f_k = 1873$ and $f_{K/k} := f_K/f_k = 3$).*

Proof. It is known that if the exponent of the ideal class group of K of odd conductor f_K is ≤ 2, then $f_k \equiv 1 \pmod 4$ is prime and

$$h_K^- = 2^{t_{K/k}-1}, \tag{8}$$

where $t_{K/k}$ denotes the number of prime ideals of k which are ramified in K/k (see [Lou95, Theorems 1 and 2]). Conversely, for a given real quadratic field k of prime conductor $f_k \equiv 1 \pmod 4$, the conductors f_K of the imaginary cyclic quartic fields K of odd conductors and containing k are of the form $f_K = f_k f_{K/k}$ for some positive square-free integer $f_{K/k} \geq 1$ relatively prime with f_k and such that

$$(f_k - 1)/4 + (f_{K/k} - 1)/2 \text{ is odd} \tag{9}$$

(in order to have $\chi_K(-1) = -1$, i.e. in order to guarantee that K is imaginary). Moreover, for such a given k and such a given $f_{K/k}$, there exists only one imaginary cyclic quartic field K containing k and of conductor $f_K = f_k f_{K/k}$, and for this K we have

$$t_{K/k} = 1 + \sum_{p \mid f_{K/k}} (3 + (\frac{p}{f_k}))/2, \tag{10}$$

where $(\frac{\cdot}{f_k})$ denote the Legendre's symbol. Finally, if we let ϕ_k denote any one of the two conjugate quartic characters modulo a prime $f_k \equiv 1 \pmod 4$, then $\chi_K(n) = \phi_k(n)(\frac{n}{f_{K/k}})$, where $(\frac{\cdot}{f_{K/k}})$ denote the Jacobi's symbol, and

$$\chi_K(2) = \begin{cases} \phi_k(2) = 1 & \text{if } f_k \equiv 1 \pmod 8 \text{ and } 2^{\frac{f_k-1}{4}} \equiv 1 \pmod{f_k} \\ \phi_k(2) = -1 & \text{if } f_k \equiv 1 \pmod 8 \text{ but } 2^{\frac{f_k-1}{4}} \not\equiv 1 \pmod{f_k} \\ -\phi_k(2) = \pm i & \text{if } f_k \equiv 5 \pmod 8. \end{cases} \tag{11}$$

Hence, we may easily compute κ_k, c_K and $t_{K/k}$ from f_k and $f_{K/k}$. In particular, we easily obtain that there are 1 377 361 imaginary cyclic fields K of odd conductors $f_K \leq 10^7$ and such that $f_k \leq 1889$, and that $c_K = 32$ for 149 187 out of them, $c_K = 32/5$ for 938 253 out of them, and $c_K = 32/9$ for 289 921 out of them. Now, let P_n denote the product of the first n odd primes $3 = p_1 < 5 = p_2 < \cdots < p_n < \cdots$ (hence, $P_0 = 1$, $P_1 = 3$, $P_2 = 15$, $\cdots$). There are two cases to consider:

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1. If $\chi_k(2) = +1$. Then, $f_k \equiv 1 \pmod 8$ is prime, $\kappa_k < 0.05$, $c_K \geq 32/9$, $f_K = f_k f_{K/k}$ where $f_{K/k}$ is a product of $n \geq 0$ distinct odd primes. Hence, $f_{K/k} \geq P_n$, $t_{K/k} \leq 1 + 2n$, $h_K^- = 2^{t_{K/k}-1} \leq 4^n$ and using (7) we obtain

$$F_k(n) := \frac{32 f_k P_n}{9 e \pi^2 4^n (\log f_k + 0.05) \log(f_k^3 P_n^2)} \leq 1.$$

Assume that $f_k \geq 36$. Then $3 f_k^{3/2} \geq 5^4$ and for $n \geq 1$ we have $p_{n+1} \geq p_2 = 5$, $P_n \geq P_1 = 3$ and

$$\frac{F_k(n+1)}{F_k(n)} = \frac{p_{n+1} \log(f_k^{3/2} P_n)}{4 \log(p_{n+1} f_k^{3/2} P_n)} \geq \frac{5 \log(f_k^{3/2} P_n)}{4 \log(5 f_k^{3/2} P_n)} \geq \frac{5 \log(3 f_k^{3/2})}{4 \log(15 f_k^{3/2})} \geq 1.$$

Since we clearly have $F_k(1) \leq F_k(0)$, we obtain $\min_{n \geq 0} F_k(n) = F_k(1)$ and

$$\frac{8 f_k}{3 e \pi^2 (\log f_k + 0.05) \log(9 f_k^3)} = F_k(1) \leq F_k(n) \leq 1,$$

which implies $f_k \leq 1899$, hence $f_k \leq 1889$ (for $f_k \equiv 1 \pmod 8$ must be prime). Hence, using (7), we obtain

$$h_K^- \geq \frac{32 f_K}{9 e \pi^2 (\log(1889) + 0.05) \log(1889 f_K^2)}.$$

Let now n denote the number of distinct prime divisors of f_K. Then $f_K \geq P_n$, $t_{K/k} \leq 2(n-1) + 1$ and $h_K^- = 2^{t_{K/k}-1} \leq 4^{n-1}$. Hence, using (7), we obtain

$$4^{n-1} \geq \frac{32 P_n}{9 e \pi^2 (\log(1889) + 0.05) \log(1889 P_n^2)},$$

which implies $n \leq 7$, $h_K^- \leq 4^6$,

$$4^6 \geq \frac{32 f_K}{9 e \pi^2 (\log(1889) + 0.05) \log(1889 f_K^2)}$$

and yields $f_K \leq 10^7$.

2. If $\chi_k(2) = -1$. Then $f_k \equiv 5 \pmod 8$ is prime, $\kappa_k \leq 2.82$, $c_K \geq 32/5$ and we follow the previous case. We obtain $f_k \leq 1329$, hence $f_k \leq 1301$ (for $f_k \equiv 5 \pmod 8$ must be prime), $n \leq 7$, $h_K^- \leq 4^6$ and $f_K \leq 7 \cdot 10^6$.

Hence, the first assertion Proposition 9 is proved. Now, for a given odd prime $f_k \leq 1889$ equal to 1 modulo 4, and for a given odd square-free integer $f_{K/k} \leq 10^7/f_k$ relatively prime with f_k, we compute κ_k, $t_{K/k}$ (using (10)), c_K (using (11)) and use (7) and (8) to deduce that if the exponent of the ideal class group of K is less than or equal to 2, then

$$2^{t_{K/k}-1} \geq \frac{c_K f_k f_{K/k}}{e \pi^2 (\log f_k + \kappa_k) \log(f_k^3 f_{K/k}^2)}. \tag{12}$$

Now, an easy calculation yields that only 400 out of 1 377 361 imaginary cyclic fields K of odd conductors and such that $f_k \leq 1889$ and $f_K \leq 10^7$ satisfy (12), and the second assertion of the Proposition is proved. •

5 The non-abelian case

We showed in [Lou03] how taking into account the behavior of the prime 2 in CM-fields can greatly improve upon the upper bounds on the root numbers of the normal CM-fields with abelian maximal totally real subfields of a given (relative) class number. We now explain how we can improve upon previously known upper bounds for residues of Dedekind zeta functions of non-necessarily abelian number fields by taking into accound the behavior of the prime 2. Let K be a number field of degree $m \geq 1$. We set

$$\Pi_K(2, s) := \prod_{\mathcal{P}|2}(1 - (N(\mathcal{P}))^{-s})^{-1}$$

(which is ≥ 1 for $s > 0$) and $\Pi_K(2) := \Pi_K(2, 1)$. In particular, $\Pi_K(2)/\Pi_\mathbf{Q}^m(2) \leq 1$. However, if 2 is inert in K, then $\Pi_K(2)/\Pi_\mathbf{Q}^m(2) = 1/(2^m - 1)$ is small.

Theorem 10 *Let K be a number field of degree $m \geq 3$ and root discriminant $\rho_K = d_K^{1/m}$. Set $v_m = (m/(m-1))^{m-1} \in [9/4, e)$, and $E(x) := (e^x - 1)/x = 1 + O(x)$ for $x > 0$. Then,*

$$\mathrm{Res}_{s=1}(\zeta_K(s)) \leq (e/2)^{m-1} v_m \frac{\Pi_K(2)}{\Pi_\mathbf{Q}^m(2)} \left(\log \rho_K + (\log 4) E(\frac{\log 4}{\log \rho_K}) \right)^{m-1} \quad (13)$$

Moreover, $0 < \beta < 1$ and $\zeta_K(\beta) = 0$ imply

$$\mathrm{Res}_{s=1}(\zeta_K(s)) \leq (1 - \beta)(e/2)^m \frac{\Pi_K(2)}{\Pi_\mathbf{Q}^m(2)} \left(\log \rho_K + (\log 4) E(\frac{\log 4}{\log \rho_K}) \right)^m. \quad (14)$$

Proof. We only prove (13), the proof of (14) being similar. According to [Lou01, Section 6.1] but using the bound

$$\zeta_K(s) \leq \frac{\Pi_K(2, s)}{\Pi_\mathbf{Q}^m(2, s)} \zeta^m(s)$$

instead of the bound $\zeta_K(s) \leq \zeta^m(s)$, we have

$$
\begin{aligned}
\mathrm{Res}_{s=1}(\zeta_K(s)) &\leq \frac{\Pi_K(2)}{\Pi_\mathbf{Q}^m(2)} \left(\frac{e \log d_K}{2(m-1)} \right)^{m-1} g(s_K) \\
&= (e/2)^{m-1} v_m \frac{\Pi_K(2)}{\Pi_\mathbf{Q}^m(2)} (\log \rho_K)^{m-1} g(s_K),
\end{aligned}
$$

where $s_K = 1 + 2(m-1)/\log d_K \in [1,6]$ and

$$g(s) := \frac{\Pi_K(2,s)/\Pi_K(2)}{\Pi_{\mathbf{Q}}^m(2,s)/\Pi_{\mathbf{Q}}^m(2)} \leq h(s) := \Pi_{\mathbf{Q}}^m(2)/\Pi_{\mathbf{Q}}^m(2,s)$$

(for $\Pi_K(2,s) \leq \Pi_K(2,1) = \Pi_K(2)$ for $s \geq 1$). Now, $\log h(1) = 0$ and $(h'/h)(s) = \frac{m\log 2}{2^s - 1} \leq m\log 2$ for $s \geq 1$. Hence,

$$\log h(s_K) \leq (s_K - 1)m\log 2 = \frac{(m-1)\log 4}{\log \rho_K},$$

$$g(s_K) \leq h(s_K) \leq \left(\exp(\frac{\log 4}{\log \rho_K})\right)^{m-1},$$

and (13) follows. •

Corollary 11 *(Compare with [Lou01, Theorems 12 and 14] and [Lou03, Theorems 9 and 22]). Set $c = 2(\sqrt{3} - 1)^2 = 1.07 \cdots$ and $v_m := (m/(m-1))^{m-1} \in [2, e)$. Let N be a normal CM-field of degree $2m > 2$, relative class number h_N^- and root discriminant $\rho_N = d_N^{1/2m} \geq 650$. Assume that N contains no imaginary quadratic subfield (or that the Dedekind zeta functions of the imaginary quadratic subfields of N have no real zero in the range $1 - (c/\log d_N) \leq s < 1$). Then,*

$$h_N^- \geq \frac{c}{2mv_m e^{c/2-1}} \left(\frac{4\sqrt{\rho_N}}{3\pi e(\log \rho_N + (\log 4)E(\frac{\log 4}{\log \rho_N}))}\right)^m. \qquad (15)$$

Hence, $h_N^- > 1$ for $m \geq 5$ and $\rho_N \geq 14610$, and for $m \geq 10$ and $\rho_N \geq 9150$. Moreover, $h_N^- \to \infty$ as $[N : \mathbf{Q}] = 2m \to \infty$ for such normal CM-fields N of root discriminants $\rho_N \geq 3928$.

Proof. To prove (15), follow the proof of [Lou01, Theorems 12 and 14] and [Lou03, Theorems 9 and 22], but now make use of Theorem 10 instead of [Lou01, Theorem 1] and finally notice that

$$\frac{\Pi_N(2)}{\Pi_K(2)/\Pi_{\mathbf{Q}}^m(2)} = 2^m \Pi_N(2)/\Pi_K(2) = 2^m \prod_{\mathcal{P}|(2)} \left(1 - \frac{\chi(\mathcal{P})}{N(\mathcal{P})}\right)^{-1} \geq (4/3)^m$$

(χ is the quadratic character associated with the quadratic extension N/K, and $\mathcal{P}$ ranges over the primes ideals of K lying above the rational prime 2). •

References

[BHM]   Y. Bugeaud, G. Hanrot and M. Mignotte, Sur l'équation diophantienne $(x^n - 1)/(x - 1) = y^q$. III. *Proc. London Math. Soc. (3)* **84** (2002), 59–78.

[CL]   H. Cohen and H. W. Lenstra. Heuristics on class groups of number fields. *Lecture Notes in Math.* **1068** (1984), 33–62. MR **85j**:11144.

Applications of bounds taking into account the behavior of small primes

[CW]   G. Cornell and L. C. Washington. Class numbers of cyclotomic fields. *J. Number Theory* **21** (1985), 260–274. MR **87d**:11079.

[Laz1]   A. J. Lazarus. Class numbers of simplest quartic fields. *Number theory (Banff, AB, 1988)*, 313–323, *Walter de Gruyter, Berlin*, 1990. MR **92d**:11119.

[Laz2]   A. J. Lazarus. On the class number and unit index of simplest quartic fields. *Nagoya Math. J.* **121** (1991), 1–13. MR **92a**:11129.

[Le]   M. Le. Upper bounds for class numbers of real quadratic fields. *Acta Arith.* **68** (1994), 141–144.

[Lou95]   S. Louboutin. Determination of all non-quadratic imaginary cyclic number fields of 2-power degrees with ideal class groups of exponents $\leq$ 2. *Math. Comp.* **64** (1995), 323–340.

[Lou98]   S. Louboutin. Computation of relative class numbers of imaginary abelian number fields. *Experimental Math.* **7** (1998), 293–303.

[Lou01]   S. Louboutin. Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields. *Canad. J. Math.* **53** (2001), 1194–1222.

[Lou02a] S. Louboutin. Majorations explicites de $|L(1,\chi)|$ (quatrième partie). *C. R. Acad. Sci. Paris Sér. I Math.* **334** (2002), 625–628.

[Lou02b] S. Louboutin. Efficient computation of class numbers of real abelian number fields. *Lectures Notes in Computer Science* **2369** (2002), 625–628.

[Lou02c] S. Louboutin. Computation of class numbers of quadratic number fields. *Math. Comp.* **71** (2002), 1735–1743.

[Lou02d] S. Louboutin. The exponent three class group problem for some real cyclic cubic number fields. *Proc. Amer. Math. Soc.* **130** (2002), 353–361.

[Lou03]   S. Louboutin. Explicit lower bounds for residues at $s = 1$ of Dedekind zeta functions and relative class numbers of CM-fields. *Trans. Amer. Math. Soc.* **355** (2003), 3079–3098.

[Lou04a] S. Louboutin. Explicit upper bounds for values at $s = 1$ of Dirichlet L-series associated with primitive even characters. *J. Number Theory* **104** (2004), 118–131.

[Lou04b] S. Louboutin. The simplest quartic fields with ideal class groups of exponents $\leq$ 2. *J. Math. Soc. Japan* **56** (2004), to appear.

[Lou04c] S. Louboutin. Class numbers of real cyclotomic fields. *Publ. Math. Debrecen*, to appear.

[Lou0?]   S. Louboutin. Explicit upper bounds for $|L(1,\chi)|$ for primitive characters χ. *Q. J. Math.*, to appear.

[Mos]   C. Moser Nombre de classes d'une extension cyclique réelle de $\mathbf{Q}$ de degré 4 ou 6 et de conducteur premier. *Math. Nachr.* **102** (1981), 45–52.

[MP]   C. Moser et J.-J. Payan. Majoration du nombre de classes d'un corps cubique de conducteur premier. *J. Math. Soc. Japan* **33** (1981), 701–706.

[MR]   M. Mignotte M. and Y. Roy, Minorations pour l'équation de Catalan. *C. R. Acad. Sci. Paris* **324** (1997), 377–380.

Applications of bounds taking into account the behavior of small primes

[Ram1]   O. Ramaré. Approximate formulae for $L(1,\chi)$. *Acta Arith.* **100** (2001), 245–256.

[Ram2]   O. Ramaré. Approximate formulae for $|L(1,\chi)|$. II. *Acta Arith.*, to appear.

[SSW]   R. G. Stanton, C. Sudler and H. C. Williams. An upper bound for the period of the simple continued fraction for $\sqrt{D}$. *Pacific J. Math.* **67** (1976), 525–536.

[Ste]   R. Steiner. *Class number bounds and Catalan's equation.* Math. Comp. **67** (1998), 1317–1322.

[SWW]   E. Seah, L. C. Washington and H. C. Williams. The calculation of a large cubic class number with an application to real cyclotomic fields. *Math. Comp.* **41** (1983), 303–305. MR **84m:12008**.

[Wa]   L. C. Washington. Class numbers of the simplest cubic fields. *Math. Comp.* **48** (1987), 371–384. MR **88a:11107**.

[WB]   H. C. Williams and J. Broere. A computational technique for evaluating $L(1,\chi)$ and the class number of a real quadratic field. *Math. Comp.* **30** (1976), 887–893. MR54# 2623.